Cofinitely homogeneous multi- and hypergraphs

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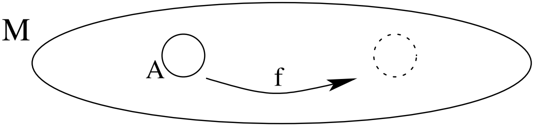
We only consider finite relational languages.

Definition

For a structure $\mathcal{M}$ and a substructure $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}$ is called $\mathcal{A}-$**homogeneous** if for each embedding $f_0 : \mathcal{A} \to \mathcal{M}$, there is an automorphism $f : \mathcal{M} \to \mathcal{M}$ such that f extends f_0
i.e. $\forall a \in \mathcal{A}, f_0(a) = f(a)$.



$\mathcal{M}$ is **homogeneous** if it is $\mathcal{A}-$homogeneous for each finite $\mathcal{A} \subseteq \mathcal{M}$.

Let **K** be a class of structures.

- **K** has the **hereditary property** (HP) if for each $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \in \mathbf{K}$.

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- **K** has the **hereditary property** (HP) if for each $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B} \in \mathbf{K}$.
- $\mathcal{A} \in \mathbf{K}$ is an **amalgamation base** for **K** if for each $\mathcal{B}, \mathcal{C} \in \mathbf{K}$ and $f_0 : \mathcal{A} \to \mathcal{B}, g_0 : \mathcal{A} \to \mathcal{C}$ there is $\mathcal{D} \in \mathbf{K}$ and $f_1 : \mathcal{B} \to \mathcal{D}, g_1 : \mathcal{C} \to \mathcal{D}$ such that for each $a \in \mathcal{A}$, $f_1(f_0(a)) = g_1(g_0(a))$.

$$\begin{tikzcd} & \mathcal{B} \arrow[dr, dashed, "f_1"] & \\ \mathcal{A} \arrow[ur, "f_0"] \arrow[r, "g_0"'] & \mathcal{C} \arrow[r, dashed, "g_1"'] & \mathcal{D} \end{tikzcd}$$

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- **K** satisfies the **amalgamation property** (AP) if each $\mathcal{A} \in \mathbf{K}$ is an amalgamation base.

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- $\mathbf{K}$ satisfies the **amalgamation property** (AP) if each $\mathcal{A} \in \mathbf{K}$ is an amalgamation base.
- $\mathbf{K}$ satisfies the **joint embedding property** (JEP) if for each $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is $\mathbf{C} \in \mathbf{K}$ such that both $\mathcal{A}$ and $\mathcal{B}$ embeds into $\mathcal{C}$

$$Age(\mathcal{M}) = \{\mathcal{A} : \mathcal{A} \hookrightarrow \mathcal{M}, \mathcal{A} \text{ is finite}\}$$

Theorem (Fraïssé 1953)

Let $\boldsymbol{K}$ be a class of finite structures closed under isomorphism satisfying HP, JEP and AP. Then there is a unique countable homogeneous structure $\mathcal{M}$ such that $Age(\mathcal{M}) = \boldsymbol{K}$.

In the relational context JEP can be excluded

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If $\mathbf{C}$ is a set of structures let

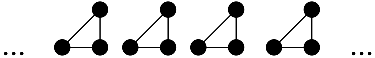
$$Forb(\mathbf{C}) = \{\mathcal{A} : \forall \mathcal{C} \in \mathbf{C}, \mathcal{C} \not\hookrightarrow \mathcal{A}\}$$

If $Forb(\mathbf{C})$ satisfies AP call the unique homogeneous structure $\mathcal{M}$ such that $Age(\mathcal{M}) = Forb(\mathbf{C})$ **the generic $\mathbf{C}-$free structure.**

Let K_n be the complete graph on n vertices.

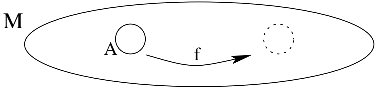
Theorem (Lachlan and Woodrow 1980)

If $\mathcal{M}$ is a countably infinite homogeneous graph then for some $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to either the generic $\emptyset-$free graph, the generic K_n-free graph or a disjoint union of K_n.



Note that the generic $\emptyset-$free graphs is the random graph.

For $k \in \mathbb{Z}^+$, $\mathcal{M}$ is ($>k-$) $k-$**homogeneous** if for each $\mathcal{A} \subseteq \mathcal{M}$ such that $|\mathcal{A}| = k$ ($>k$), $\mathcal{M}$ is $\mathcal{A}-$homogeneous.



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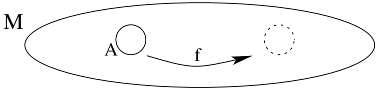

$\mathcal{M}$ is **cofinitely homogeneous** if $\mathcal{M}$ is $> k-$homogeneous for some k.

Note: If $\mathcal{M}$ is cofinitely homogeneous there are a finite amount of isomorphism classes of $\mathcal{A} \subseteq \mathcal{M}$ such that $\mathcal{M}$ is *not* $\mathcal{A}-$homogeneous.

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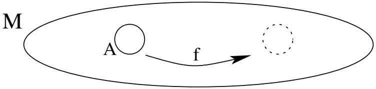

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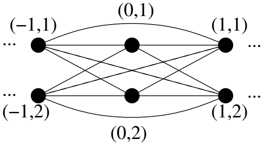
Proposition

Let $\boldsymbol{K}$ be a class of structures closed under isomorphism satisfying HP and JEP such that for some $k \in \mathbb{Z}^+$ each $\mathcal{A} \in \boldsymbol{K}$ with $|\mathcal{A}| > k$ is an amalgamation base.
Then there exists a unique cofinitely homogeneous structure $\mathcal{M}$ such that $Age(\mathcal{M}) = \boldsymbol{K}$.

For $n \in \mathbb{Z}^+$ let $\mathcal{G}_n$ be the graph with vertex set $V = \mathbb{Z} \times \{1, \ldots, n\}$ and edge set $E = \{\{(a,i),(b,j)\} : a \neq b\}$. Note that $\mathcal{G}_n \restriction \mathbb{Z} \times \{i\} \cong K_\infty$ and $\mathcal{G}_n$ is the homogeneous graph isomorphic to the infinite disjoint union of K_n.

The Graph G_2



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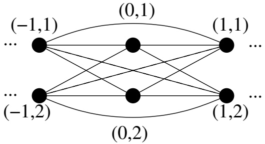
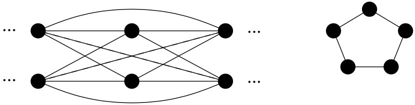


Lemma

*Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is cofinitely homogeneous but **not** $1-$homogeneous if and only if for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{G}_n \dot\cup \mathcal{H}$.*

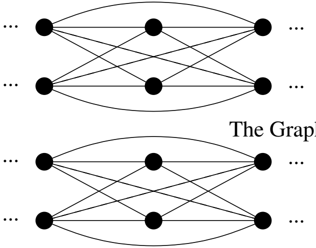
The Graph $G_2 \dot\cup C_5$



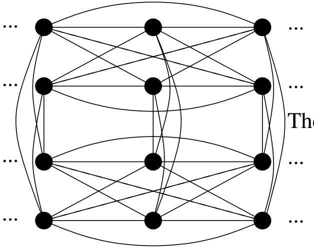
If $t \geq 2$ let $\mathcal{H}_{t,1}$ be the graph with universe $H_{t,1} = \mathbb{Z} \times \{1, ..., t\} \times \{1, 2\}$ such that the inclusion map $\iota : \mathcal{H}_{t,1} \to \mathcal{G}_t \times \{1, 2\}$ is an isomorphism. Let $\mathcal{H}_{t,2}$ have the same universe as $\mathcal{H}_{t,1}$ but with edge set

$$E_{t,2} = E_{G_{t,1}} \cup \{\{(a, i, z), (b, j, w)\} : z \neq w \text{ and } a = b\}.$$

Lastly define the graph $\mathcal{H}_{1,2}$ as having universe $\mathbb{Z} \times \{1, 2\}$ and edge set $E = \{\{(a, i), (b, j)\} : i = j \text{ or } a = b \text{ but } i \neq j\}$.



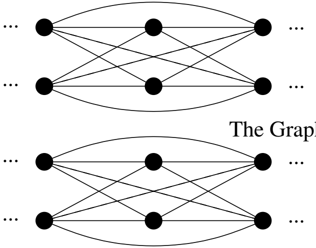
The Graph $H_{2,1}$



The Graph $H_{2,2}$

Lemma

*Let $\mathcal{M}$ be a countably infinite graph. $\mathcal{M}$ is cofinitely homogeneous, $1-$homogeneous but **not** $2-$homogeneous if and only if $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to $\mathcal{H}_{t,1}$ or $\mathcal{H}_{t,2}$ for some t.*



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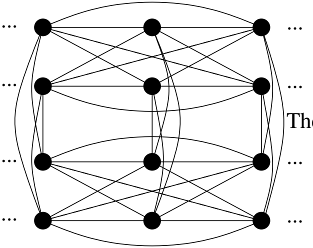

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Lemma

If $\mathcal{M}$ is a cofinitely homogeneous infinite graph which is $1-$homogeneous and $2-$homogeneous, then $\mathcal{M}$ is homogeneous.

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Theorem (A. 2018)

If $\mathcal{M}$ is a cofinitely homogeneous countably infinite graph then $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to one of the following

- *A homogeneous graph.*
- $\mathcal{G}_n \dot{\cup} \mathcal{H}$ *for some* $n \in \mathbb{Z}^+$ *and finite homogeneous graph* $\mathcal{H}$.
- $\mathcal{H}_{t,1}$ *or* $\mathcal{H}_{t,2}$ *for some* t.

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Note: The non-homogeneous cofinitely homogeneous graphs are all $\omega-$stable.
They also have relational complexity 1 and 2 respectively.

Remember that for $A \subseteq \mathcal{M}$,

$$\begin{aligned} acl(A) = \{b \in M : \text{There is } \varphi(x, \bar{y}) \text{ such that for some } \bar{a} \in A \\ \mathcal{M} \models \varphi(b, \bar{a}) \text{ and } \varphi(x, \bar{a}) \text{ is algebraic}\} \\ = \{b \in M : tp(b/A) \text{ is algebraic}\} \end{aligned}$$

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Definition

A structure $\mathcal{M}$ is **pseudotrivial** if for each $a \in \mathcal{M}$, $acl(a) \neq \{a\}$.

Does this make sense?

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No, but...

Theorem (Lachlan and Woodrow 1980)

If $\mathcal{M}$ is a countably infinite homogeneous graph then for some $n \in \mathbb{Z}^+ \cup \{\infty\}$, $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to either the random graph, the generic K_n−free graph or a disjoint union of K_n.

The pseudotrivial homogeneous graphs are all the infinite disjoint unions of K_n for finite n.

Note: K_∞ and disjoint unions of K_∞ are not pseudotrivial.

Theorem (A. 2018)

Let $k \in \mathbb{Z}^+$. If $\mathcal{M}$ is a non-homogeneous cofinitely homogeneous countably infinite graph then $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to one of the following

- $\mathcal{G}_n \dot\cup \mathcal{H}$ *for some $n \in \mathbb{Z}^+$ and finite homogeneous graph $\mathcal{H}$.*
- $\mathcal{H}_{t,1}$ *or* $\mathcal{H}_{t,2}$ *for some t.*

Non-homogeneous cofinitely homogeneous graphs are pseudotrivial. (unless $n = 1$)

Thus *pseudotrivial*, in the context, makes sense.

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Can we classify the cofinitely homogeneous $r-$hypergraphs?

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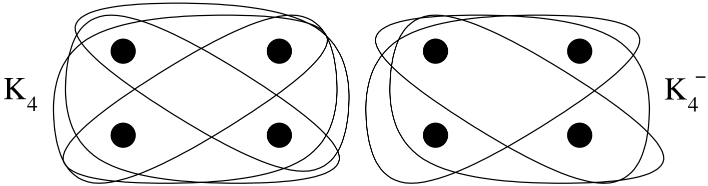
Can we classify the cofinitely homogeneous $r-$hypergraphs?

Warning!

The homogeneous $r-$hypergraphs are not (even close to being) classified and there are uncountably many isomorphism classes.

Let K_4 be the complete $3-$hypergraph on 4 elements.
Let K_4^- be K_4 but with one edge removed.



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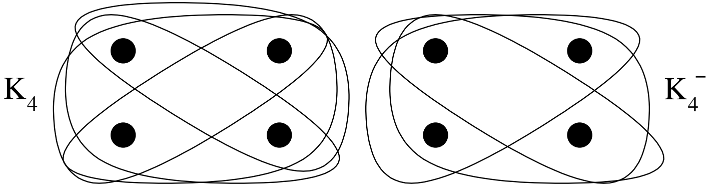

Important Example: $Forb(K_4, K_4^-)$ satisfy HP and AP. Thus the generic (K_4, K_4^-)-free hypergraph $\mathcal{M}$ exists and is homogeneous. Note that $\mathcal{M}$ is not simple.

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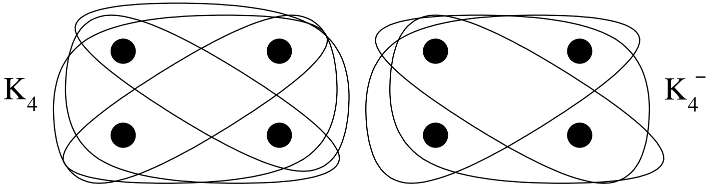

Important Example: $Forb(K_4, K_4^-)$ satisfy HP and AP. Thus the generic (K_4, K_4^-)-free hypergraph $\mathcal{M}$ exists and is homogeneous. Note that $\mathcal{M}$ is not simple.

$\mathcal{M}^c \dot{\cup} \mathcal{M}^c$ is a cofinitely homogeneous hypergraph which is not simple.
Remember that all cofinitely homogeneous graphs are $\omega-$stable.

For *any* homogeneous 3-hypergraph $\mathcal{M}$ such that $Age(\mathcal{M}) \subseteq Forb(K_4, K_4^-)$, $\mathcal{M}^c \dot{\cup} \mathcal{M}^c$ is[1] cofinitely homogeneous.

Question

The generic (K_4, K_4^-)−free hypergraph and the empty hypergraph both satisfy the above condition. Are there more? Are there uncountably many?

[1] We may have "a few" edges between the two parts.

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What about the *pseudotrivial* cofinitely homogeneous hypergraphs?

Lemma

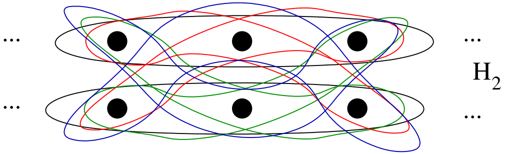
For $r > 2$ there are no pseudotrivial homogeneous $r-$hypergraphs.

[1] We may have "a few" edges between the two parts.

For $n \in \mathbb{Z}^+$ let $\mathcal{H}_n$ be the 3−hypergraph with vertex set $V = \mathbb{Z} \times \{1, \dots, n\}$ and edge set

$$E = \{\{(a,i),(b,j),(c,k)\} : a \neq b, b \neq c, a \neq c\}.$$

Note that $\mathcal{H}_n \restriction \mathbb{Z} \times \{i\} \cong K_\infty$.



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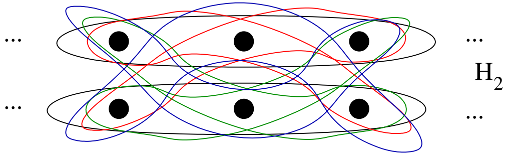
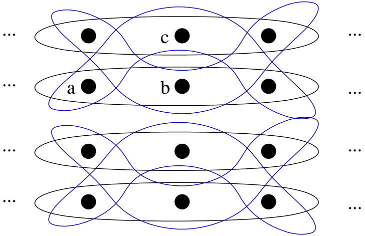


If $\mathcal{A}$ is a hypergraph such that $|\mathcal{A}| = n$ then $\mathcal{H}_n(\mathcal{A})$ is $\mathcal{H}_n$ but with $\mathcal{H}_n(\mathcal{A}) \restriction \{i\} \times \{1, \dots, n\} \cong \mathcal{A}$. i.e. $\mathcal{A}$ is on each column. $\mathcal{H}_n(\mathcal{A})$ is **not** homogeneous unless $n = 1$, it is however cofinitely homogeneous.

Note that $\mathcal{H}_n \dot\cup \mathcal{H}_n$ is **not** cofinitely homogeneous.
a, b in the same row and b, c in the same column are not distinguishable after choosing only elements in the other part.



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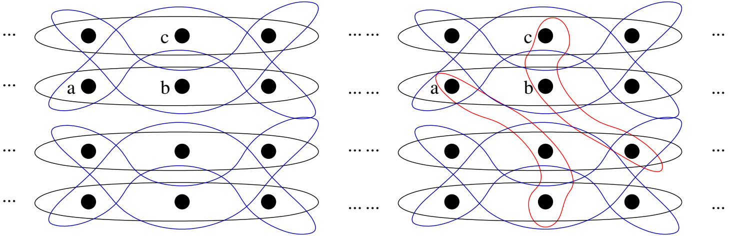
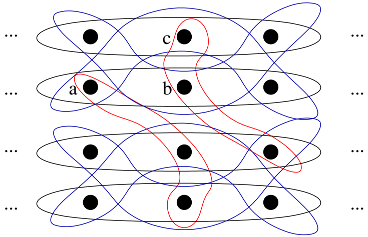


Let $\mathcal{H}_n \otimes \mathcal{H}_n$ be $\mathcal{H}_n \dot{\cup} \mathcal{H}_n$ but $R(a, b, c)$ also hold for each a in one component and b, c in the other component which are in the same column.

Lemma

Let $\mathcal{M}$ be a pseudotrivial cofinitely homogeneous $3-$hypergraph which is $1-$homogeneous but not $2-$homogeneous.
There is a homogeneous finite $3-$hypergraph $\mathcal{A}$, $n = |\mathcal{A}|$, such that $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to

$$\mathcal{H}_n(\mathcal{A}) \otimes \mathcal{H}_n(\mathcal{A}) \text{ or } \mathcal{H}_n(\mathcal{A})$$



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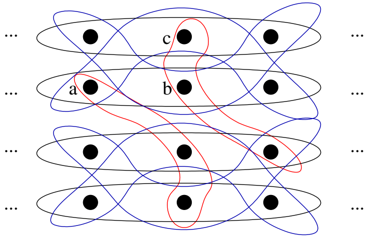
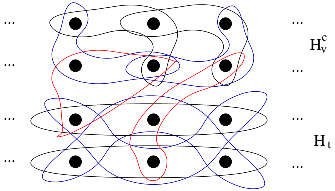


Note: The homogeneous finite $3-$hypergraphs were classified by Lachlan and Tripp (1995). Still open for $r-$hypergraphs.

Let $\mathcal{H}_t \oplus \mathcal{H}_v^c$ be $\mathcal{H}_t \dot\cup \mathcal{H}_v^c$ but with extra $R(a,b,c)$ for

- each $a \in \mathcal{H}_t^c$ and $b, c \in \mathcal{H}_v$ such that b, c are in the same column.
- each $a \in \mathcal{H}_v$ and $b, c \in \mathcal{H}_t^c$ such that b, c are **not** in the same column.



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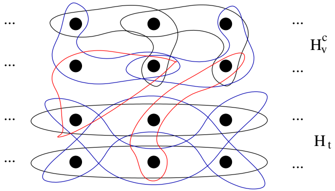


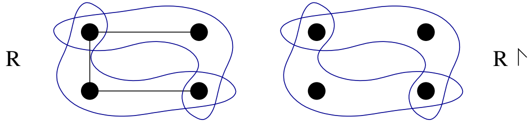
Lemma

Let $\mathcal{M}$ be a cofinitely homogeneous pseudotrivial $3-$hypergraph which is ***not*** *1-homogeneous, and have at least 2 infinite $1-$types.*
Then, for some homogeneous $\mathcal{A}$, $|\mathcal{A}| = t$ and $\mathcal{B}$, $|\mathcal{B}| = v$, we have that $\mathcal{M} \cong \mathcal{H}_t(\mathcal{A}) \oplus \mathcal{H}_v^c(\mathcal{B})$.

A cat

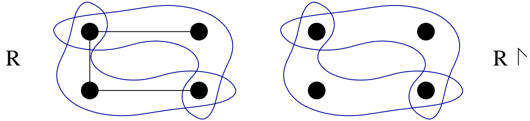
A $2,3-$hypergraph is a $3-$hypergraph with an extra binary relation E which is symmetric and anti-reflexive.

If $\mathcal{R}$ is a $2,3-$hypergraph then $\mathcal{R} \upharpoonright$ is the reduct of $\mathcal{R}$ to a $3-$hypergraph.

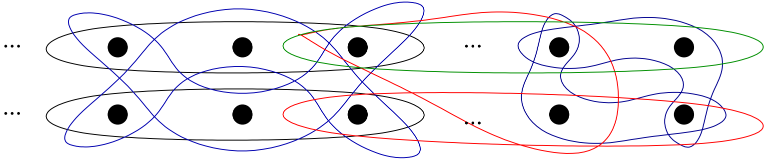


A $2,3-$hypergraph is a $3-$hypergraph with an extra binary relation E which is symmetric and anti-reflexive.

If $\mathcal{R}$ is a $2,3-$hypergraph then $\mathcal{R}\restriction$ is the reduct of $\mathcal{R}$ to a $3-$hypergraph.



If $\mathcal{A} = \mathcal{R}\restriction$, then define $\mathcal{H}_t \otimes_{\mathcal{R}} \mathcal{A}$ as $\mathcal{H}_t \dot{\cup} \mathcal{A}$, but also with $R(a,b,c)$ if $a \in \mathcal{H}_t$ and $\mathcal{R} \models E(b,c)$.



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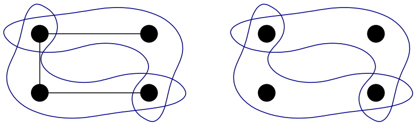
R



R $\restriction$

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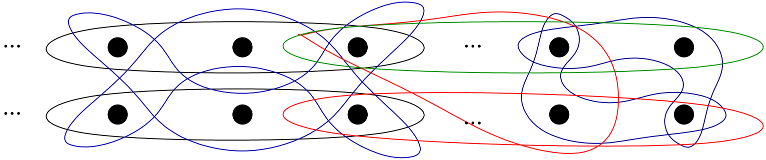
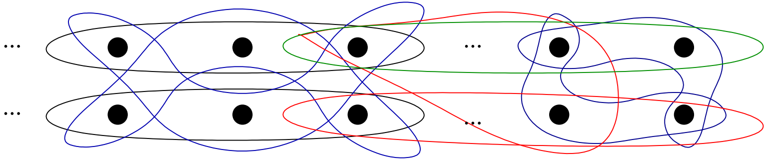


Further more let $\mathcal{H}_t \widehat{\otimes}_{\mathcal{R}} \mathcal{A}$ be $\mathcal{H}_t \otimes_{\mathcal{R}} \mathcal{A}$ but with $R(a,b,c)$ for each $a,b \in \mathcal{H}_t$ in the same column and $c \in \mathcal{A}$.



Lemma

Let $\mathcal{M}$ be a cofinitely homogeneous pseudotrivial $3-$hypergraph which is not $1-$homogeneous with at least one finite $1-$type. Then there is a homogeneous $2,3-$hypergraph $\mathcal{R}$ with $\mathcal{A} = \mathcal{R} \restriction$ and a homogeneous $3-$hypergraph $\mathcal{B}$ with $|\mathcal{B}| = t$ such that $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to

$$\mathcal{H}_t(\mathcal{B}) \otimes_{\mathcal{R}} \mathcal{A} \textit{ or } \mathcal{H}_t(\mathcal{B}) \widehat{\otimes}_{\mathcal{R}} \mathcal{A}$$

There is no classification of homogeneous $2,3-$hypergraphs.

Trivial Lemma

No $\omega-$categorical pseudotrivial $2-$homogeneous $r-$hypergraph, with $r > 2$, exists.

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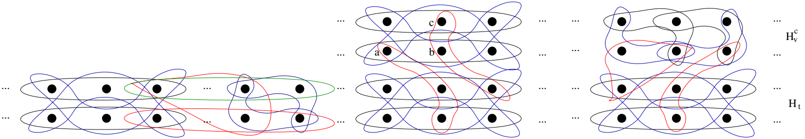
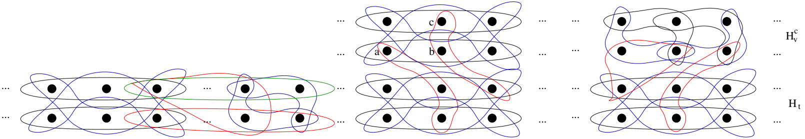


Theorem (A. 2018)

Let $\mathcal{M}$ be a pseudotrivial cofinitely homogeneous $3-$hypergraph. Then $\mathcal{M}$ (or $\mathcal{M}^c$) is isomorphic to

- $\mathcal{H}_t(\mathcal{B}) \otimes_{\mathcal{R}} \mathcal{A}$ *or* $\mathcal{H}_t(\mathcal{B}) \widehat{\otimes}_{\mathcal{R}} \mathcal{A}$.
- $\mathcal{H}_t(\mathcal{A}) \oplus \mathcal{H}_v^c(\mathcal{B})$.
- $\mathcal{H}_t(\mathcal{A}) \otimes \mathcal{H}_t(\mathcal{A})$ *or* $\mathcal{H}_t(\mathcal{A})$

Very similar for pseudotrivial $r-$hypergraphs.



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Note: All pseudotrivial cofinitely homogeneous $r-$hypergraphs are $\omega-$stable.
They also have relational complexity at most $r-1$.

For $r \in \mathbb{Z}^+$ an $r-$**multigraph** $\mathcal{G}$ is a structure $(G, E_1, \ldots, E_{r-1})$ where G is the universe and each E_i is a binary relation which is symmetric anti-reflexive. Also, if $i \neq j$ then $E_i \cap E_j = \emptyset$.

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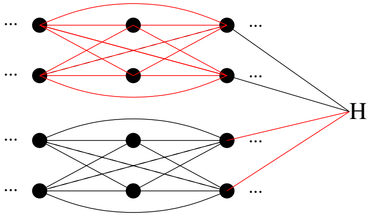
Can we classify the cofinitely homogeneous multigraphs?

Warning!

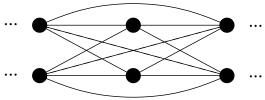
The homogeneous $r-$multigraphs are not (for $r > 2$) classified and there are uncountably many isomorphism classes.

Discouraging example:
Let $\mathcal{M}$ be a homogeneous $r-$multigraph using only the edges $E_1, \ldots, E_r$.
Let $\mathcal{N}$ be a homogeneous $r-$multigraph using only the edges $F_1, \ldots, F_r$.
Then $\mathcal{M} \dot{\cup} \mathcal{N}$ is a cofinitely homogeneous $(2r+1)-$multigraph.



Lemma

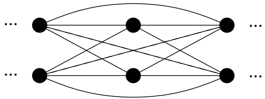
Let $\mathcal{M}$ cofinitely homogeneous $3-$multigraph with 2 infinite $1-$types and 1 finite $1-$type, then $\mathcal{M} \cong G_t(\mathcal{A}) \otimes G_v(\mathcal{B}) \otimes \mathcal{H}$, for finite homogeneous graphs $\mathcal{A}, \mathcal{B}$ and $3-$multigraphs $\mathcal{H}$.



H

Lemma

Let $\mathcal{M}$ cofinitely homogeneous $3-$multigraph with 1 infinite $1-$type and 1 finite $1-$type, then $\mathcal{M} \cong \mathcal{G}_t(\mathcal{A})\dot{\cup}\mathcal{H}$ or $\mathcal{G}(\mathcal{A})\dot{\cup}\mathcal{H}$ for finite homogeneous graph $\mathcal{A}$ and $3-$multigraph $\mathcal{H}$ and cofinitely homogeneous, $1-$homogeneous graph $\mathcal{G}$.



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Lemma

A classification of the cofinitely homogeneous $3-$multigraphs which are not $1-$homogeneous.

Summary

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[2] Classified up to some finite homogeneous structures

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- Cofinitely homogeneous $r-$hypergraphs
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- Cofinitely homogeneous digraphs, tournaments and other structures. Very open

[2] Classified up to some finite homogeneous structures

Thank you!