Simple structures axiomatized by almost sure theories

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Special session on model theory and limit structures

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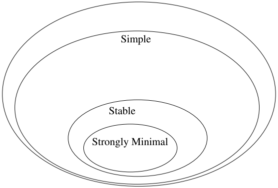
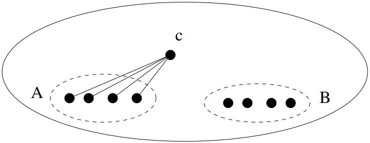


The first example of simple structure which is not stable is usually the Rado graph $\mathcal{G}$ axiomatized by the following extension properties:
For each finite disjoint $A, B \subseteq \mathcal{G}$ there exists $c \in \mathcal{G}$ such that:



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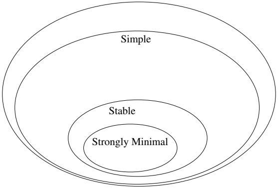

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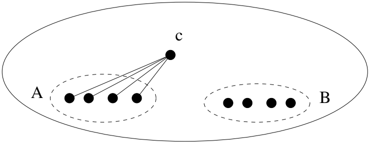
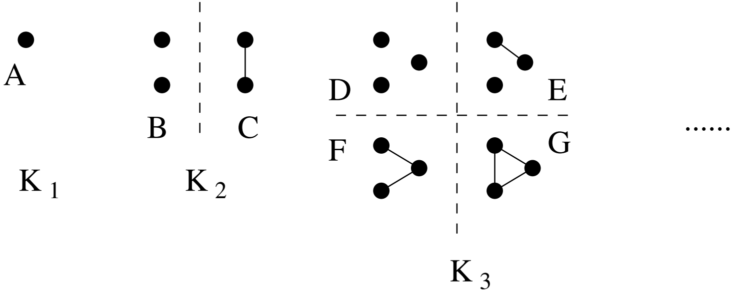
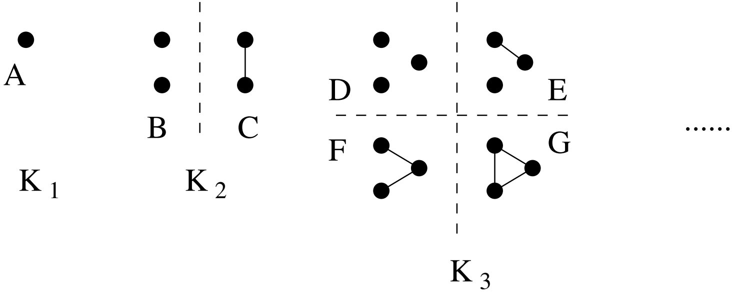


The Rado graph has an other characterization by almost sure theories.



For each $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a finite set of finite structures and let μ_n be the probability measure on $\mathbf{K}_n$ such that $\mu_n(\mathcal{M}) = 1/|\mathbf{K}_n|$. Let $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$. A property $\mathbf{P}$ is **almost sure** for $\mathbf{K}$ if

$$\lim_{n\to\infty} \mu_n(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N} \text{ satisfies } \mathbf{P}\}) = 1$$



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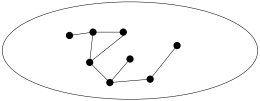
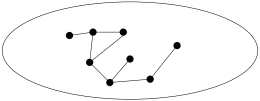
The **almost sure theory** for $\mathbf{K}$, $T_{\mathbf{K}}$, is the set of all sentences (in the language) which are almost sure. $\mathbf{K}$ has a $0-1$ law if for each sentence φ, either φ or $\neg\varphi$ is almost sure and thus $T_{\mathbf{K}}$ is complete.



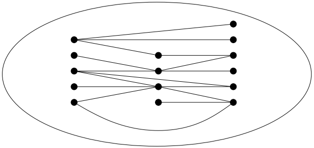
- $\mathbf{K}_n$ consist of all graphs with universe $\{1, \ldots, n\}$. Then $\mathbf{K}$ has a $0-1$ law [1]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random graph.** This model is isomorphic to the Rado graph.

[1] Fagin (1976) and Glebskii et. al. (1969) independently

[2] Kolaitis et. al. (1987)



- $\mathbf{K}_n$ consist of all graphs with universe $\{1, \ldots, n\}$. Then $\mathbf{K}$ has a $0-1$ law [1]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random graph.** This model is isomorphic to the Rado graph.
- For $t \in \mathbb{N}$, $\mathbf{K}_n$ consist of all $t-$partite graphs with universe $\{1, \ldots, n\}$. Then $\mathbf{K}$ has a $0-1$ law[2]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random $t-$partite graph.**



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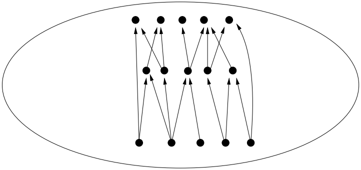
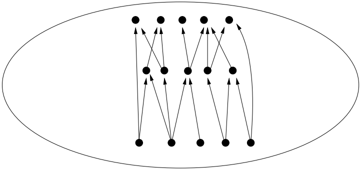


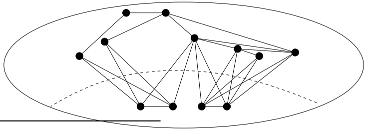
- For $\mathbf{K}_n$ consist of all partial orders with universe $\{1, \ldots, n\}$. Then $\mathbf{K}$ has a $0-1$ law[3]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random partial order.**

[3] Compton (1988)

[4] A. and Koponen (2015)



- For $\mathbf{K}_n$ consist of all partial orders with universe $\{1, \ldots, n\}$. Then $\mathbf{K}$ has a $0-1$ law[3]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random partial order.**
- Let $\mathcal{A}$ be a graph, H a group and let $\mathbf{K}_n$ be all graphs $\mathcal{G}$ with universe $\{1, \ldots, n\}$ where $H \leq Aut(\mathcal{G})$ and $\mathcal{A} \hookrightarrow spt(Aut(\mathcal{G}))$. Then $\mathbf{K}$ has a $0-1$ law [4]. Call the infinite countable model for $T_{\mathbf{K}}$ **the random nonrigid graph.**



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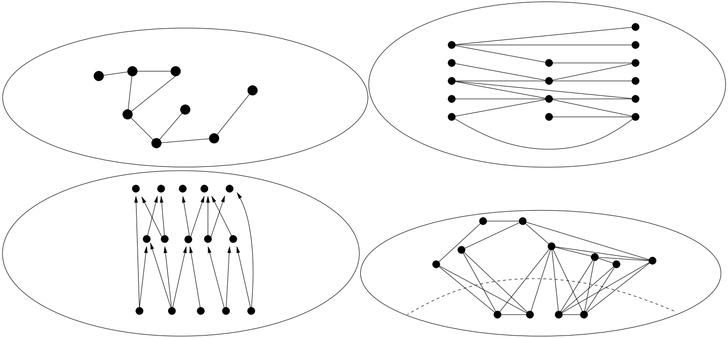
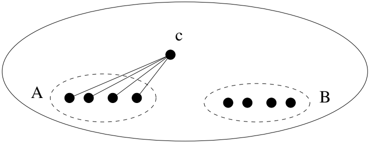
These almost sure theories are all simple, $\omega-$categorical with SU-rank 1 and trivial algebraic closure. $(acl(X) = acl(\emptyset) \cup X)$



The structures are however also axiomatized by extension properties which depend on the partitions.



The random graph

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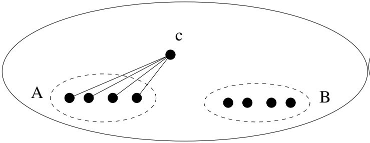

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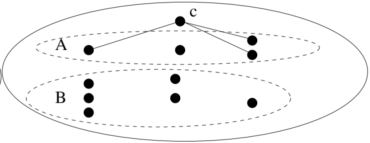

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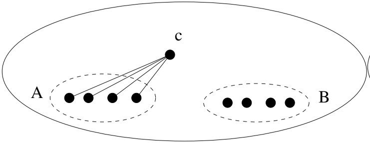

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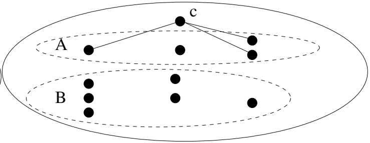

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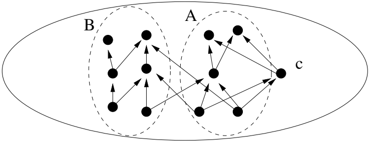

The random partial order

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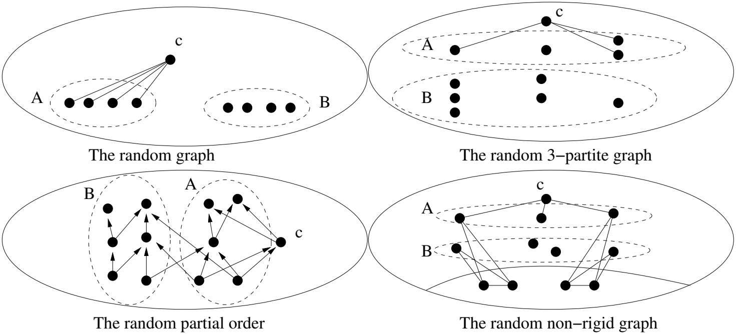

Theorem (A. 2016)

If T is a simple, $\omega-$categorical theory with $SU-$ rank 1 and trivial algebraic closure over a binary vocabulary then there are sets of finite structures $\mathbf{K}_n$ with probability measures μ_n such that if $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$ then $T_{\mathbf{K}} = T$.

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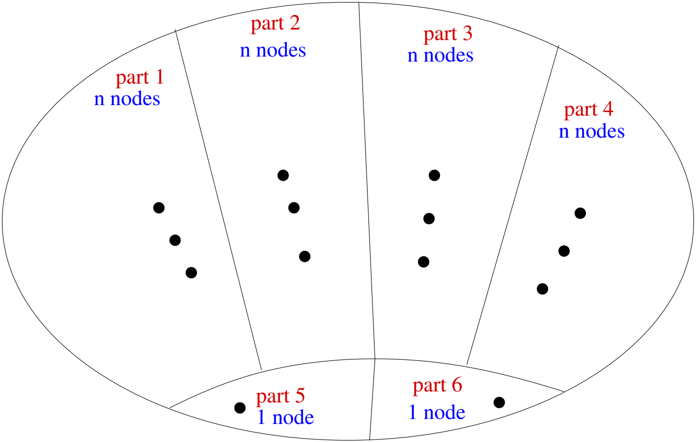
Theorem (A. 2016)

T is simple, $\omega-$categorical with SU-rank 1 and trivial algebraic closure over a binary vocabulary if and only if T is axiomatized by $\xi-$extension properties.

For $0 \leq t < l$, let $\mathbf{K}_n$ consist of all graphs with l parts where t are of size 1:



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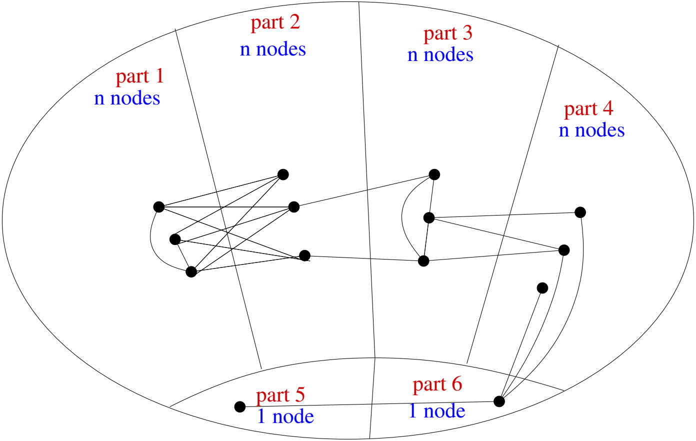
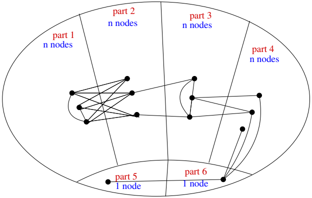


Between nodes in part i and j of size n we may choose among only edges, only non-edges or both.

Between part i and a 1 node part we have a unique choice.

Each simple $\omega-$categorical theory with $SU-rank$ 1 with trivial algebraic closure over a binary vocabulary is the almost sure theory of such sets.



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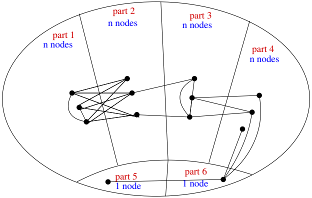
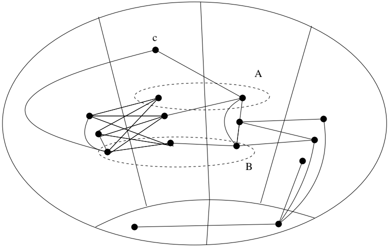


Similarly we have extension properties (which are a bit tricky to describe).



Let

$$\mathbf{K}_n = \{\mathcal{A} : A = \{1, \ldots, n\}, \mathcal{A} \hookrightarrow \mathcal{M}\}$$

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Not all structures with the above properties are random structures, for instance choose $\mathcal{M}$ as the disjoint union of two random graphs.

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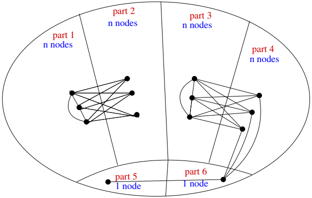
Theorem (A. 2016)

If $\mathcal{M}$ is simple $\omega-$categorical with SU$-$rank 1 and trivial algebraic closure, $acl(\emptyset) = \emptyset$ over a binary vocabulary then $\mathcal{M}$ is a reduct of a binary random structure.

Stable $\omega-$categorical with $SU-$rank 1 and trivial algebraic closure:



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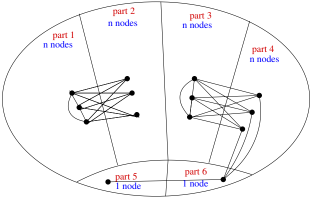
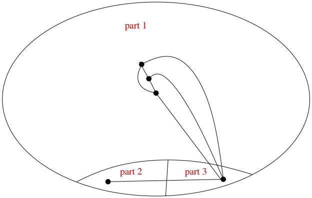


Strongly minimal $\omega-$categorical with trivial algebraic closure:



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The algebraic closure is degenerate if $acl(X) = \bigcup_{x \in X} acl(x)$.

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- **3-ary relations?** Open for Ultrahomogeneous structures.
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- **Non-trivial but degenerate algebraic closure?** Yes, a quite similar classification is possible.

The algebraic closure is degenerate if $acl(X) = \bigcup_{x \in X} acl(x)$.

- **Non-degenerate algebraic closure?** Trivially hold for Ultrahomogeneous structures.

Open Question

Does there exist a binary simple $\omega-$categorical structure with SU$-$rank 1 and without degenerate algebraic closure?

[5] Hrushovski (1997)

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An $\omega-$categorical, simple, SU-rank 1 structure without degenerate algebraic closure (not even 1-based) **using a 3-ary relation**.[5]

One can easily create a **SU-rank 2** structure without degenerate algebraic closure using binary relations.

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One can easily create a **SU-rank 2** structure without degenerate algebraic closure using binary relations.

Lemma

If $\mathcal{M}$ is binary simple $\omega-$categorical with SU-rank 1 and model-complete then $\mathcal{M}$ has degenerate algebraic closure.

This means that we may not create a counter example using the Hrushovski-construction.

[5] Hrushovski (1997)

Thank you!

A., *Simple structures axiomatized by almost sure theories*, Annals of Pure and Applied Logic 167 (2016) 435-456.