Countably categorical almost sure theories

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Generalized to finite relational first order structures $\mathcal{M} = (M, R_1, \ldots, R_k)$. 

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Generalized to finite relational first order structures $\mathcal{M} = (M, R_1, \ldots, R_k)$.

For each $n \in \mathbb{N}$ let $K_n$ be a finite set of finite structures and $\mu_n$ a probability measure on $K_n$. If $\varphi$ is a formula let

$$\mu_n(\varphi) = \mu_n(\{\mathcal{N} \in K_n : \mathcal{N} \models \varphi\})$$

$K = \bigcup_{n=1}^{\infty} K_n$ has a convergence law if for each first order formula $\varphi$, $\lim_{n \to \infty} \mu_n(\varphi)$ converges.
If we let $\varphi$ be the formula $\exists x \exists y (xEy)$ then

$$\mu_1(\varphi) = 0 \quad \mu_2(\varphi) = 2/3 \quad \mu_3(\varphi) = 5/6$$

$$\lim_{n \to \infty} \mu_n(\varphi)$$ converges if the sequence $0, 2/3, 5/6, \ldots$ converges.
0-1 laws

If for each formula $\varphi$

$$\lim_{n \to \infty} \mu_n(\varphi) = 1 \quad \text{or} \quad \lim_{n \to \infty} \mu_n(\varphi) = 0$$

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$$\lim_{n \to \infty} \mu_n(\varphi) = 1 \quad \text{or} \quad \lim_{n \to \infty} \mu_n(\varphi) = 0$$

then $K$ has $0 - 1$ law.

Let $K_n$ consisting of all structures with universe $\{1, \ldots, n\}$ (over a fixed vocabulary) with $\mu_n(\mathcal{N}) = \frac{1}{|K_n|}$. Fagin (1976) and independently Glebskii et. al.(1969) proved that this $K$ has a $0 - 1$ law.
More 0 − 1 Laws

Let \( K \) consist of all partial orders and let \( \mu_n(M) = \frac{1}{|K_n|} \). Compton (1988): \( K \) has a 0 − 1 law.
More 0 – 1 Laws

Let $\mathcal{K}$ consist of all partial orders and let $\mu_n(\mathcal{M}) = \frac{1}{|\mathcal{K}_n|}$. Compton (1988): $\mathcal{K}$ has a 0 – 1 law.

Let $\mathcal{K}$ consist of all graphs but let $\mu_n$ give high probability to sparse (few edges) graphs. Shelah and Spencer (1988) showed that $\mathcal{K}$ has a 0 – 1 law.
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Let $K$ consist of all $d$—regular graphs and $\mu_n$ a certain, edge depending, probability measure. Haber and Krivelevich (2010) proved that $K_n$ has a $0—1$ law.
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Let $K$ consist of all $l$–coloured structures with a vectorspace pregeometry. Koponen (2012) proved a 0 – 1 law for $K$ under both uniform (the normal $\frac{1}{|K_n|}$) and dimension conditional measure.
Fagins method of proving 0 – 1 laws

$\mathcal{N}$ satisfies the k-extension property $\varphi_k$ (for graphs) if:

\[ A, B \subseteq \mathcal{N}, A \cap B = \emptyset, |A \cup B| \leq k \Rightarrow \exists z : \]

\[ aEz \text{ and } \neg bEz \text{ for each } a \in A, b \in B \]
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If $K$ consist of all structures, then $\lim_{n \to \infty} \mu_n(\varphi_k) = 1$. We say that $\varphi_k$ is an almost sure property.
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$$T_\mathcal{K} = \{ \varphi : \lim_{n \to \infty} \mu_n(\varphi) = 1 \}$$

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If \( \mathbf{K} \) consist of all structures, then \( \lim_{n \to \infty} \mu_n(\varphi_k) = 1 \). We say that \( \varphi_k \) is an almost sure property.

\[ T_\mathbf{K} = \{ \varphi : \lim_{n \to \infty} \mu_n(\varphi) = 1 \} \]

is called the almost sure theory.

Note: \( T_\mathbf{K} \) is complete iff \( \mathbf{K} \) has a 0–1 law.
Let $\kappa \geq \aleph_0$. For $\kappa$-categorical theories completeness is equivalent with not having any finite models.

**Theorem**

$T_K$ is $\aleph_0$–categorical.

Hence this will prove that $K$ has a 0 – 1 law.
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**Theorem**

$T_K$ is $\aleph_0$–categorical.

Hence this will prove that $K$ has a $0 – 1$ law.

**Proof.**

Take $N, M \models T_K$. Build partial isomorphisms back and forth by using the extension properties to help.
The proof method with extension properties has been used in multiple articles proving $0 - 1$ laws. In general we get the following
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**Theorem**

\( K \) has a 0–1 law and \( T_K \) is \( \aleph_0 \)-categorical

iff

\( K \) almost surely satisfies all extension properties

Extension properties may be very complicated.
$\mathcal{M}^{eq}$ is constructed from a structure $\mathcal{M}$ by for each $\emptyset$–definable $r$–ary equivalence relation $E$:

- Add unique element $e \in M^{eq} - M$ for each $E$–equivalence class.
- Add new unary relation symbol $P_E$ such that $e$ represents an $E$–equivalence class iff $\mathcal{M}^{eq} \models P_E(e)$
- Add a $r + 1$-ary relation symbol $R_E(y, \bar{x})$ such that $\bar{a} \in M$ is in the equivalence class of $e$ iff $\mathcal{M}^{eq} \models R_E(e, \bar{a})$. 

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- Add new unary relation symbol \( P_E \) such that \( e \) represents an \( E \)-equivalence class iff \( \mathcal{M}^{eq} \models P_E(e) \).
- Add a \( r + 1 \)-ary relation symbol \( R_E(y, \bar{x}) \) such that \( \bar{a} \in M \) is in the equivalence class of \( e \) iff \( \mathcal{M}^{eq} \models R_E(e, \bar{a}) \).

Could be thought of as an “Anti-quotient”. A very important structure in infinite model theory.
If $E = \{E_1, \ldots, E_n\}$ is a finite set of $\emptyset -$definable equivalence relations then let $K^E$ be $K$ where we add the $M^{eq}$ structure for only the equivalence relations in $E$ to each $\mathcal{N} \in K$. 

Theorem

Let $K$ be a set of finite relational structures with almost sure theory $T_K$, then $K$ has a $0$-$1$ law and $T_K$ is $\omega$-categorical.

iff $K^E$ has a $0$-$1$ law and $T_{K^E}$ is $\omega$-categorical.

Proof:

An application of the previous theorem.
If $E = \{E_1, \ldots, E_n\}$ is a finite set of $\emptyset$–definable equivalence relations then let $K^E$ be $K$ where we add the $M^{eq}$ structure for only the equivalence relations in $E$ to each $\mathcal{N} \in K$.

**Theorem**

Let $K$ be a set of finite relational structures with almost sure theory $T_K$, then

$K$ has a 0–1 law and $T_K$ is $\omega$–categorical.

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$K^E$ has a 0–1 law and $T_{K^E}$ is $\omega$–categorical.
If \( E = \{ E_1, ..., E_n \} \) is a finite set of \( \emptyset \)-definable equivalence relations then let \( K^E \) be \( K \) where we add the \( M^{eq} \) structure for only the equivalence relations in \( E \) to each \( N \in K \).

**Theorem**

*Let \( K \) be a set of finite relational structures with almost sure theory \( T_K \), then*

\( K \) *has a 0–1 law and \( T_K \) is \( \omega \)-categorical.*

iff

\( K^E \) *has a 0–1 law and \( T_{K^E} \) is \( \omega \)-categorical.*

**Proof:** An application of the previous theorem.
A theory $T$ is strongly minimal if for each $M \models T$, formula $\varphi(x, \bar{y})$ and $\bar{a} \in M$.

$$\varphi(M, \bar{a}) = \{ b \in M : M \models \varphi(b, \bar{a}) \} \text{ or } \neg \varphi(M, \bar{a})$$

is finite.
Theorem

Assume $\mathcal{K}$ has a $0-1$ law and $\mathcal{N} \in \mathcal{K}_n$ implies $|\mathcal{N}| = n$. Then

$T_\mathcal{K}$ is strongly minimal and $\omega$–categorical

$\iff$

There exists $m \in \mathbb{N}$ such that $\lim_{n \to \infty} \mu_n(\{\mathcal{M} \in \mathcal{K}_n : \text{there is } X \subseteq M, |X| \leq m, \text{Sym}_X(M) \leq \text{Aut}(\mathcal{M})\}) = 1$
Questions?

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