To infinity and back

Logical limit laws and almost sure theories

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Opponent: Kerkko Luosto

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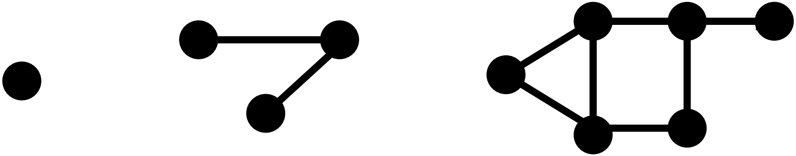
Introduktion

A finite graph $\mathcal{G} = (G, E)$ is a finite set G with a binary "edge" relation E.



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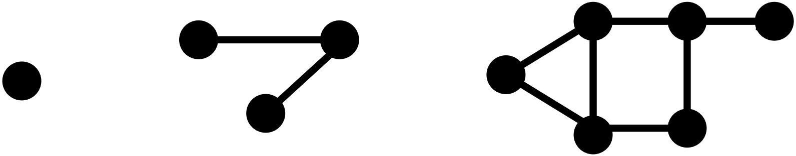
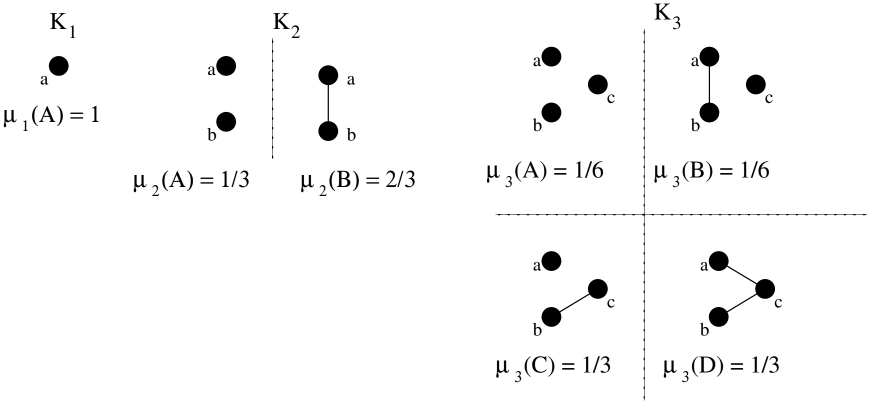


Generalized to finite relational structures $\mathcal{M} = (M, R_1, ..., R_k)$.

For each $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a finite set of finite structures and μ_n a probability measure on $\mathbf{K}_n$. If φ is a formula let

$$\mu_n(\varphi) = \mu_n(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N} \models \varphi\})$$

$\mathbf{K} = \bigcup_{n=1}^{\infty} \mathbf{K}_n$ has a convergence law if for each formula φ, $\lim_{n\to\infty} \mu_n(\varphi)$ converges.



If we let φ be the formula $\exists x \exists y (xEy)$ then

$$\mu_1(\varphi) = 0 \qquad \mu_2(\varphi) = 2/3 \qquad \mu_3(\varphi) = 5/6$$

$\lim_{n\to\infty} \mu_n(\varphi)$ converges if the sequence $0, 2/3, 5/6, \ldots$ converges.

0-1 laws

If for each formula φ

$$\lim_{n\to\infty} \mu_n(\varphi) = 1 \quad \text{or} \quad \lim_{n\to\infty} \mu_n(\varphi) = 0$$

then **K** has $0-1$ law.

[1] R. Fagin, Probabilities on finite models, J. Symbolic Logic 41(1976), no. 1

[2] Glebskii et. al. Volume and fraction of satisfieability of formulas over the lower predicate calculus, Kibernetyka Vol. 2 (1969)

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then $\mathbf{K}$ has $0-1$ law.

Let $\mathbf{K}_n$ consisting of all structures with universe $\{1, ..., n\}$ (over a fixed vocabulary) with $\mu_n(\mathcal{N}) = \frac{1}{|\mathbf{K}_n|}$. Fagin[1] and independently Glebksii et. al.[2] proved that this $\mathbf{K}$ has a $0-1$ law.
Since then many other $0-1$ laws have been proven.

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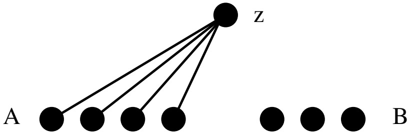
Fagins method of proving $0-1$ laws

$\mathcal{N}$ satisfies the k-extension property φ_k (for graphs) if:

$$A, B \subseteq N, A \cap B = \emptyset, |A \cup B| \leq k \Rightarrow \exists z :$$

$$aEz \text{ and } \neg bEz \text{ for each } a \in A, b \in B$$



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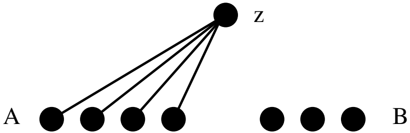

If **K** consist of all structures, then $\lim_{n\to\infty} \mu_n(\varphi_k) = 1$. We say that φ_k is an almost sure property.

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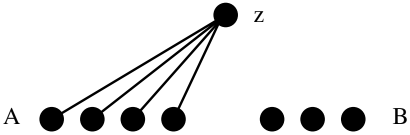

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$$T_{\mathbf{K}} = \{\varphi : \lim_{n\to\infty} \mu_n(\varphi) = 1\}$$

is called the almost sure theory.

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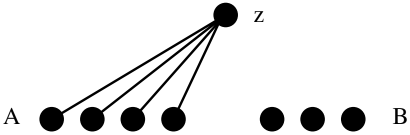

If **K** consist of all structures, then $\lim_{n\to\infty} \mu_n(\varphi_k) = 1$. We say that φ_k is an almost sure property.

$$T_{\mathbf{K}} = \{\varphi : \lim_{n\to\infty} \mu_n(\varphi) = 1\}$$

is called the almost sure theory.
Note: $T_{\mathbf{K}}$ is complete iff **K** has a $0-1$ law.

For categorical theories completeness is equivalent with not having any finite models.

Theorem

$T_{\mathbf{K}}$ is $\aleph_0$−categorical.

Hence this will prove that $\mathbf{K}$ has a $0-1$ law.

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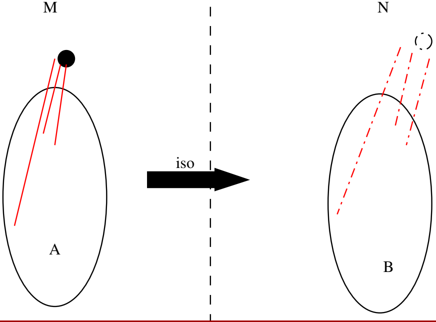
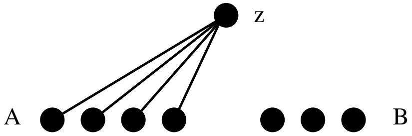
Proof.

Take $\mathcal{N}, \mathcal{M} \models T_{\mathbf{K}}$. Build partial isomorphisms back and forth by using the extension properties to help.



$0 - 1$ law proof Summary

1. Find extension properties which are almost sure.



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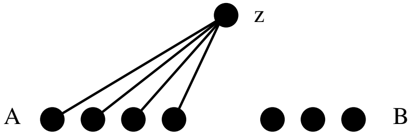

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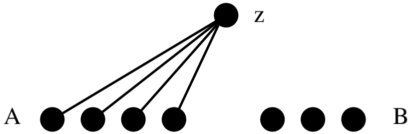

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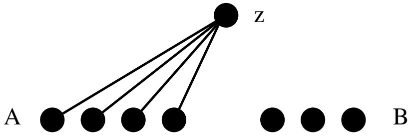

2. Collect these into a theory T. T is complete iff **K** has a 0 − 1 law.
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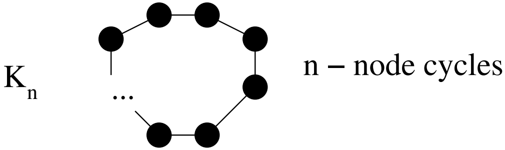
Note: This method is not allways useable to prove 0 − 1 laws.



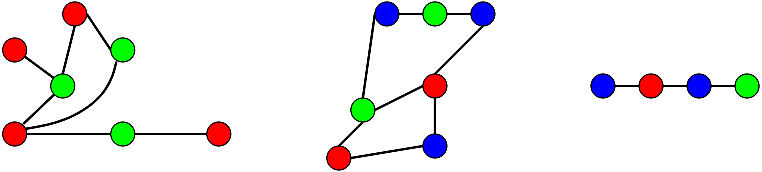
Paper I: Random $l-$colourable structures with a pregeometry

Paper I in one sentence:

Having a vector space "as universe" we show that $\mathbf{K}_n$ consisting of colourable structures has a $0 - 1$ law.

Coloured graphs

An $l-$colouring of a graph $\mathcal{G} = (V, E)$ is a function $\gamma : V \to \{1, ..., l\}$ such that whenever aEb, $\gamma(a) \neq \gamma(b)$.



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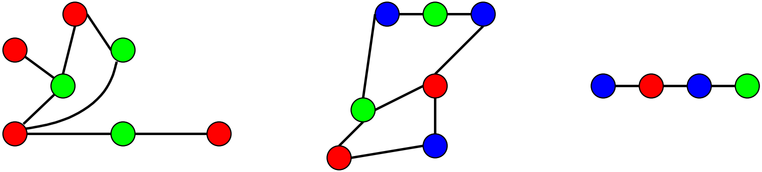
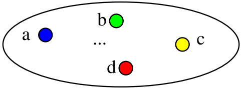
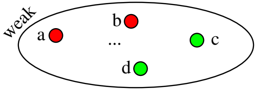


With higher arities on relations we have a problem.
Either: $R(a_1, ..., a_n)$ implies $\gamma(a_i) \neq \gamma(a_j)$ for each i, j.
or (weak coloring) $R(a_1, ..., a_n)$ implies $\gamma(a_i) \neq \gamma(a_j)$ for some i, j.



R(a,b,c,...,d)



R(a,b,c,...,d)

If $\mathcal{M}$ has the *span* operator of a vector space then:

$$a \in Span(b) \Rightarrow \gamma(a) = \gamma(b)$$

$$R(a_1, ..., a_n) \Rightarrow \forall a, b \in Span(a_1, ..., a_n), a \notin span(b) : \gamma(a) \neq \gamma(b)$$

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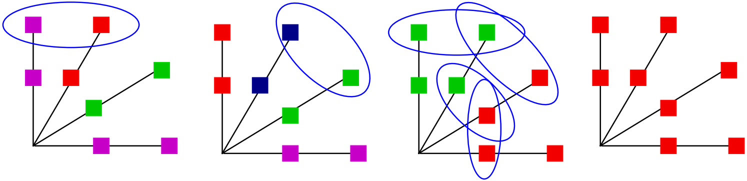
$$R(a_1, ..., a_n) \Rightarrow \forall a, b \in Span(a_1, ..., a_n), a \notin span(b) : \gamma(a) \neq \gamma(b)$$

or for weak colourings

$$R(a_1, ..., a_n) \Rightarrow \exists a, b \in Span(a_1, ..., a_n) : \gamma(a) \neq \gamma(b)$$



Describing a colouring

Fixate a vocabulary V, $l \in \mathbb{N}$ and finite vector spaces $\mathcal{G}_n$. Let $\mathbf{K}_n$ be all $l-$colourable $V-$structures with underlying span operator from $\mathcal{G}_n$. Associate a probability measure δ_n to each $\mathbf{K}_n$ called the dimension conditional measure.

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Theorem

There is a $V-$formula $\xi(x, y)$ such that with asymptotic probability 1 in $\mathcal{N} \in \mathbf{K}_n$ for each $a, b \in \mathcal{N}$, $\mathcal{N} \models \xi(a, b)$ if $\gamma(a) \neq \gamma(b)$ for each $l-$colouring γ.

The proof is quite different depending on if we have strong colourings or not.

A 0-1 law

Theorem

K *has a* $0-1$ *law for the measure* δ.

Proof, use Fagins extension axiom method:

A 0-1 law

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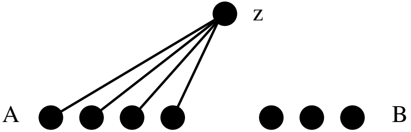
1. Build almost sure extension axioms using $\xi(x, y)$ to make sure the edges are in the right places.



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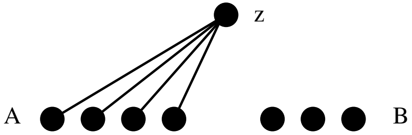


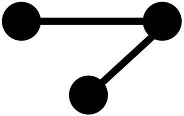
2. Collect the extension axioms into a theory T (together with some more things).
3. Prove that T is $\omega-$categorical and hence complete.

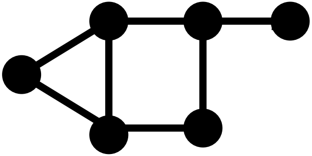
Paper II: Limit laws and automorphism groups of random nonrigid structures

Paper **II** in one sentence:

Having $\mathbf{K}_n$ consisting of structures with non-trivial automorphism group, we show that $\mathbf{K}$ has a convergence law.

For **K** consisting of all $V-$structures we have that almost surely for $\mathcal{N} \in \mathbf{K}$, $Aut(\mathcal{N}) = \{id_{\mathcal{N}}\}$ ($\mathcal{N}$ is rigid).





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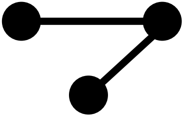

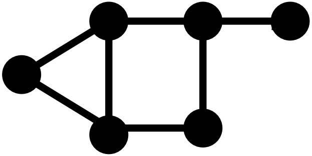

Def: If $f : \mathcal{M} \to \mathcal{M}$ an automorphism then the support

$$Spt(f) = \{a \in M : f(a) \neq a\}$$

$$spt(\mathcal{M}) = max\{spt(g) : g \in Aut(\mathcal{M})\}$$

$$Spt^*(\mathcal{M}) = \{a \in M : \exists g \in Aut(\mathcal{M}), g(a) \neq a\}$$

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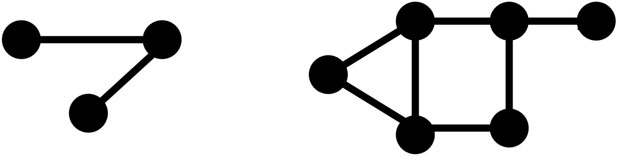

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$$Spt^*(\mathcal{M}) = \{a \in M : \exists g \in Aut(\mathcal{M}), g(a) \neq a\}$$

The above result is that almost surely $Spt^*(\mathcal{N}) = \emptyset$.

Proposition

For each $m \in \mathbb{N}$ *there is* $t > m$ *such that*

$$\lim_{n \to \infty} \frac{|\{\mathcal{M} \in \mathbf{K}_n : spt(\mathcal{M}) \geq t\}|}{|\{\mathcal{M} \in \mathbf{K}_n : spt(\mathcal{M}) \geq m\}|} = 0$$

Proven through tedious computations.

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Proven through tedious computations.

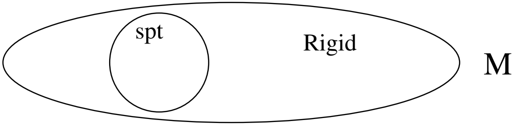
Proposition

For $\mathcal{M} \in \mathbf{K}$ is such that $spt(\mathcal{M}) \leq k$ then $|Spt^*(\mathcal{M})| \leq k^{k+2}$.

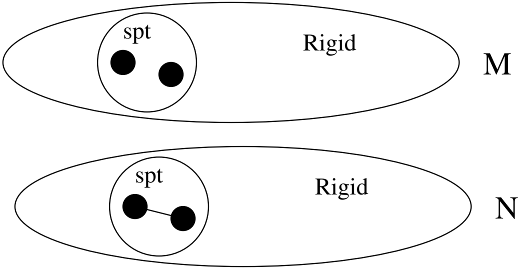
Proven by using a ramsey theoretic argument.

Corollary

When studying asymptotic properties of $\mathcal{M} \in \mathbf{K}_n$ restricted to $spt(\mathcal{M}) \geq m$ or to $Aut(\mathcal{M}) = \mathcal{G}$ for some group $\mathcal{G}$, we may without loss of generality restrict only to $\mathcal{M}$ such that $spt^*(\mathcal{M}) \leq t$ with t independent of n.



The structure of the support is also important.



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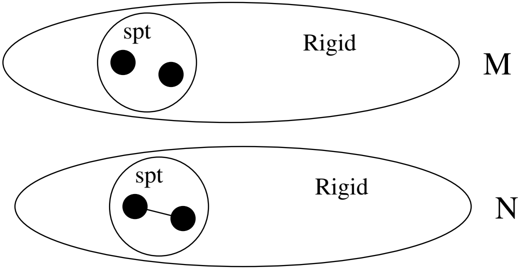

We may describe the support, and both of M and N above are "common". Hence

$$\exists x, y(x, y \in spt \wedge xEy)$$

will not go to 0 or 1, i.e. we do not have a $0-1$ law by just studying $spt^* = k$.

For $\mathcal{A}$ structure without fixed point and $H < Aut(\mathcal{A})$, $\mathbf{S}_n(\mathcal{A}, H)$ be structures with universe $\{1, ..., n\}$ where $\mathcal{A}$ is embeddable and with H a subgroup of the automorphism group of the support.

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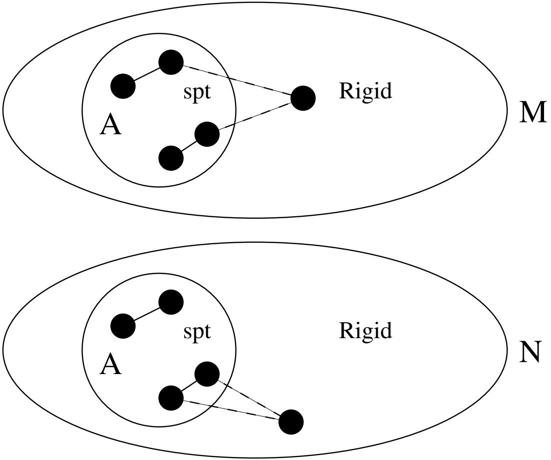

So $M \in \mathbf{S}_n(\mathcal{A}, H)$ and $N \in \mathbf{S}_n(\mathcal{A}, H')$ where $H \neq H'$.

$\mathbf{S}_n(\mathcal{A}, H)$ is a basic enough set to get some nice results.

Proposition

For $\mathcal{A}, \mathcal{A}'$ without fixpoints and $H \leq Aut(\mathcal{A}), H' \leq Aut(\mathcal{A}')$ then:

$$\lim_{n\to\infty} \frac{\mathbf{S}_n(\mathcal{A}, H)}{\mathbf{S}_n(\mathcal{A}', H')}$$

exist and go to $0, \infty$ or some number $a \in \mathbb{Q}^+$, depending on the size of the automorphism groups and the number of orbits.

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Proposition

G, G' finite groups then the following limit exists in $\mathbb{Q}^+ \cup \{\infty\}$

$$\lim_{n\to\infty} \frac{|\{\mathcal{M} \in \mathbf{K}_n : G' \leq Aut(\mathcal{M})\}|}{|\{\mathcal{M} \in \mathbf{K}_n : G \leq Aut(\mathcal{M})\}|}$$

Theorem

$\mathbf{S}_n(\mathcal{A}, H)$ has a $0-1$ law.

Theorem

If G is a finite group then $\{\mathcal{M} \in \mathbf{K} : G \cong Aut(\mathcal{M})\}$, $\{\mathcal{M} \in \mathbf{K} : G \leq Aut(\mathcal{M})\}$, $\{\mathcal{M} \in \mathbf{K} : |spt^*(\mathcal{M})| = m\}$ and $\{\mathcal{M} \in \mathbf{K} : spt(\mathcal{M}) \geq m\}$ all have convergence laws.

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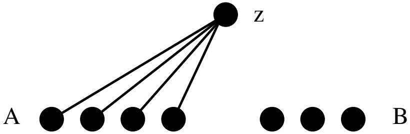
Proof: Extension properties



Which now need to take special care of the support.

Paper III: Countably categorical almost sure theories

Paper **III** in one sentence:

Assuming a $0-1$ law on $\mathbf{K}$ we show how the almost sure theory affect and is affected by $\mathbf{K}$

Let $\mathbf{K} = \cup_{n=1}^{\infty} \mathbf{K}_n$ with each $\mathbf{K}_n$ having a probability μ_n associated.
Recall that $T_{\mathbf{K}}$ is the theory of all sentences φ such that

$$\lim_{n\to\infty} \mu_n(\varphi) = 1$$

“The almost sure theory”

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"The almost sure theory"

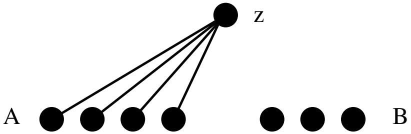
We have shown that using extension properties we may often show $0-1$ laws. In general we get the following

Theorem

K *has a* $0-1$ *law and* $T_{\mathbf{K}}$ *is* $\aleph_0$*−categorical*

iff

K *almost surely satisfies extension properties*



Extension properties may be very complicated (like Paper I).

$\mathcal{M}^{eq}$ is the structure $\mathcal{M}$ where for each $\emptyset-$definable $r-$ary equivalence relation E:

- There is a unique element $e \in M^{eq} - M$ for each $E-$equivalence class.
- There is a new unary relation symbol P_E such that e represents an $E-$equivalence class iff $\mathcal{M}^{eq} \models P_E(e)$
- There is a $r+1$-ary relation symbol $R_E(y, \bar{x})$ such that $\bar{a} \in M$ is in the equivalence class of e iff $\mathcal{M}^{eq} \models R_E(e, \bar{a})$.

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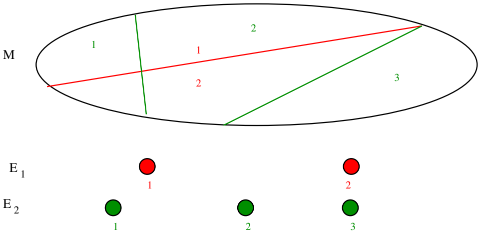

Could be thought of as an "Anti-quotient". A very important structure in infinite model theory.

If $E = \{E_1, ..., E_n\}$ is a finite set of $\emptyset-$definable equivalence relations then let $\mathbf{K}^E$ be $\mathbf{K}$ where we add the $\mathcal{M}^{eq}$ structure for only the equivalence relations in E to each $\mathcal{N} \in \mathbf{K}$.

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Theorem

Let $\mathbf{K}$ *be a set of finite relational structures with almost sure theory* $T_{\mathbf{K}}$*, then*
$\mathbf{K}$ *has a* $0-1$ *law and* $T_{\mathbf{K}}$ *is* $\omega-$*categorical.*
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Proof: An application of the previous theorem.

Strongly minimal countably categorical almost sure theories

A theory T is strongly minimal if for each $\mathcal{M} \models T$, formula $\varphi(x, \bar{y})$ and $\bar{a} \in M$.

$$\varphi(\mathcal{M}, \bar{a}) = \{b \in M : \mathcal{M} \models \varphi(b, \bar{a})\} \text{ or } \neg\varphi(\mathcal{M}, \bar{a})$$

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Theorem

Assume $\mathbf{K}$ *has a* $0-1$ *law and* $\mathcal{N} \in \mathbf{K}_n$ *implies* $|N| = n$ *then*

$$\text{There exists } m \in \mathbb{N} \text{ such that } \lim_{n \to \infty}$$

$$\mu_n(\{\mathcal{M} \in \mathbf{K}_n : \text{there is } X \subseteq M, |X| \leq m, Sym_X(M) \leq Aut(\mathcal{M})\}) = 1$$

$$\Leftrightarrow$$

$$T_{\mathbf{K}} \text{ is strongly minimal and } \omega\text{-categorical}$$

The End

Questions...?