A zero-one law for l-colourable structures with a vector space pregeometry

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Finite structures $\mathcal{M}$ over a language L with the universe $\{1, ..., n\}$ for some $n \in \mathbb{Z}^+$.

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Associate a probability measure μ_n with $\mathbf{K}_n$.
Extend to formulas

$$\mu_n(\varphi) = \mu_n(\{\mathcal{M} \in \mathbf{K}_n : \mathcal{M} \models \varphi\})$$

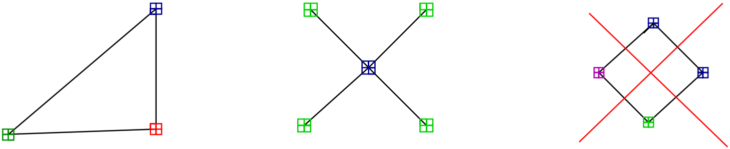
$\mathbf{K} = \bigcup_{n=1}^{\infty} \mathbf{K}_n$ has a $0 - 1$ law if for each sentence $\varphi \in L$

$$\lim_{n\to\infty} \mu_n(\varphi) = 1 \text{ or } \lim_{n\to\infty} \mu_n(\varphi) = 0$$

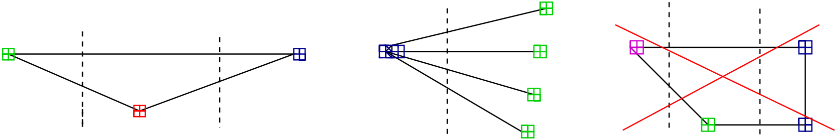
Fagin[1] and Glebskii et.al.[2] independently for $\mathbf{K}_n$ all structures.

[1] R. Fagin, Probabilities on finite model theory, J. Symbolic Logic 41 (1976), no.1
[2] Glebskii et.al. Volume and fraction of Satisfiability... , Kibernetyka Vol. 2 (1969)

Graph $\mathcal{G} =< V, E >$, V - Universe of vertices, E - Binary adjacency relation on V. For some $l \in \mathbb{Z}^+$ a graph $\mathcal{G}$ is $l-$coloured if adjacent vertices implies different colour.



May view as partitions without edges.



Kolaitis et. al.[3] showed that for $l > 2$

$$\mathbf{K}_n = \{\mathcal{G} : \mathcal{G} \text{ is an } l-\text{colourable graph with universe } \{1, ..., n\}\}$$

has a $0-1$ law using the uniform measure i.e. $\mu_n(\mathcal{G}) = \frac{1}{|\mathbf{K}_n|}$.
How is this generalised?

[3] Ph. G. Kolaitis, H. J. Prömel, B. L. Rothschild, $K_l + 1$ -free graphs: Asymptotic structure and a 0-1 law, Transactions of the AMS, Oct 1987

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Colouring for structures whose relations have arities ≥ 3 may be generalised in two ways:

- Strong colourings
- Weak colourings

[3] Ph. G. Kolaitis, H. J. Prömel, B. L. Rothschild, K_{l+1} -free graphs: Asymptotic structure and a 0-1 law, Transactions of the AMS, Oct 1987

Koponen[4] showed for both strong and weak colourings that if

$$\mathsf{K}_n = \{\mathcal{M} : \mathcal{M} \text{ is } l-\text{coloured with universe } \{1, ..., n\}\}$$

then K_n has a $0-1$ law for both the uniform measure and the dimension conditional measure.

Applications in extremal hypergraph theory: $\mathcal{G}$ is $\mathcal{H}-$free if $\mathcal{G}$ has no (possibly weak) subgraph which is isomorphic to $\mathcal{H}$.

This sometimes results in a partitionable graph ex. Person et. al[5] which hence is $l-$colourable, and so has a $0-1$ law.

[4] V. Koponen, Asymptotic probabilities of extension properties and random l-colourable structures, Annals of Pure and Applied Logic, Vol. 163 (2012) 391-438.

[5] Y. Person, M. Schacht, Almost all hypergraphs without Fano planes are bipartite, SODA 09 Proceedings of the twentieth annual ACM-SIAM Symposium on discrete mathematics, Society for industrial and applied mathematics, Philadelphia, 2009.

Vector spaces may be generalised into pregeometries:

For a finite dimensional vector space $\mathcal{V}$ over a finite field let $\mathcal{G} = (V, cl_{\mathcal{G}})$ be s.t. $cl_{\mathcal{G}} : \mathcal{P}(V) \to \mathcal{P}(V)$ and for

$$X \subseteq V \qquad cl_{\mathcal{G}}(X) := Span_{\mathcal{V}}(X).$$

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We call $\mathcal{G}$ a (vector space) pregeometry. *This notion may be generalised further.*

A structure $\mathcal{M}$ has an underlying vector space pregeometry $\mathcal{G}$ if there are multiple formulas θ such that

$$a \in cl_{\mathcal{G}}(A) \quad \Leftrightarrow \quad \theta(a, A).$$

so we may define the pregeometry in $\mathcal{M}$.

What if such a structure was coloured?

For some $l \in \mathbb{Z}^+$ a structure $\mathcal{M}$ with an underlying vector space pregeometry is $l-$painted if

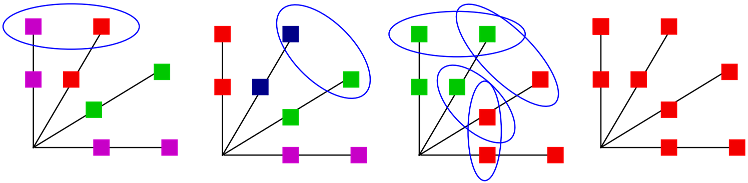
- Each element (except zero) has a colour (out of l possible).
- Elements in the same one-dimensional subspace have the same colour.

If $\mathcal{M}$ is $l-$painted then

- $\mathcal{M}$ is weakly $l-$coloured if for each relation R and $\bar{x} \in R^{\mathcal{M}}$ there are $x, y \in cl(\bar{x})$ such that x and y have different colour.
- $\mathcal{M}$ is strongly $l-$coloured if for each relation R and $\bar{x} \in R^{\mathcal{M}}$ then for each $x, y \in cl(\bar{x})$, s.t. $x \notin cl(y)$, then x and y have different colour.

Notice that if R is binary and is the only relation and $cl(X) = X$ for each $X \subseteq M$ then being weakly or strongly coloured is the same as a graph being coloured.

Bellow we have $\mathbb{Z}_3^2$ coloured with the blue areas representing a binary relation.



Koponen[4] showed that each (weakly or strongly) $l-$coloured set of structures with a common pregeometry has a $0-1$ law.

The uniform measure on a set $\mathbf{K}_n$ for any $X \subseteq \mathbf{K}_n$ is $\mu_n(X) = \frac{|X|}{|\mathbf{K}_n|}$.

But how easy is it to determine $|\mathbf{K}_n|$?

How does μ_n reflect the creation process of $\mathcal{M}$?

These are not trivialities and makes the uniform measure a bit bad when used in constructions.

One possible solution is the dimension conditional measure δ_n.

The uniform measure on a set K_n for any $X \subseteq \mathsf{K}_n$ is $\mu_n(X) = \frac{|X|}{|\mathsf{K}_n|}$.

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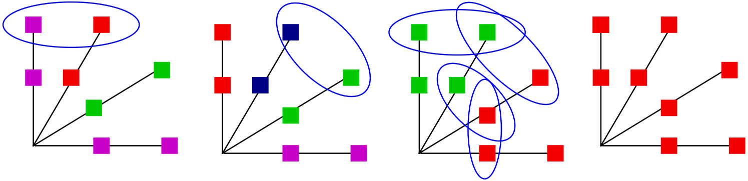
One possible solution is the dimension conditional measure δ_n. Let

K_n is the set of all l-coloured structures over a language L with an underlying vector space pregeometry with dimension n. Then:

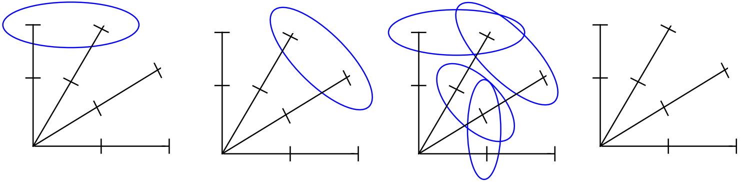
$$\delta_n'(\mathcal{M}) = \frac{1}{|\text{Different colourings of structures in } \mathsf{K}_n|}$$

$$\delta_n(\mathcal{M}) = \delta_n'(\mathcal{M}) * \frac{1}{|\text{Structures in } \mathsf{K}_n \text{ with the same colouring as } \mathcal{M}|}$$

These are coloured structures.



But these are colourable structures i.e. no explicit colours.



Colourable is much weaker than coloured, but it is a more applicable case. Let

$$C_n = \{\mathcal{M} : M = \{1, ..., n\}, \mathcal{M} \text{ has a pregeometry \& weakly colourable}\}$$

$$S_n = \{\mathcal{M} : M = \{1, ..., n\}, \mathcal{M} \text{ has a pregeometry \& strongly colourable}\}$$

Koponen[4] showed that for the trivial pregeometry ($cl_{\mathcal{M}}(X) = X$) C_n and S_n have a $0-1$ law for both δ_n and μ_n.

Theorem

If C_n (or S_n) has a vector space, affine or projective pregeometry then C_n (or S_n) has a $0-1$ law.

may have even weaker assumptions in the case of S_n.

The following more specific results have also been proved

Theorem

If C_n (or S_n) has a vector space, affine or projective pregeometry then:

- *The sentences which are almost surely true has an explicit axiomatisation.*
- *There is a formula ξ such that almost surely if $\mathcal{M} \in C_n$ (or S_n) then for each $x, y \in \mathcal{M}$, x and y have the same colour iff $\mathcal{M} \models \xi(x, y)$.*
- *For each structure there is almost surely a unique colouring.*

Where "almost surely" means that the property has asymptotic probability one. All is with regard to the dimension conditional measure δ_n and holds for both C_n and S_n, with some weaker assumptions possible for S_n.

Proof method according according to Fagin[1] which is summarised as follows:

1. Create so called extension axioms, which can describe how to form bigger structures from smaller ones.
2. Prove that the extension axioms are true with asymptotic probability 1.
3. Create a theory from the extension axioms together with some more necessary sentences.
4. Prove that this theory is countably categorical and consistent.
5. Conclude that the theory is complete and hence φ or $\neg\varphi$ has probability one following from part 2.

Much work is put into creating a formula ξ which describes how a colouring will look like without mentioning colours.

Questions?

O. Ahlman, V. Koponen, Random l-colourable structures with a pregeometry, Preprint, http://arxiv.org/abs/1207.4936

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2. Y. V Glebskii. D. I . Kogan, M.I. Liogonkii, V.A. Talanov, Volume and fraction of Satisfiability of formulas of the lower predicate calculus, Kibernetyka Vol. 2 (1969) 17-27.
3. Ph. G. Kolaitis, H. J. Prömel, B. L. Rothschild, K_{l+1} -free graphs: Asymptotic structure and a $0-1$ law, Transactions of the American Mathematical Society, volume 303, number 2, Oct 1987, 637-671.
4. V. Koponen, Asymptotic probabilities of extension properties and random l-colourable structures, Annals of Pure and Applied Logic, Vol. 163 (2012) 391-438.
5. Y. Person, M. Schacht, Almost all hypergraphs without Fano planes are bipartite, SODA 09 Proceedings of the twentieth annual ACM-SIAM Symposium on discrete mathematics, Society for industrial and applied mathematics, Philadelphia, 2009.