Category Theory and Structuralism

Erik Palmgren

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The term *structuralism* occurred in several branches of the humanities and the sciences in the period 1929 – 1970: in Linguistics (Ferdinand de Saussure, Roman Jakobson), Anthropology (Claude Lévi-Strauss), Developmental psychology (Jean Piaget), Literature (Workshop for potential literature, Raymond Queneau) and in Mathematics (Nicolas Bourbaki). To the layman the structuralist movement in mathematics was perhaps most visible the form of *New Math,* which was strongly influenced by the Bourbaki school. It has been argued in (Aubin 1997) that there were cultural connections between these movements. (See also A. Aczel 2007.) Some of these interactions may be regarded as rather superficial. The epistemologist Piaget however was very much influenced by Bourbaki, and seems to have suggested that those patterns of thought used to explain cognitive development were closely related to the mathematical "mother structures" found by Bourbaki.

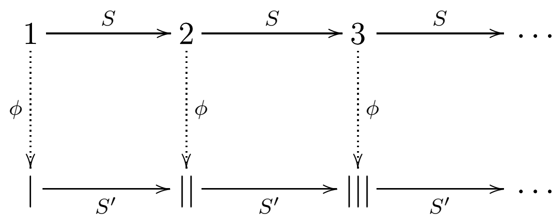
On a very general level, structuralism refers to a mode of thinking involving abstraction from specifics and systematic identification and naming of common patterns. It is the relation of objects under study to each other that is of importance rather than their specific appearance, or "nature". In mathematics, Richard Dedekind may be said to be the first structuralist. He described the positive integers $(1, 2, 3, \ldots)$ as *positions* in an infinite progression of elements (a so-called simply infinite system)

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \longrightarrow \cdots$$

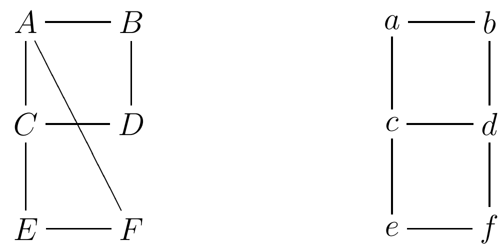
Here it is the relative position rather the objects themselves that represent the numbers. He showed (Dedekind 1888) that any two such systems have the same structure, i.e. are *isomorphic.* For the purpose of mathematics it is irrelevant which one is used. Another system is for instance

$$| \longrightarrow || \longrightarrow ||| \longrightarrow \cdots \longrightarrow |\,|\ldots| \longrightarrow \cdots .$$

In the case of Dedekind's example the isomorphism ϕ is given by $\phi(1) = |$ and $\phi(S(x)) = S'(\phi(x))$:



Further examples of isomorphisms: Two isomorphic graphs (undirected graphs) are



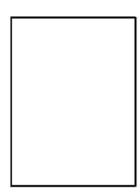
The isomorphism is specified by the following list of arrows

$A \dashrightarrow d \qquad B \dashrightarrow b$
$C \dashrightarrow c \qquad D \dashrightarrow a$
$E \dashrightarrow e \qquad F \dashrightarrow f$

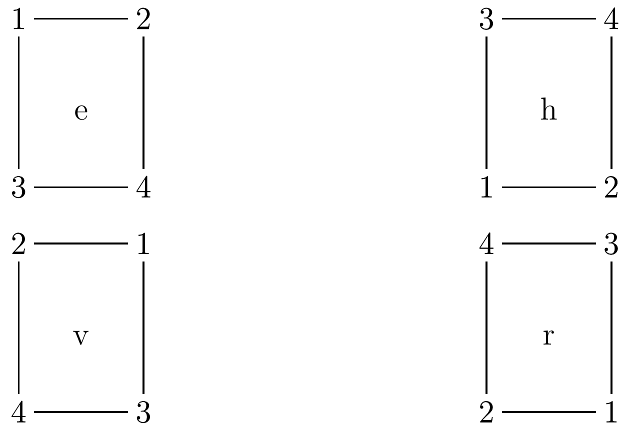
Unlike the natural numbers in Dedekind's simply infinite system, it is not possible to uniquely define any node in the graphs above by specifying how many neighbours the node should have (an example of a structural description with respect to graphs). For instance c and d, in the graph on the right, are indistinguishable. On the other hand this can be done fully in a simply infinite system: 1 is described as the element who has no predecessor; 2 has 1 as it unique predecessor; 3 has 2 as its unique predecessor, etc.

A very important structure in mathematics is that of a *group.* It arises naturally in physics and geometry.

"Theory of paper turning." Consider a blank A4 paper placed on a table.



The task is to see what kind of flips and turns can be made so that the corners are permuted.



The actions (transformations) are thus: e (do nothing), h (flip the paper along the horizontal line), v (flip the paper along vertical line) and r (rotate the paper 180 degrees). These transformations constitute a so-called group structure which is exhibted by the multiplication table below. For instance *horizontal flip* followed by *vertical flip* is the same as a *rotation*, in symbols

$$\mathrm{v} \cdot \mathrm{h} = \mathrm{r}.$$

The full table of possible transformations and their combinations is

V :

·	e	h	v	r
e	e	h	v	r
h	h	e	r	v
v	v	r	e	h
r	r	v	h	e

It satisfies the so-called group-laws

$$e \cdot x = x = x \cdot \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

$$x \cdot x^{-1} = e \qquad x^{-1} \cdot x = e.$$

An abstract group is any set G with operations $\cdot$ and $(\ )^{-1}$ and a neutral element e satisfying these laws. Further examples of groups are (1) the non-zero rational numbers with usual multiplication, (2) $\mathbb{Z}_n$ integers modulo n with addition as the group operation ($n = 12$ or $n = 24$ is “clock arithmetic”), (3) in case we instead of A4 sheets consider pieces of paper that have the shape of regular polygons (isosceles triangles, squares, pentagons, etc.) we get the so-called *dihedral groups*. These give important examples of non-commutative groups, i.e. where it may happen that $x \cdot y \neq y \cdot x$.

The fact that all the actions in the group V are determined by the two kinds of flips is embodied in the group isomorphism

$$V \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In his study of kinship in aboriginal tribes in Australia the anthropologist Lévi-Strauss enlisted André Weil, one of the collaborators of Bourbaki, to help him analyze the complicated intermarriage patterns. Weil discovered that one could use the mathematical group structure of marriage rules to analyze whether subpopulations would eventually be (genetically) isolated. Such a marriage rule could be that males of certain tribes were required to join another tribe to marry. Weil was proud of this discovery (A. Aczel 2007) which appeared in appendix to one of Lévi-Strauss' seminal books on kinship.

The early nineteenth century saw the development of alternative geometries, Non-euclidean geometries. For this reason the defects of Euclid's classical axiomatic account of geometry — *The Elements* — became more and more visible. In 1899 David Hilbert published the first fully precise treatment of geometry, *Grundlagen der Geometrie.* The proofs therein did not rely on any intuition or preconception about space, but used only the axioms and logical deduction. To make this point he quipped in a private conversation that the inferences of his geometry would be equally valid if he replaced the terms *point, line* and *plane* with the words *beer jug, chair* and *table,* respectively. This work became a model for the modern axiomatic method.

A group of leading mathematicians, using the pen name *Nicolas Bourbaki*, undertook in 1934 the task of making a unified development of central parts of mathematics according to this new standard. This resulted in a long series of books *Eléments de Mathématique* that became very influential. In a manifesto (Bourbaki 1950) some main principles of their structuralist view of mathematics were presented. Some "mother structures" were proposed that dealt with order, algebraic operations and topology (theory of continuous transformation). These were supposed to occur in various combinations in a given mathematical structure. In Leo Corry's account of the evolution of structures in mathematics (Corry 2004) it is noted that there was actually a great discrepancy between Bourbaki's informal use of the word structure and his formal logical theory of structures.

The rather complicated formal theory of structures was hardly used in the actual development of the *Eléments de Mathématique.* Each structure (in the formal sense) is based on a system of sets, operations and relations, and to each type of structure a notion of *isomorphism* is associated. (These structures are thus akin to the familiar models associated with first-order logic, but allow for higher-order relations to accommodate topologies and similar structures.)

Bourbaki's formal notion of structure was indeed inadequate. In areas like algebraic geometry one dealt with structures (e.g. sheaves of abelian groups) that were not based on sets. The mathematical notion of a *category* developed by Samuel Eilenberg and Saunders MacLane in the 1940s turned out to be able encompass such structures as well. A category $\mathcal{C}$ is basically a graph with arrows between a collection of nodes called *objects*. These arrows are considered as the admissible transformations (*morphisms*) between the objects. Two morphisms f and g may be composed to form a new morphism $f \circ g$ provided the arrow f begins where g ends. The composition operation $\circ$ obeys the so-called monoid laws

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad f \circ 1 = f \qquad 1 \circ f = f$$

1 is a special identity morphism that is assumed to exist for each object.

Examples. A basic example of a category is the category **Ens** of sets. Its objects are sets and its arrows are functions and $\circ$ is the usual function composition. From this category, the categories of groups, rings and topological spaces can be obtained by imposing appropriate structures on the sets and requiring the functions to respect these structures. Categories so obtained are called *concrete categories.*

In an arbitrary (abstract) category an *isomorphism* is now a morphism $A \xrightarrow{f} B$ for which there exists a morphism $B \xrightarrow{g} A$ with the property that $f \circ g = 1$ and $g \circ f = 1$. Intuitively, we can transform A into B using f and then using g we can come back to A. We may also proceed in the opposite direction starting out with B.

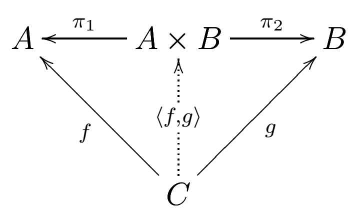
In contrast to Bourbaki's definition we have a notion of isomorphism that do not refer to the internal structure of the objects. Indeed, the objects may just be positions or place holders. Remarkably, by stating conditions on the "web" of arrows we may impose internal structure on the objects. These structures are determined only up to isomorphism, just as for Dedekind's natural numbers. For instance, if we have a category $\mathcal{C}$ with Cartesian products we may define what a group object is in $\mathcal{C}$ (cf. MacLane 1997). This gives a vast generalization of the usual notion of group, as it need not be based on a set of group elements.

Defintion: A *terminal object* in a category $\mathcal{C}$ is an object $\mathbf{1}$, such that for any object A of $\mathcal{C}$ there is a unique arrow from A to $\mathbf{1}$, which we denote by $!_A$.

Definition: A category $\mathcal{C}$ has *(binary) Cartesian products* if for any pair of objects A and B there is an object $A \times B$ and two arrows $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$, such that if C is any other object with arrows $f : C \longrightarrow A$ and $g : C \longrightarrow B$ there is a unique arrow $\langle f, g \rangle : C \longrightarrow A \times B$ so that

$$\begin{aligned} \pi_1 \circ \langle f, g \rangle &= f \\ \pi_2 \circ \langle f, g \rangle &= g \end{aligned}$$

Diagrammatically, this is expressed as



Thus arrows with same starting point can be paired together to form a new arrow $\langle f, g \rangle$. The orginal arrows can be recovered by composition with the projections π_1 and π_2.

For any category $\mathbb{C}$ with binary Cartesian products and a terminal object $\mathbf{1}$ we may define a *group object* in $\mathbb{C}$ to consist of an object G and three arrows

$$\mathbf{1} \xrightarrow{e} G \qquad G \times G \xrightarrow{m} G \qquad G \xrightarrow{i} G$$

(neutral element, multiplication and inverse) which satisfies the (rather) straightforward translations of the usual group laws using $\langle\ ,\ \rangle$ and $!_{(\,)}$. We refer to, e.g., (McLarty 1992) for further details.

In the category of sets the group objects are just the usual groups (of which we have seen some examples of). But the definition is now much more general, and we can investigate the structure of groups in many different categories. For instance, in the category of topological spaces **Top**, a group object is what was already known as a topological group, i.e. a topological space which is a group with continuous multiplication and inversion. Various matrix groups provide good examples.

The notion of a simply infinite system can be captured in category theory, faithfully to Dedekind's idea. In a (cartesian closed) category a *natural number object* is a pair of arrows

$$\mathbf{1} \xrightarrow{0} N \xrightarrow{S} N,$$

which is such that if for any other pair with

$$\mathbf{1} \xrightarrow{b} A \xrightarrow{f} A$$

there is a unique arrow $r : N \longrightarrow A$ with

$$\begin{aligned} r \circ 0 &= b, \\ r \circ S &= f \circ r. \end{aligned}$$

It is easy to show that two natural numbers objects must be isomorphic.

William Lawvere proposed in the beginning of 1960s that the theory of categories could serve as a foundation for the whole of mathematics in place of set theory (Lawvere 1964). Together with Myles Tierney he developed, at the end of the decade, a more general foundation, the notion of *elementary topos*, which connected it to Brouwer's intuitionistic mathematics. See McLarty (1992, 2002) and Awodey (1996). Awodey and McLarty have advocated category theory as a structuralist foundation for mathematics.

An elementary topos is a category which incorporate some features from set theory, which can informally (and incompletely) described be as follows

- It has binary cartesian products and a terminal object ("one element set").
- It has a natural number object.
- The solutions to an equation of arrows is a object:

  $$\{x \in X : f(x) = g(x)\}$$

  (Equalizers exists)
- The arrows from one object A to another object B form an object B^A (Cartesian closure).
- There an object Ω of truth-values and a bijective correspondence between subobjects ("subsets") of an object X and arrows ("characteristic functions") $X \longrightarrow \Omega$. Power objects are defined by $P(X) = \Omega^X$.

In category theory we can, at least in a formal way, realize Brouwer's dictum about mathematics being prior to logic. It is possible to formulate many properties of a category in a very rudimentary logic for equation solving, in which all the logical formulas have the form:

> If the system A of equations has a solution, then the system B of equations also has a solution.

These equations may posed between completely abstract entities, which need not be numbers. In such an equational language we can formulate the properties of an elementary topos.

There is a debate whether category theory is an adequate or natural foundation for mathematics (Feferman 1977, Awodey 1996, Hellman 2003, McLarty 2004). Does it (secretly) require classical ZF set theory as a foundation or motivation? Studies like (Joyal and Moerdijk 1995), (Moerdijk and Palmgren 2002) and (Moerdijk and van den Berg 2006) shows that it is not tied to classical foundations. One may indeed build predicative versions of elementary toposes upon constructive type theories like Martin-Löf's theory (Martin-Löf 1975), or constructive set theories, like that of P. Aczel (1978).

It category theory a natural foundation for mathematics?

- It describes well the practice of certain disciplines of mathematics.
- It is less clear that it explains the foundations as simple as possible.

It is seems to be a general experience among practioneers that a category-theoretic framework tends to favour constructive modes of reasoning. Why is this so?

- Many categorical constructions can be expressed in fragments of logic (regular logic or geometric logic). For these fragments uses of the principle of excluded middle can always be eliminated (Barr's Theorem).
- Fundamental constructions are defined by universal properties, which tends to give unique and canonical constructions.
- Fundamental constructions are made to work in sheaves, which is akin to Kripke models, and thus forces use of intutionistic logic.

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