Almost sure theories approximating simple structures

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Workshop on Logic and Random Graphs
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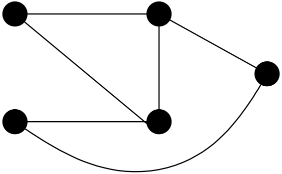
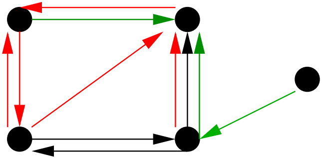
Consider binary relational structures $\mathcal{G} = (V, E_1, ..., E_k)$ over a fixed vocabulary. Thus if $k = 1$ and E_1 is symmetric and non-reflexive $\mathcal{G} = (V, E_1)$ is a graph.





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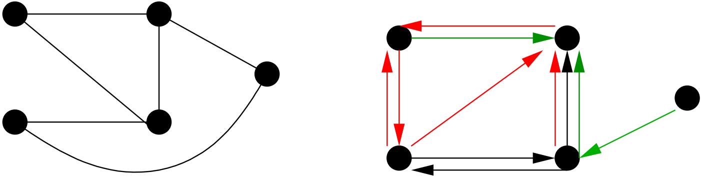

For each $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a set of finite structures and let μ_n be a probability measure on $\mathbf{K}_n$. Let $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$. A property $\mathbf{P}$ is **almost sure** for $\mathbf{K}$ if

$$\lim_{n\to\infty} \mu_n(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N} \text{ satisfies } \mathbf{P}\}) = 1$$

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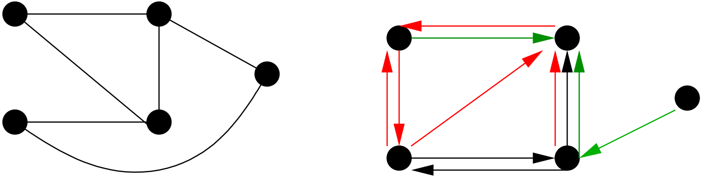

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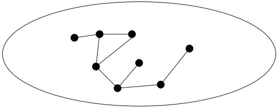
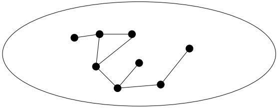
The **almost sure theory** for $\mathbf{K}$, $T_{\mathbf{K}}$ is the set of all sentences (in the language) which are almost sure. $\mathbf{K}$ has a $0-1$ law if for each sentence φ, either φ or $\neg\varphi$ is almost sure and thus $T_{\mathbf{K}}$ is complete.



Consider only the uniform measure over each respective set.

- $\mathbf{K}_n$ consist of all graphs with node set $[n]$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random graph.**



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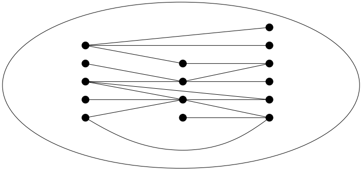
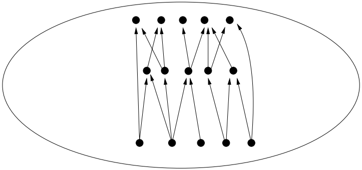
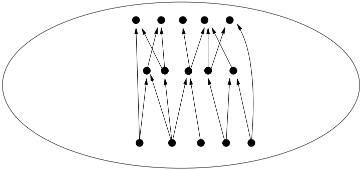
- $\mathbf{K}_n$ consist of all graphs with node set $[n]$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random graph.**
- For $t \in \mathbb{N}$, $\mathbf{K}_n$ consist of all $t-$partite graphs with node set $[n]$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random $t-$partite graph.**

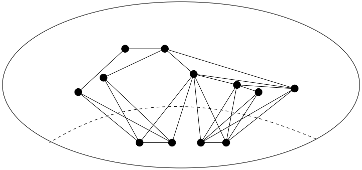




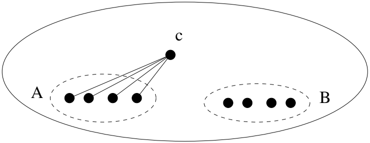
- For $\mathbf{K}_n$ consist of all partial orders with universe $[n]$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random partial order.**



- For $\mathbf{K}_n$ consist of all partial orders with universe $[n]$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random partial order.**
- Let $\mathcal{A}$ be a graph, H a group and let $\mathbf{K}_n$ be all graphs $\mathcal{G}$ with universe $[n]$ where $H \leq Aut(\mathcal{G})$ and $\mathcal{A} \hookrightarrow spt(Aut(\mathcal{G}))$. Then $\mathbf{K}$ has a $0-1$ law. Call the countable model for $T_{\mathbf{K}}$ **the random nonrigid graph.**



To prove these $0-1$ laws extension properties are crucial.



The random graph

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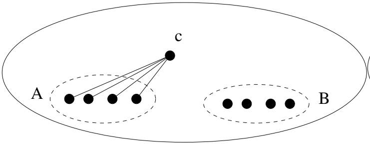
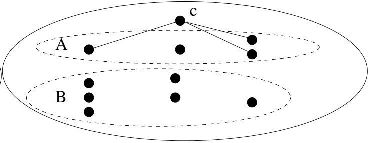


The random graph



The random 3−partite graph

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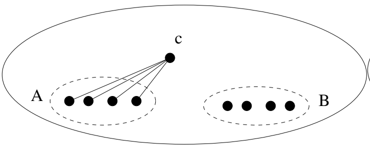

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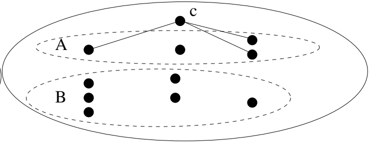

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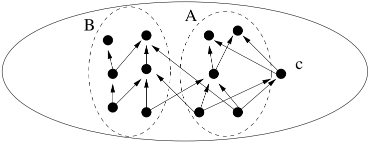

The random partial order

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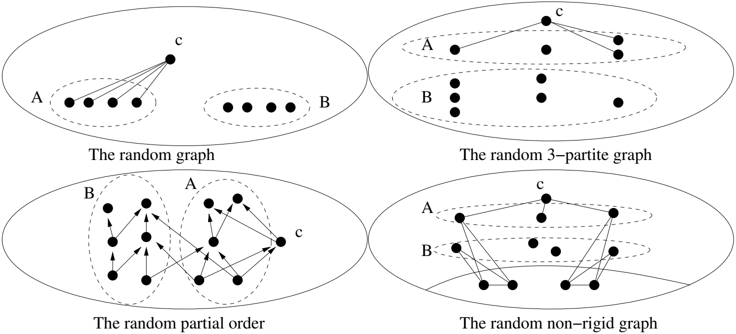
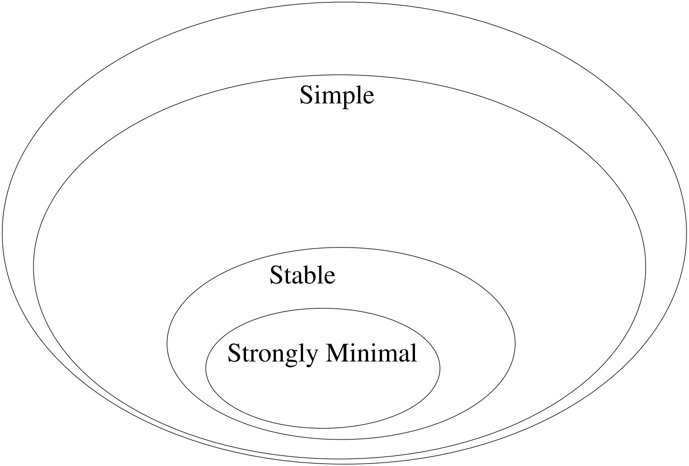
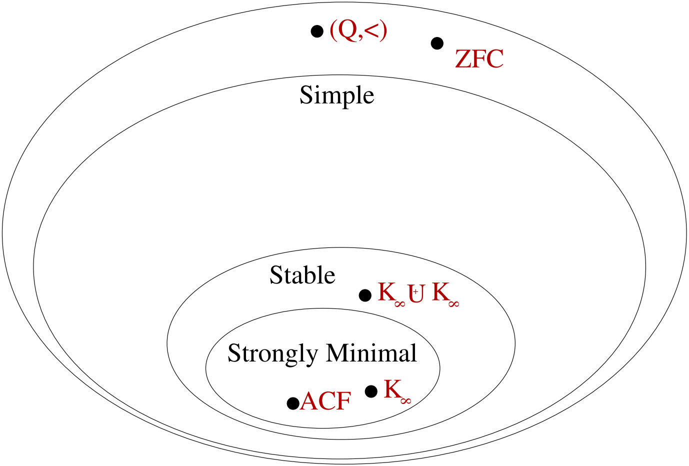


These examples all satisfy extension properties which depend on the partitioning.


Simple
Stable
Strongly Minimal


(Q,<)
ZFC
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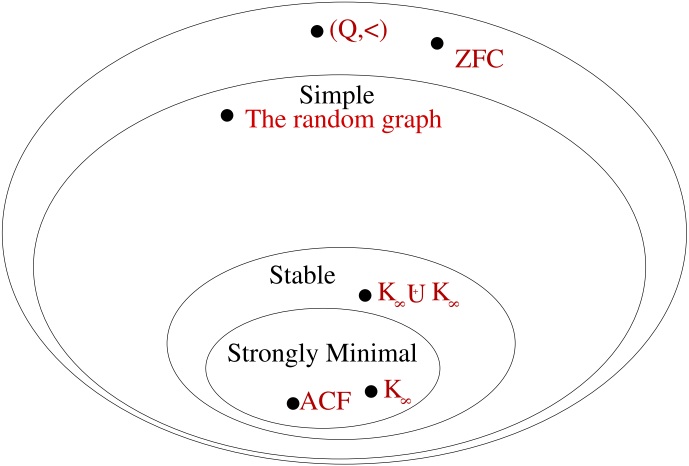
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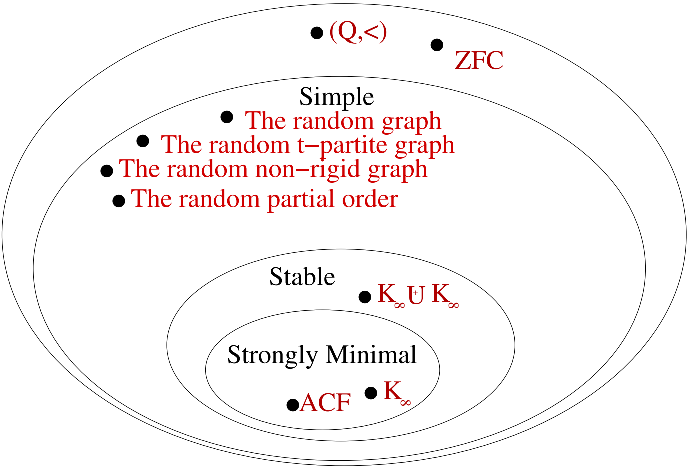

Furthermore, these examples of simple structures are $\omega-$categorical, with $SU-rank$ 1 and with trivial pregeometry.

Theorem (A. 2015)

If T is a simple, $\omega-$categorical theory with $SU-$rank 1 and trivial pregeometry over a binary vocabulary then there are sets of finite structures $\mathbf{K}_n$ with probability measures μ_n such that if $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n \in \mathbb{N}}$ then $T_{\mathbf{K}} = T$.

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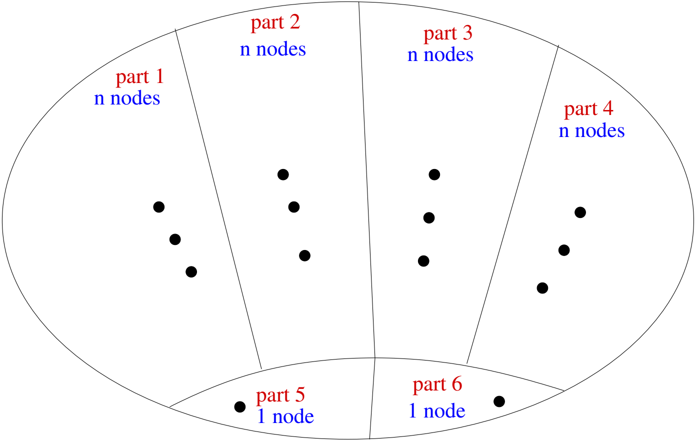
Theorem (A. 2015)

$T_{\mathbf{K}}$ is simple, $\omega-$categorical with SU-rank 1 and trivial pregeometry over a binary vocabulary if and only if $\mathbf{K}$ almost surely satisfy $\xi-$extension properties.

For $0 \leq t < l$, let $\mathbf{K}_n$ consist of all graphs with l parts where t are of size 1:



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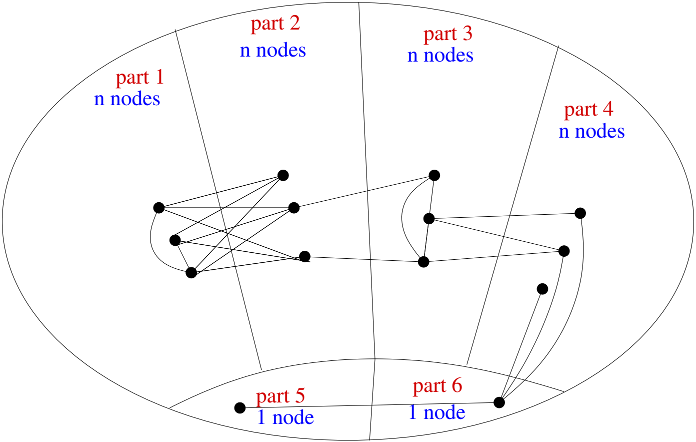
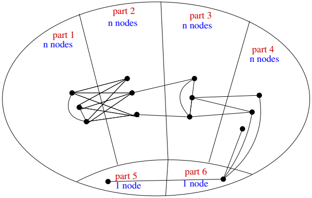


Between nodes in part i and j of size n we may choose among only edges, only non-edges or both.
Between part i and a 1 node part we have a unique choice.

Each simple $\omega-$categorical theory with $SU - rank$ 1 with trivial pregeometry over a binary vocabulary is the almost sure theory of such sets.



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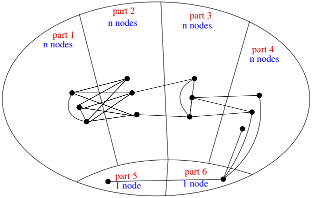
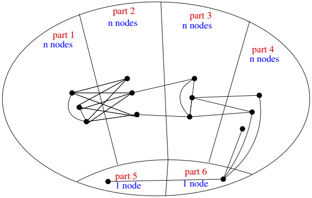


Let

$$\mathbf{K}_n = \{\mathcal{A} : A = [n], \mathcal{A} \hookrightarrow \mathcal{M}\}$$

(i.e. "all substructures") with the uniform measure μ_n. If $Th(\mathcal{M}) = T_{\mathbf{K}}$ then we call $\mathcal{M}$ a **random structure**.

Each simple $\omega-$categorical theory with $SU - rank$ 1 with trivial pregeometry over a binary vocabulary is the almost sure theory of such sets.



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Not all structures with the above properties are random structures, for instance choose $\mathcal{M}$ as the disjoint union of two random graphs.

Theorem (A. 2015)

If $\mathcal{M}$ is simple $\omega-$categorical with $SU-$rank 1 and trivial pregeometry, $acl(\emptyset) = \emptyset$ over a binary vocabulary then $\mathcal{M}$ is a reduct of a binary random structure.

Open Questions

1. What model theoretic properties are possible to get in almost sure theories of graphs? Are they all Simple?

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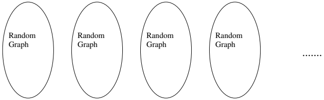

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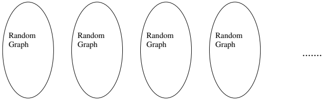

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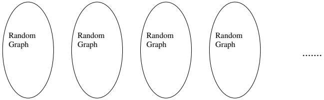
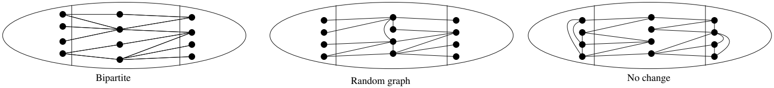


3. Which simple $\omega-$categorical binary structures with $SU-$rank 1 and trivial pregeometry over a binary vocabulary are random structures?
4. In the previous construction, when do we get the same almost sure theory?



Thank you!