Easy and hard homogenizable structures

Ove Ahlman,
Uppsala University

Logic Colloquium 2017

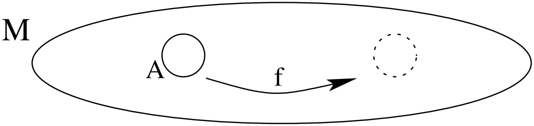
14 August 2017

Table of Contents

Throughout this talk I will only consider structures with a finite relational signature.

Definition

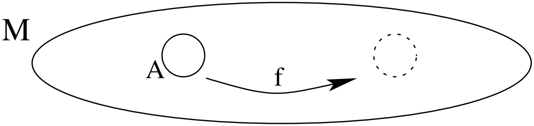
A structure $\mathcal{M}$ is called **homogeneous** if for each finite $\mathcal{A} \subseteq \mathcal{M}$ and embedding $f : \mathcal{A} \to \mathcal{M}$, f may be extended to an automorphism of $\mathcal{M}$.



Throughout this talk I will only consider structures with a finite relational signature.

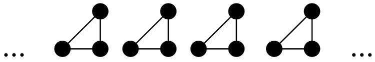
Definition

A structure $\mathcal{M}$ is called **homogeneous** if for each finite $\mathcal{A} \subseteq \mathcal{M}$ and embedding $f : \mathcal{A} \to \mathcal{M}$, f may be extended to an automorphism of $\mathcal{M}$.



Theorem (Lachlan and Woodrow 1980)

If $\mathcal{M}$ is a countably infinite homogeneous graph then for some $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to either the random graph, the generic K_n-free graph or a disjoint union of K_n.



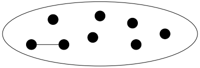
Definition

A structure $\mathcal{M}$ is **homogenizable** if for some formulas $\varphi_1(\bar{x}_1), \ldots, \varphi_n(\bar{x}_n)$ and $R_1 = \varphi_1(\mathcal{M}), \ldots, R_n = \varphi_n(\bar{x}_n)$, if $R_1, \ldots, R_n$ are added to the signature of $\mathcal{M}$ then $\mathcal{M}$ becomes homogeneous.

Definition

A structure $\mathcal{M}$ is **homogenizable** if for some formulas $\varphi_1(\bar{x}_1), \ldots, \varphi_n(\bar{x}_n)$ and $R_1 = \varphi_1(\mathcal{M}), \ldots, R_n = \varphi_n(\bar{x}_n)$, if $R_1, \ldots, R_n$ are added to the signature of $\mathcal{M}$ then $\mathcal{M}$ becomes homogeneous.

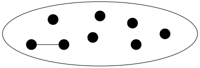
The infinite graph with 1 edge, add $\varphi(x)$: $\exists y(xEy)$



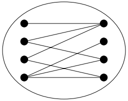
Definition

A structure $\mathcal{M}$ is **homogenizable** if for some formulas $\varphi_1(\bar{x}_1), \ldots, \varphi_n(\bar{x}_n)$ and $R_1 = \varphi_1(\mathcal{M}), \ldots, R_n = \varphi_n(\bar{x}_n)$, if $R_1, \ldots, R_n$ are added to the signature of $\mathcal{M}$ then $\mathcal{M}$ becomes homogeneous.

The infinite graph with 1 edge, add $\varphi(x)$: $\exists y(xEy)$



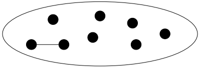
The random bipartite graph, add $\varphi(x, y)$: $\exists z(xEz \wedge yEz)$



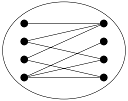
Definition

A structure $\mathcal{M}$ is **homogenizable** if for some formulas $\varphi_1(\bar{x}_1), \dots, \varphi_n(\bar{x}_n)$ and $R_1 = \varphi_1(\mathcal{M}), \dots, R_n = \varphi_n(\bar{x}_n)$, if $R_1, \dots, R_n$ are added to the signature of $\mathcal{M}$ then $\mathcal{M}$ becomes homogeneous.

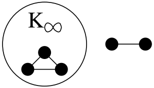
The infinite graph with 1 edge, add $\varphi(x)$: $\exists y(xEy)$



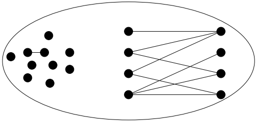
The random bipartite graph, add $\varphi(x, y)$: $\exists z(xEz \wedge yEz)$



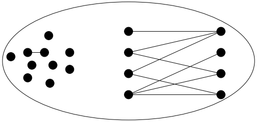
The infinite complete graph with a disjoint K_2, $\varphi(x)$: $\exists y_1, y_2(xEy_1 \wedge xEy_2)$



The random bipartite graph in disjoint union with the inifinite graph with 1 edge, add:

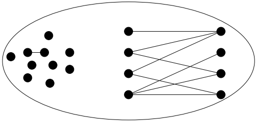


The random bipartite graph in disjoint union with the inifinite graph with 1 edge, add:



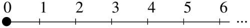
$$\begin{array}{ll} \varphi_1(x) & \exists y_1, y_2(xEy_1 \wedge xEy_2) \\ \varphi_2(x) & \neg\varphi_1(x) \wedge \exists y(xEy) \\ \psi(x,y) & \exists z(xEz \wedge yEz) \end{array}$$

The random bipartite graph in disjoint union with the inifinite graph with 1 edge, add:

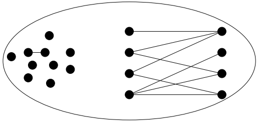


$$\begin{array}{rl} \varphi_1(x) & \exists y_1, y_2(xEy_1 \wedge xEy_2) \\ \varphi_2(x) & \neg\varphi_1(x) \wedge \exists y(xEy) \\ \psi(x,y) & \exists z(xEz \wedge yEz) \end{array}$$

The non-negative rational numbers, add $\varphi(x)$: $\forall y(x \leq y)$

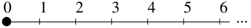


The random bipartite graph in disjoint union with the inifinite graph with 1 edge, add:



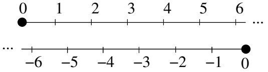
$$\begin{array}{rl} \varphi_1(x) & \exists y_1, y_2(xEy_1 \wedge xEy_2) \\ \varphi_2(x) & \neg\varphi_1(x) \wedge \exists y(xEy) \\ \psi(x,y) & \exists z(xEz \wedge yEz) \end{array}$$

The non-negative rational numbers, add $\varphi(x)$: $\forall y(x \leq y)$

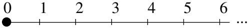


The non-negative rational numbers in disjoint union with the non-positive rational numbers, add:

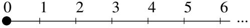
$$\begin{array}{ll} \varphi_1(x) & \exists y(x < y) \\ \varphi_2(x) & \exists y(y < x) \\ \varphi_3(x) & \forall w(\varphi_1(w) \rightarrow w < x) \end{array}$$

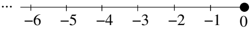


$$\forall y(x \leq y)$$

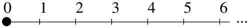


$$\begin{array}{ll} \varphi_1(x) & \exists y(x < y) \\ \varphi_2(x) & \exists y(y < x) \\ \varphi_3(x) & \forall w(\varphi_1(w) \rightarrow w < x) \end{array}$$

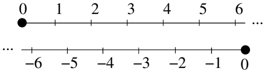




$$\forall y(x \leq y)$$



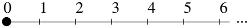
$$\begin{array}{ll} \varphi_1(x) & \exists y(x < y) \\ \varphi_2(x) & \exists y(y < x) \\ \varphi_3(x) & \forall w(\varphi_1(w) \rightarrow w < x) \end{array}$$



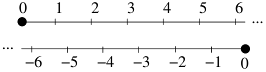
Question: How complicated can the homogenization get?

- Number of formulas?

$\forall y(x \leq y)$



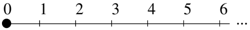
$$
\begin{array}{ll}
\varphi_1(x) & \exists y(x < y) \\
\varphi_2(x) & \exists y(y < x) \\
\varphi_3(x) & \forall w(\varphi_1(w) \rightarrow w < x)
\end{array}
$$



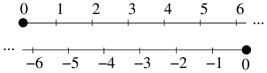
Question: How complicated can the homogenization get?

- Number of formulas?
- Number of free variables?

$\forall y(x \leq y)$



$$\begin{array}{ll} \varphi_1(x) & \exists y(x < y) \\ \varphi_2(x) & \exists y(y < x) \\ \varphi_3(x) & \forall w(\varphi_1(w) \to w < x) \end{array}$$

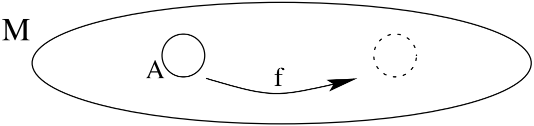


Question: How complicated can the homogenization get?

- Number of formulas?
- Number of free variables?
- At least $\exists$, $\forall, \forall\exists$, $\exists\forall$ etc. ?

Definition

$\mathcal{M}$ is $(>k)$ $k-$**homogeneous** if for each finite $\mathcal{A} \subseteq \mathcal{M}$ such that $|\mathcal{A}| = k$ $(|\mathcal{A}| > k)$ and embedding $f : \mathcal{A} \to \mathcal{M}$, may be extended to an automorphism.

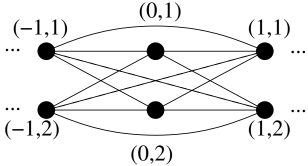


Fact

A $>k-$homogeneous structure is homogenizable.

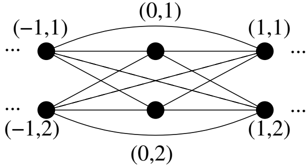
For $n \in \mathbb{Z}^+$ let $\mathcal{G}_n$ be the graph with vertex set $V = \mathbb{Z} \times \{1, \ldots, n\}$ and edge set $E = \{\{(a,i),(b,j)\} : a \neq b\}$. Note that $\mathcal{G}_n \restriction \mathbb{Z} \times \{i\} \cong K_\infty$ and $\mathcal{G}_n$ is the homogeneous graph isomorphic to the infinite disjoint union of K_n.

The Graph G_2



For $n \in \mathbb{Z}^+$ let $\mathcal{G}_n$ be the graph with vertex set $V = \mathbb{Z} \times \{1, \ldots, n\}$ and edge set $E = \{\{(a,i),(b,j)\} : a \neq b\}$. Note that $\mathcal{G}_n \restriction \mathbb{Z} \times \{i\} \cong K_\infty$ and $\mathcal{G}_n$ is the homogeneous graph isomorphic to the infinite disjoint union of K_n.

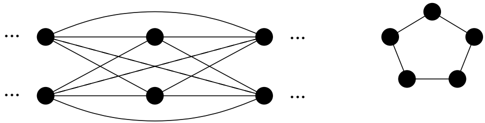
The Graph G_2



Lemma

Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is $>k-$homogeneous but **not** $1-$*homogeneous if and only if for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{G}_n \dot\cup \mathcal{H}$.*

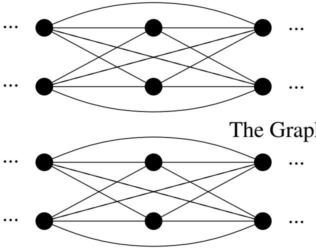
The Graph $G_2 \dot\cup C_5$



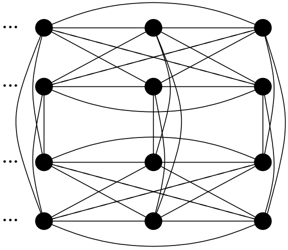
If $t \geq 2$ let $\mathcal{H}_{t,1}$ be the graph with universe $H_{t,1} = \mathbb{Z} \times \{1, ..., t\} \times \{1, 2\}$ such that the inclusion map $\iota : \mathcal{H}_{t,1} \to \mathcal{G}_t \times \{1, 2\}$ is an isomorphism. Let $\mathcal{H}_{t,2}$ have the same universe as $\mathcal{H}_{t,1}$ but with edge set

$$E_{t,2} = E_{G_{t,1}} \cup \{\{(a, i, z), (b, j, w)\} : z \neq w \text{ and } a = b\}.$$

Lastly define the graph $\mathcal{H}_{1,2}$ as having universe $\mathbb{Z} \times \{1, 2\}$ and edge set $E = \{\{(a, i), (b, j)\} : i = j \text{ or } a = b \text{ but } i \neq j\}$.



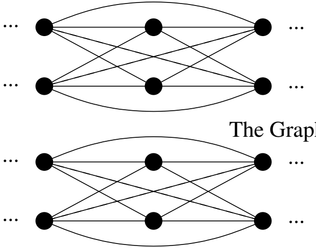
The Graph $H_{2,1}$



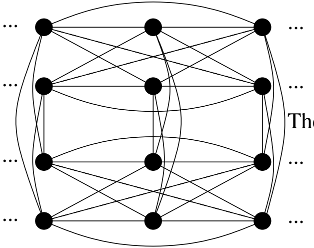
The Graph $H_{2,2}$

Lemma

Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is $>$k-homogeneous, 1−homogeneous but **not** *2−homogeneous if and only if $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{H}_{t,1}$ or $\mathcal{H}_{t,2}$ for some t.*



The Graph $H_{2,1}$



The Graph $H_{2,2}$

Lemma

If $\mathcal{M}$ is a $>k-$homogeneous infinite graph which is $1-$homogeneous and $2-$homogeneous, then $\mathcal{M}$ is homogeneous.

Lemma

If $\mathcal{M}$ is a $>k-$homogeneous infinite graph which is $1-$homogeneous and $2-$homogeneous, then $\mathcal{M}$ is homogeneous.

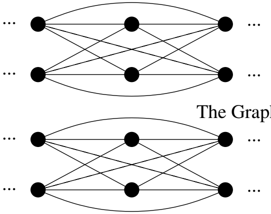
Theorem (A. 2016)

Let $k \in \mathbb{Z}^+$. If $\mathcal{M}$ is a $>k-$homogeneous countably infinite graph then $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to one of the following

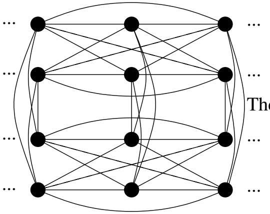
- *A homogeneous graph.*
- $\mathcal{G}_n \dot{\cup} \mathcal{H}$ *for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$.*
- $\mathcal{H}_{t,1}$ *or $\mathcal{H}_{t,2}$ for some t.*

Corollary

$\mathcal{G}_n \dot{\cup} \mathcal{H}$ is homogenizable using a unary formula.
$\mathcal{H}_{t,1}$ and $\mathcal{H}_{t,2}$ are homogenizable using a single binary formula.

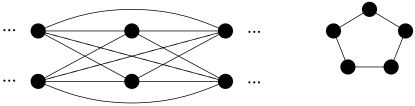


The Graph $H_{2,1}$



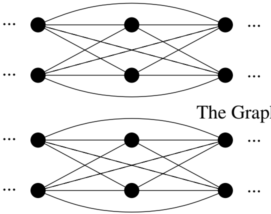
The Graph $H_{2,2}$

The Graph $G_2 \dot{\cup} C_5$

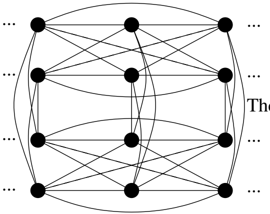


Questions:

- How does this work for more complicated vocabularies?
- Especially, can we find a nice classification without classifying the homogeneous ones?



The Graph $H_{2,1}$



The Graph $H_{2,2}$

Thank you!