Automorphism groups and limit laws of random nonrigid structures.

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Let $\{R_1, ..., R_\rho\}$ be a finite set of relation symbols, and L the corresponding language.

$$\mathbf{S}_n := \{\mathcal{M} : \mathcal{M} \text{ is an } L\text{−structure with universe } M = \{1, ..., n\}\}$$

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For an L−structure $\mathcal{M}$.

$$Aut(\mathcal{M}) := \{f : f \text{ is an isomorphism from } \mathcal{M} \text{ to } \mathcal{M}\}$$

Note that $Aut(\mathcal{M})$ is a group under composition. If $Aut(\mathcal{M}) = \{Id_{\mathcal{M}}\}$ we say that $Aut(\mathcal{M})$ is trivial, and $\mathcal{M}$ is called rigid.

Fagin[1] proved that almost all structures in $\mathbf{S}$ are rigid i.e.

$$\lim_{n\to\infty} \frac{|\{\mathcal{M} \in \mathbf{S}_n : Aut(\mathcal{M}) = \{Id_{\mathcal{M}}\}\}|}{|\mathbf{S}_n|} = 1$$

[1] R. Fagin, *The number of finite relational structures*, Discrete Mathematics (1977)
[2] R. Fagin, *Probabilities on finite model theory*, J. Symbolic Logic 41 (1976), no.1
[3] Glebskii et.al. *Volume and fraction of Satisfiability...* , Kibernetyka Vol. 2 (1969)

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On the other hand Fagin[2] and independently Glebskii et. al.[3] proved that $\mathbf{S}$ has a zero-one law i.e.

$$\text{For each sentence } \varphi \in L, \lim_{n\to\infty} \frac{|\{\mathcal{M} \in \mathbf{S}_n : \mathcal{M} \models \varphi\}|}{|\mathbf{S}_n|} = 1 \text{ or } 0$$

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Combined, these results give us that the rigid structures in $\mathbf{S}$ has a zero-one law.

Is it possible generalize this?

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For $f \in Aut(\mathcal{M})$ let $Spt(f) = \{a \in M : f(a) \neq a\}$.

What if we restrict $\mathbf{S}_n$ to only $\mathcal{M}$ s.t. $|Aut(\mathcal{M})| > 1$, or more general; there is $f \in Aut(\mathcal{M})$ s.t. $|Spt(f)| > t$ for some $t \in \mathbb{N}$. Call this set $\mathbf{S}_n(spt > t)$

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- What does the group $Aut(\mathcal{M})$ look like asymptotically (trivial in the rigid case)?

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What if we instead restrict $\mathbf{S}_n$ to $\mathcal{M}$ such that $G \cong Aut(\mathcal{M})$ for some group G.

We have found many answers. In short:
The automorphism groups and their support are as small and simple as allowed, and we have a convergence law.

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Zero-One law in more deconstructed cases; For $\mathcal{A} \in \mathbf{S}$ without fix-points and $H \leq Aut(A)$ not fixating $\mathcal{A}$ let:

$$\mathbf{S}_n(\mathcal{A}, H) := \{\mathcal{M} \in \mathbf{S}_n : \mathcal{A} \subseteq \mathcal{M}, H \leq Aut(\mathcal{M})\}$$

Theorem
$\boldsymbol{S}(\mathcal{A}, H)$ has a zero-one law.

Some convergence laws then follows from finite unions over $\mathcal{A}$ and H.

Previous work has been done by Cameron[4] who studied nonrigid unlabeled undirected graphs. He showed, among other things that

$$\lim_{n\to\infty} \frac{|\{\mathcal{M} \in \mathbf{S}_n : G \cong Aut(\mathcal{M})\}|}{|\{\mathcal{M} \in \mathbf{S}_n : G \leq Aut(\mathcal{M})\}|}$$

always converges to a rational number, something we now have generalized to a general vocabulary (labeled and unlabeled).

[4] P. J. Cameron, *On graphs with given Automorphism Group*, European Journal of Combinatorics (1980)

$$Spt^*(\mathcal{M}) = \{a \in \mathcal{M} : a \in Spt(f) \text{ for some } f \in Aut(\mathcal{M})\}$$

Define (for any $t \in \mathbb{N}$):

$$\mathbf{S}_n(spt \geq t) := \{\mathcal{M} \in \mathbf{S}_n : \exists f \in Aut(\mathcal{M}), |spt(f)| > t\}$$

$$\mathbf{S}_n(spt^* \geq t) := \{\mathcal{M} \in \mathbf{S}_n : |Spt^*| \geq t\}$$

similarly define $\mathbf{S}_n(spt = t), \mathbf{S}_n(t_0 \leq spt \leq t_1), \mathbf{S}_n(spt^* = t)$ etc.

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For $\mathcal{A} \in \mathbf{S}$ such that $Aut(\mathcal{A})$ has no fixed point, and $H \leq Aut(A)$ such that H has no fixed point. Then

$$\mathcal{M} \in \mathbf{S}_n(\mathcal{A}, H)$$

if $\mathcal{M} \in \mathbf{S}_n$ such that there is an embedding $f : \mathcal{A} \to \mathcal{M}, f(A) = Spt^*(\mathcal{M})$ and $\{f\sigma f^{-1} : \sigma \in H\}$ is a subgroup of $Aut(\mathcal{M}) \restriction Spt^*(\mathcal{M})$.