$>k-$homogeneous graphs and homogenizable structures

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Workshop on finite and pseudofinite structures

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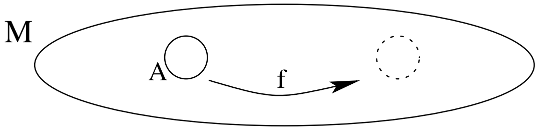
Throughout this talk I will only consider structures with a finite relational signature.

Definition

A structure $\mathcal{M}$ is called $k-$**homogeneous** if for each $\mathcal{A} \subseteq \mathcal{M}$ such that $|\mathcal{A}| = k$ and embedding $f : \mathcal{A} \to \mathcal{M}$, f may be extended to an automorphism of $\mathcal{M}$.



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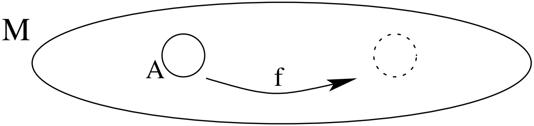

Definition

If $\mathcal{M}$ is $i-$homogeneous for each $i > k$ ($i \leq k$) we say that $\mathcal{M}$ is $>k-$**homogeneous** ($\leq k-$**homogeneous**).
If $\mathcal{M}$ is both $>k-$homogeneous and $\leq k-$homogeneous, $\mathcal{M}$ is just called **homogeneous** (or ultrahomogeneous).

Homogeneous Graphs

For $n \in \mathbb{Z}^+$ let K_n be the complete graph on n vertices and K_∞ be the countably infinite complete graph.

Theorem (Gardiner 1976 independently of Golfand and Klin 1978)

If $\mathcal{G}$ is a finite homogeneous graph then for some $n \in \mathbb{Z}^+$, $\mathcal{G}$ (or $\mathcal{G}^c$) is isomorphic to either the pentagon, the 3×3-Rook graph or a disjoint union of K_n.

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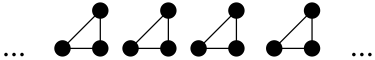
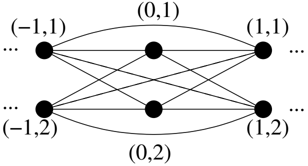


Theorem (Lachlan and Woodrow 1980)

If $\mathcal{M}$ is a countably infinite homogeneous graph then for some $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to either the random graph, the generic K_n-free graph or a disjoint union of K_n.

For $n \in \mathbb{Z}^+$ let $\mathcal{G}_n$ be the graph with vertex set $V = \mathbb{Z} \times \{1, \ldots, n\}$ and edge set $E = \{\{(a,i),(b,j)\} : a \neq b\}$. Note that $\mathcal{G}_n \restriction \mathbb{Z} \times \{i\} \cong K_\infty$ and $\mathcal{G}_n$ is the homogeneous graph isomorphic to the infinite disjoint union of K_n.

The Graph G_2



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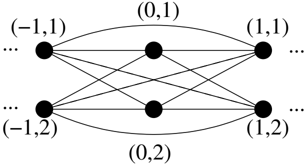
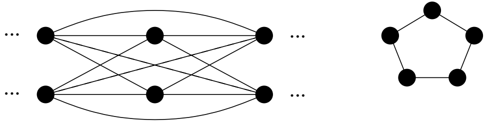


Lemma

Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is $>k-$homogeneous but **not** $1-$*homogeneous if and only if for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{G}_n \dot{\cup} \mathcal{H}$.*

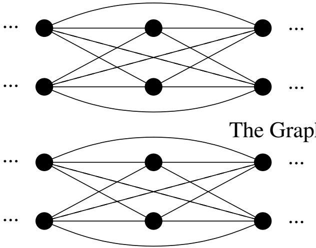
The Graph $G_2 \dot{\cup} C_5$



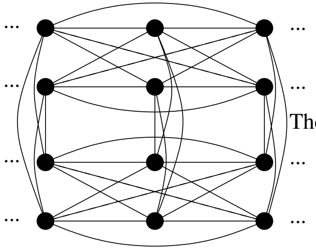
If $t \geq 2$ let $\mathcal{H}_{t,1}$ be the graph with universe $H_{t,1} = \mathbb{Z} \times \{1, ..., t\} \times \{1, 2\}$ such that the inclusion map $\iota : \mathcal{H}_{t,1} \to \mathcal{G}_t \times \{1, 2\}$ is an isomorphism. Let $\mathcal{H}_{t,2}$ have the same universe as $\mathcal{H}_{t,1}$ but with edge set

$$E_{t,2} = E_{G_{t,1}} \cup \{\{(a, i, z), (b, j, w)\} : z \neq w \text{ and } a = b\}.$$

Lastly define the graph $\mathcal{H}_{1,2}$ as having universe $\mathbb{Z} \times \{1, 2\}$ and edge set $E = \{\{(a, i), (b, j)\} : i = j \text{ or } a = b \text{ but } i \neq j\}$.



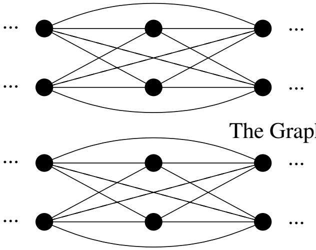
The Graph $H_{2,1}$



The Graph $H_{2,2}$

Lemma

Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is $>k$-homogeneous, $1-$homogeneous but **not** *$2-$homogeneous if and only if $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{H}_{t,1}$ or $\mathcal{H}_{t,2}$ for some t.*



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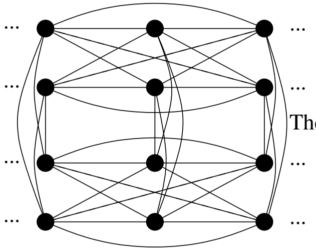

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If $\mathcal{M}$ is a $>k-$homogeneous infinite graph which is $1-$homogeneous and $2-$homogeneous, then $\mathcal{M}$ is homogeneous.

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Theorem (A. 2016)

Let $k \in \mathbb{Z}^+$. If $\mathcal{M}$ is a $>k-$homogeneous countably infinite graph then $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to one of the following

- *A homogeneous graph.*
- *$\mathcal{G}_n \dot\cup \mathcal{H}$ for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$.*
- *$\mathcal{H}_{t,1}$ or $\mathcal{H}_{t,2}$ for some t.*

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This classification also work for finite graphs when the size of $\mathcal{M}$ is large with respect to k.

Definition

A structure $\mathcal{M}$ is **homogenizable** if for some formulas $\varphi_1(\bar{x}_1), \ldots, \varphi_n(\bar{x}_n)$ and $R_1 = \varphi_1(\mathcal{M}), \ldots, R_n = \varphi_n(\bar{x}_n)$, if $R_1, \ldots, R_n$ are added to the signature of $\mathcal{M}$ then $\mathcal{M}$ becomes homogeneous.

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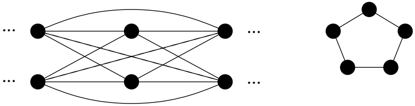

Homogenizable by $\varphi(x)$ defined by $\exists y_1, y_2, y_3(xEy_1 \wedge xEy_2 \wedge xEy_3)$.

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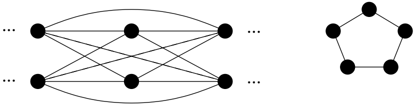

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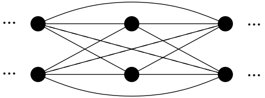

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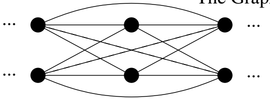

Homogenizable by $\varphi(x, y)$ defined by $\exists z(xEz \wedge yEz)$

If **K** is a set of finite structures, **K** satisfy the Hereditary property (**HP**) if for any $\mathcal{B} \in \mathbf{K}$ and $\mathcal{A} \subseteq \mathcal{B}$ we have $\mathcal{A} \in \mathbf{K}$.
$\mathcal{A} \in \mathbf{K}$ is an **amalgamation base** if for each $\mathcal{B}, \mathcal{C} \in \mathbf{K}$ and embeddings $f : \mathcal{A} \to \mathcal{B}, g : \mathcal{A} \to \mathcal{C}$ there is $\mathcal{D} \in \mathbf{K}$ and embeddings $f' : \mathcal{B} \to \mathcal{D}$ and $g' : \mathcal{C} \to \mathcal{D}$ such that for each $a \in A$, $f'(f(a)) = g'(g(a))$.

$$\begin{tikzcd} & B \arrow[dr, "f'"] & \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & & D \\ & C \arrow[ur, "g'"'] & \end{tikzcd}$$

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\end{tikzcd}$$

K satisfy the amalgamation property (**AP**) if each structure in **K** is an amalgamation base.
Let $Age(\mathcal{M})$ be the class of all finite structures embeddable in $\mathcal{M}$

Theorem (Fraïssé 1953)

If a class **K** *is closed under isomorphism and satisfy HP and AP then there is a unique homogeneous structure* $\mathcal{M}$ *such that* $Age(\mathcal{M}) = \mathbf{K}$.

Because of the Fraïssé correspondence, generalizations to homogenizable structures often focus on generation from a class **K** with something weaker than AP.
Covington (1990), Hartman, Hubicka, Nesetril (2014), Atserias, Torunczyk (2016) all found *necessary* conditions for a class of structures to generate a homogenizable structure.

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Theorem (A. 2016)

Let $\mathcal{M}$ be a countably infinite structure which is model-complete and $\omega-$categorical. Age($\mathcal{M}$) satisfies SEAP if and only of $\mathcal{M}$ is homogenizable.

Where SEAP is a *weaker* version of AP.

Definition

Let $\mathcal{M}$ be a homogenizable structure. $\mathcal{M}$ is **boundedly homogenizable** if for each finite $\mathcal{A} \subseteq \mathcal{M}$ there is finite $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$ such that each embedding $f : \mathcal{B} \to \mathcal{M}$ is extendable to an automorphism.

Examples include the "random $l-$partite graph", "the random partial order", "the $>k-$homogeneous structures", etc.

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Lemma

If $\mathcal{M}$ is a homogenizable structure, the following are equivalent

- *$\mathcal{M}$ is boundedly homogenizable.*
- *For each $\bar{a} \in M$ there is a supertuple $\bar{a} \subseteq \bar{b} \in M$ such that $tp(\bar{b})$ is isolated by a quantifier free formula.*
- *$\mathcal{M}$ is model-complete and for each $\mathcal{A} \subseteq \mathcal{M}$ there is an amalgamation base $\mathcal{B}$ for $Age(\mathcal{M})$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$.*

Theorem

If $\mathcal{M}$ is $\omega-$stable, model-complete and homogenizable then $\mathcal{M}$ is boundedly homogenizable.

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Question: Are all model-complete homogenizable structures boundedly homogenizable?

Definition

Let $\mathcal{M}$ be a homogenizable structure. $\mathcal{M}$ is **uniformly homogenizable** if there is a finite $\mathcal{N} \subseteq \mathcal{M}$ such that for each finite $\mathcal{A} \subseteq \mathcal{M}$, if $\mathcal{N} \subseteq \mathcal{A}$ then each embedding $f : \mathcal{A} \to \mathcal{M}$ is extendable to an automorphism.

Definition

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Lemma

If $\mathcal{M}$ is a homogenizable structure, the following are equivalent

- *$\mathcal{M}$ is uniformly homogenizable.*
- *There is a tuple $\bar{c} \in M$ such that if $\bar{a} \in M$ and $\bar{c} \subseteq \bar{a}$ then $tp(\bar{a})$ is isolated by a quantifier free formula.*
- *$\mathcal{M}$ is model-complete and there is a finite $\mathcal{N} \subseteq \mathcal{M}$ such that if $\mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{M}$ then $\mathcal{A}$ is an amalgamation base for $Age(\mathcal{M})$.*

Examples include all algebraic homogenizable structures, $> k-$homogeneous structures, $l-$partite graphs with part-witness.

Theorem

If $\mathcal{M}$ is boundedly homogenizable using only new unary relations and have trivial algebraic closure then $\mathcal{M}$ is an infinite union of uniformly homogenizable structures of only a finite amount of isomorphism types.

Thank you!

- A., *$>k-$homogeneous infinite graphs*, arXiv 1601.07307 (2016).
- A., *Homogenizable strucures and model completeness*, Accepted for publication in AFML, arXiv 1601.07304 (2016).