Countably categorical almost sure theories

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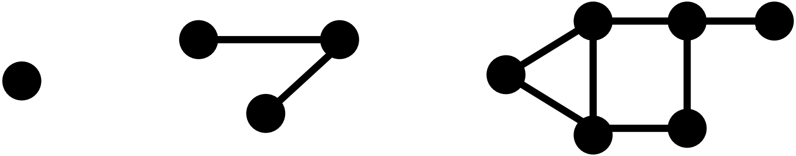
Introduction

A finite graph $\mathcal{G} = (G, E)$ is a finite set G with a binary "edge" relation E.



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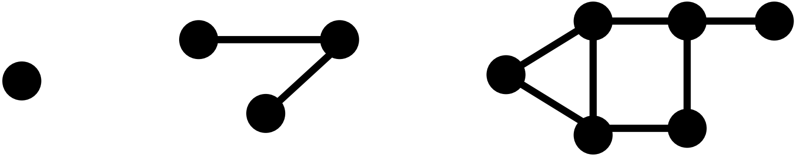
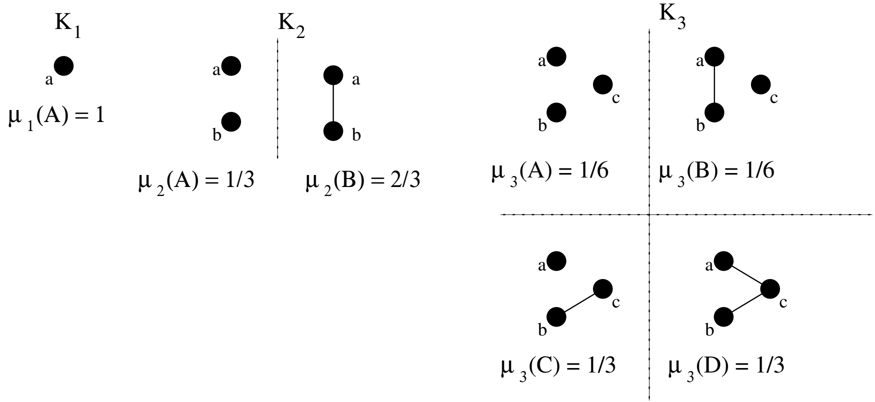


Generalized to finite relational first order structures $\mathcal{M} = (M, R_1, ..., R_k)$.

For each $n \in \mathbb{N}$ let $\mathbf{K}_n$ be a finite set of finite structures and μ_n a probability measure on $\mathbf{K}_n$. If φ is a formula let

$$\mu_n(\varphi) = \mu_n(\{\mathcal{N} \in \mathbf{K}_n : \mathcal{N} \models \varphi\})$$

$\mathbf{K} = \bigcup_{n=1}^{\infty} \mathbf{K}_n$ has a convergence law if for each first order formula φ, $\lim_{n\to\infty} \mu_n(\varphi)$ converges.



If we let φ be the formula $\exists x \exists y (xEy)$ then

$$\mu_1(\varphi) = 0 \qquad \mu_2(\varphi) = 2/3 \qquad \mu_3(\varphi) = 5/6$$

$\lim_{n\to\infty} \mu_n(\varphi)$ converges if the sequence $0, 2/3, 5/6, \ldots$ converges.

0-1 laws

If for each formula φ

$$\lim_{n\to\infty} \mu_n(\varphi) = 1 \quad \text{or} \quad \lim_{n\to\infty} \mu_n(\varphi) = 0$$

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Let $\mathbf{K}_n$ consisting of all structures with universe $\{1, ..., n\}$ (over a fixed vocabulary) with $\mu_n(\mathcal{N}) = \frac{1}{|\mathbf{K}_n|}$. Fagin (1976) and independently Glebksii et. al.(1969) proved that this $\mathbf{K}$ has a $0 - 1$ law.

More $0-1$ Laws

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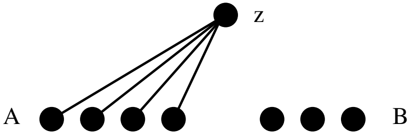
Let $\mathbf{K}$ consist of all $l-$coloured structures with a vectorspace pregeometry. Koponen (2012) proved a $0-1$ law for $\mathbf{K}$ under both uniform (the normal $\frac{1}{|\mathbf{K}_n|}$) and dimension conditional measure.

Fagins method of proving $0 - 1$ laws

$\mathcal{N}$ satisfies the k-extension property φ_k (for graphs) if:

$$A, B \subseteq N, A \cap B = \emptyset, |A \cup B| \leq k \Rightarrow \exists z :$$

$$aEz \text{ and } \neg bEz \text{ for each } a \in A, b \in B$$



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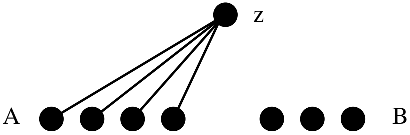

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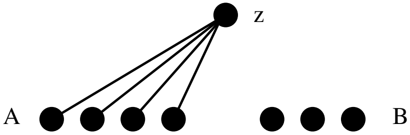

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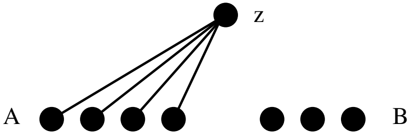

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Note: $T_{\mathbf{K}}$ is complete iff $\mathbf{K}$ has a $0 - 1$ law.

Let $\kappa \geq \aleph_0$. For κ-categorical theories completeness is equivalent with not having any finite models.

Theorem

$T_{\mathbf{K}}$ is $\aleph_0$−categorical.

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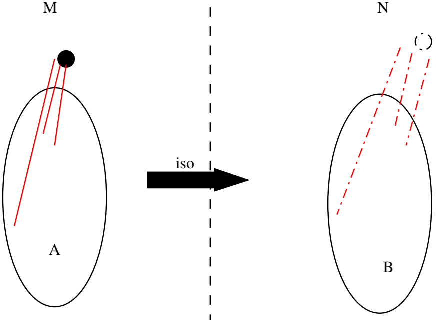
Proof.

Take $\mathcal{N}, \mathcal{M} \models T_{\mathbf{K}}$. Build partial isomorphisms back and forth by using the extension properties to help.



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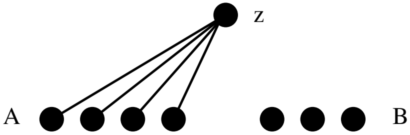
Theorem

K *has a* $0-1$ *law and* $T_{\mathbf{K}}$ *is* $\aleph_0-$*categorical*

iff

K *almost surely satisfies all extension properties*



Extension properties may be very complicated.

$\mathcal{M}^{eq}$ is constructed from a structure $\mathcal{M}$ by for each $\emptyset-$definable $r-$ary equivalence relation E:

- Add unique element $e \in M^{eq} - M$ for each $E-$equivalence class.
- Add new unary relation symbol P_E such that e represents an $E-$equivalence class iff $\mathcal{M}^{eq} \models P_E(e)$
- Add a $r+1$-ary relation symbol $R_E(y, \bar{x})$ such that $\bar{a} \in M$ is in the equivalence class of e iff $\mathcal{M}^{eq} \models R_E(e, \bar{a})$.

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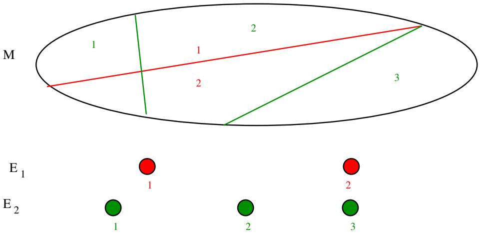

Could be thought of as an "Anti-quotient". A very important structure in infinite model theory.

If $E = \{E_1, ..., E_n\}$ is a finite set of $\emptyset-$definable equivalence relations then let $\mathbf{K}^E$ be $\mathbf{K}$ where we add the $\mathcal{M}^{eq}$ structure for only the equivalence relations in E to each $\mathcal{N} \in \mathbf{K}$.

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Proof: An application of the previous theorem.

Strongly minimal countably categorical almost sure theories

A theory T is strongly minimal if for each $\mathcal{M} \models T$, formula $\varphi(x, \bar{y})$ and $\bar{a} \in M$.

$$\varphi(\mathcal{M}, \bar{a}) = \{b \in M : \mathcal{M} \models \varphi(b, \bar{a})\} \text{ or } \neg\varphi(\mathcal{M}, \bar{a})$$

is finite.

Theorem

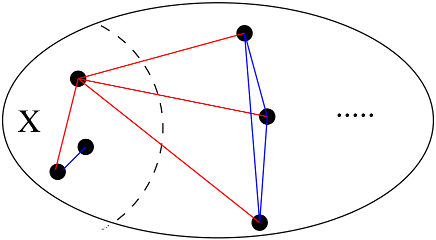
Assume $\mathbf{K}$ *has a* $0-1$ *law and* $\mathcal{N} \in \mathbf{K}_n$ *implies* $|N| = n$. *Then*

$$T_{\mathbf{K}} \textit{ is strongly minimal and } \omega\textit{-categorical}$$

$$\Leftrightarrow$$

$$\textit{There exists } m \in \mathbb{N} \textit{ such that } \lim_{n\to\infty}$$

$$\mu_n(\{\mathcal{M} \in \mathbf{K}_n : \textit{there is } X \subseteq M, |X| \leq m, Sym_X(M) \leq Aut(\mathcal{M})\}) = 1$$



Questions?

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