Constructive Aspects of Non-Standard Analysis

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An elementary model for infinitesimals

A common informal definition of infinitesimals was historically as quantities that goes to zero.

The following sequences both converge to zero, but at different rates

$$\begin{aligned} \mathbf{a} &= (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots, \frac{1}{n^2}, \ldots), \\ \mathbf{b} &= (\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots, \frac{1}{n+3}, \ldots) \end{aligned}$$

For sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ define the relation $<^\star$ (*eventually smaller than*)

$$\mathbf{x} <^\star \mathbf{y} \text{ iff for some } n \text{ it holds that for all } m \geq n\text{: } x_m < y_m.$$

So clearly: $\mathbf{a} <^\star \mathbf{b}$.

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So clearly: $\mathbf{a} <^\star \mathbf{b}$.

For each number c, we define $c^\star$ as the constant sequence

$$c^\star = (c, c, c, \ldots).$$

Then it is clear that for each positive number ε

$$\mathbf{a} = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots, \frac{1}{n^2}, \ldots) <^\star \varepsilon^\star.$$

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There are also sequences that can be considered as infinite numbers. The sequence

$$\Omega = (0, 1, 2, 3, \dots)$$

satisfies $n^\star <^\star \Omega$ for each natural number n.

The usual arithmetical operations are extended to sequences

$$\begin{aligned}
(x_1, x_2, x_3, \cdots) + (y_1, y_2, y_3, \cdots) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots) \\
(x_1, x_2, x_3, \cdots) \cdot (y_1, y_2, y_3, \cdots) &= (x_1 y_1, x_2 y_2, x_3 y_3, \dots) \\
-(x_1, x_2, x_3, \cdots) &= (-x_1, -x_2, -x_3, \dots) \\
|(x_1, x_2, x_3, \cdots)| &= (|x_1|, |x_2|, |x_3|, \dots)
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For the infinitesimal

$$\mathbf{b} = (\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{n+4}, \dots)$$

we have thus

$$\mathbf{b}\cdot(\Omega+4^{*}) = (\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots, \frac{1}{n+4}, \dots)\cdot(0+4, 1+4, 2+4, \dots, (n+4), \dots) = 1^{\star},$$

so b may be considered as the inverse $\frac{1}{\Omega+4^{\star}}$ of $\Omega + 4^{\star}$.

It is natural to expect that Ω is invertible, but

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We define the relation $=^\star$, eventually equal, by

$$\mathbf{x} =^\star \mathbf{y} \text{ iff there is some } n \text{ such that for all } m \geq n\text{: } x_m = y_m.$$

Then

$$\Omega \cdot (0, 1, 1/2, 1/3, \ldots) = (0, 1, 1, 1, \ldots) =^* 1^*.$$

In general, for any sequence of numbers $\mathbf{x}$ with $0 <^\star |x|$, we may define

$$\mathbf{y} = (0, 0, \ldots, 0, x_n^{-1}, x_{n+1}^{-1}, x_{n+2}^{-1}, \ldots)$$

where n is such that $|x_m| > 0$ for all $m \geq n$. Then

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Ω-calculus

We have here described the essential ingredients of the so-called Ω-kalkül introduced by Curt Schmieden and Detlef Laugwitz in

> Eine Erweiterung der Infinitesimalrechnung. *Mathematisches Zeitschrift.* vol. 69 (1958) pp. 1–39.

and further developed by Laugwitz in several papers and books. Independently, these ideas where rediscovered and suggested as a constructive foundation for non-standard analysis in

> P. Martin-Löf (1989). Mathematics of Infinity. In: P. Martin-Löf and G. Mints (red.), *COLOG-88 Computer Logic,* Lecture Notes in Computer Science, vol. 417, Springer.

The construction of non-standard extensions of sets, relations and functions in the sense of the Ω-calculus is as follows.

- For each set M, let the non-standard version $M^\star$ of M be the set of infinite sequences $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ where each term $x_k \in M$, i.e. $\mathbf{x}$ is function $\mathbb{N} \to M$.
- For each relation $R \subseteq M \times N$ introduce a non-standard relation $R^\star \subseteq (M \times N)^\star$ by

$$R^\star(\mathbf{x}, \mathbf{y}) \text{ iff there is } k \text{ s.t. for all } n \geq k\text{: } R(x_n, y_n)$$

Examples: $=^\star \subseteq (M \times M)^\star$ and $<^\star \subseteq (\mathbb{R} \times \mathbb{R})^\star$.

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Examples: $=^\star \subseteq (M \times M)^\star$ and $<^\star \subseteq (\mathbb{R} \times \mathbb{R})^\star$.

The set of functions $A \times B \to C$ has a non-standard extension $(A \times B \to C)^\star$ being (of course) the set of infinite sequences $\mathbf{f} = (f_0, f_1, f_2, \ldots)$ of functions $A \times B \to C$. Such a non-standard function can be applied to nonstandard elements as follows: for $\mathbf{x} \in A^\star$ and $\mathbf{y} \in B^\star$,

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = (f_0(x_0, y_0), f_1(x_1, y_1), f_2(x_2, y_2), \ldots, f_n(x_n, y_n), \ldots).$$

An important special case is when $f = g^\star$ and $g : A \times B \to C$,

$$g^\star(\mathbf{x}, \mathbf{y}) = (g(x_0, y_0), g(x_1, y_1), g(x_2, y_2), \ldots, g(x_n, y_n), \ldots).$$

This way we get nonstandard extensions of arithmetical operations, e.g. $g = + : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
These extensions are constructive in distinction to those of classical non-standard analysis (Robinson 1966).

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Note that if $\mathbf{x} \in M^\star$, then the function $f : \mathbb{N} \to M$ given by $f(n) = x_n$ yields

$$\mathbf{x} = f^\star(\Omega).$$

Thus every nonstandard object is the image of Ω under a standard function. Hence the name Ω-calculus.

Some elementary non-standard notions

Definitions. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\star$. Then

(a) $\mathbf{x}$ *infinitesimal* if $|\mathbf{x}| <^\star \varepsilon^\star$ for each positive $\varepsilon \in \mathbb{R}$.

(b) $\mathbf{x}$ and $\mathbf{y}$ are *infinitely close*, in symbols $\mathbf{x} \simeq \mathbf{y}$, if $\mathbf{x} - \mathbf{y}$ is infinitesimal.

(c) $\mathbf{x}$ is *finite*, if $|\mathbf{x}| <^\star \varepsilon^\star$ for some positive $\varepsilon \in \mathbb{R}$.

(d) $\mathbf{x}$ is *standard* if $\mathbf{x} =^\star u^\star$ for some $u \in \mathbb{R}$.

(e) $\mathbf{x}$ is *convergent* if it is infinitely close a standard number.

Note: If x and y infinitesimal, then so is $x + y$ and xy.
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Lemma 1: For $\mathbf{x} = (x_0, x_1, x_2, \ldots) \in \mathbb{R}^\star$ and $a \in \mathbb{R}$,

$$\mathbf{x} \simeq a^\star$$

if and only if the sequence (x_n) converges to a. $\square$

Derivatives can be calculated: for $x \in \mathbb{R}^\star$ finite and an infinitesimal dx with $|dx| > 0$

$$\frac{(x + dx)^3 - x^3}{dx} = 3x^2 + 3x\,dx + (dx)^2 \simeq 3x^2.$$

(Robinson suggested that paradoxes in old texts were mainly due to the conflations of the relations $=$ and $\simeq$.)

Non-standard charcterizations of standard notions

A function $f : \mathbb{R} \to \mathbb{R}$ is **sequentially continuous** at $a \in \mathbb{R}$ if for any sequence $(x_n)_n$ converging to a, the sequence $(f(x_n))_n$ converges $f(a)$.

[Classically speaking this is of course the same as continuous at the point a.]

Using Lemma 1 and the definition of application of $f^\star$ we find immediately that

Theorem 2: The function $f : \mathbb{R} \to \mathbb{R}$ is sequentually continuous at $a \in \mathbb{R}$, if and only if, for any $x \in \mathbb{R}^\star$,

$$x \simeq a^\star \implies f^\star(x) \simeq f^\star(a).$$

Non-standard charcterizations of standard notions, cont.

A function $f : \mathbb{R} \to \mathbb{R}$ is **sequentially uniformly continuous** if for any sequences (x_n) and (y_n) where $x_n - y_n$ converges to 0, the difference $f(x_n) - f(y_n)$ goes to 0.

Classically this is equivalent to uniform continuity. Constructively, it is a weaker notion.

The following is straightforward:

Theorem 3: The function $f : \mathbb{R} \to \mathbb{R}$ is sequentually uniformly continuous if and only if, for any $x, y \in \mathbb{R}^\star$,

$$x \simeq y \Longrightarrow f^\star(x) \simeq f^\star(y).$$

Example: It is now immediate why $f(x) = x^2$ is not uniformly continuous $\mathbb{R} \to \mathbb{R}$:

$$\Omega + \frac{1}{\Omega} \simeq \Omega$$

but

$$f^\star(\Omega + \frac{1}{\Omega}) - f^\star(\Omega) = (\Omega + \frac{1}{\Omega})^2 - \Omega^2 = 2 + \frac{1}{\Omega^2} \simeq 2.$$

Non-standard analysis in historical research

Cauchy had claimed 1821 — and maintained, despite criticism from Abel and others — the following erroneous theorem (expressed in modern language):

"Theorem C": Given a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ so that $f(x) = \lim_n f_n(x)$ exists for each $x \in \mathbb{R}$. Then f is continuous.

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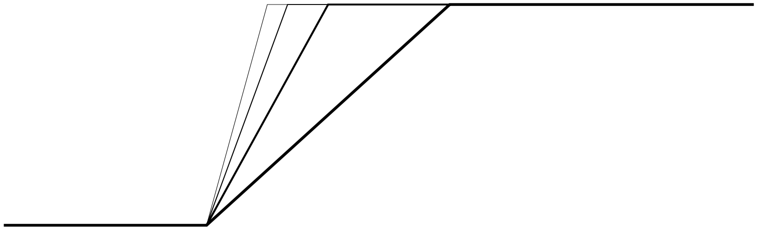
The modern correction to this result is to require the convergence to be uniform instead of point-wise.

A counter-example to "Theorem C" is given by $g_n(x) = \max(0, \min(1, nx))$, which has the step-function $g(x) = 1$ for $x > 0$, and $g(x) = 0$ for $x \leq 0$ as limit.



Laugwitz (1987a) claims that Cauchy's proof is indeed correct, but that the modern language rendering of his result is wrong. Rectified statement of result using the Ω-calculus:

Theorem C': Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n = 1, 2, 3, \ldots)$ be a sequence of continuous functions. Suppose that for each positive $\varepsilon \in \mathbb{R}$ and each *convergent* $x \in \mathbb{R}^\star$, there is n so that for all natural numbers $m \geq n$,

$$|(f_m)^\star(x) - f^\star(x)| <^\star \varepsilon^\star.$$

Then f is continuous.

Proof. Let $u \in \mathbb{R}$, $x = u^\star$ and let $\alpha \in \mathbb{R}^\star$ be infinitesimal. It is sufficient according to Theorem 2 to show

$$f^\star(x+\alpha) \simeq f^\star(x),$$

to establish the continuity of f. Take positive $\varepsilon \in \mathbb{R}$. Both x and $x+\alpha$ are convergent. Chose m large enough that

$$|f_m^\star(x) - f^\star(x)| < (\varepsilon/3)^\star \qquad |f_m^\star(x+\alpha) - f^\star(x+\alpha)| < (\varepsilon/3)^\star.$$

Now f_m is continuous, so $|f_m^\star(x) - f_m^\star(x+\alpha)| < (\varepsilon/3)^\star$. Thus:

$$\begin{aligned} |f^\star(x+\alpha) - f^\star(x)| &\leq |f^\star(x+\alpha) - f_m^\star(x+\alpha)| \\ &\quad +|f_m^\star(x+\alpha) - f_m^\star(x)| \\ &\quad +|f_m^\star(x) - f^\star(x)| \\ &< (\frac{\varepsilon}{3})^\star + (\frac{\varepsilon}{3})^\star + (\frac{\varepsilon}{3})^\star = \varepsilon^\star \end{aligned}$$

□

The counterexample to "Theorem C" fails to meet the conditions of the rectified formulation.

Recall: $g_n(x) = \max(0, \min(1, nx))$, which has the step-function $g(x) = 1$ for $x > 0$, and $g(x) = 0$ for $x \leq 0$ as point-wise limit.

Now $\frac{1}{\Omega}$ is convergent, but not standard. We have

$$g_n^*(\frac{1}{\Omega}) = \frac{n}{\Omega}$$

but

$$g^*(\frac{1}{\Omega}) = 1.$$

So

$$|g_n^*(\frac{1}{\Omega}) - g^*(\frac{1}{\Omega})| \simeq 1$$

for all $n \in \mathbb{N}$.

Interestingly the condition in the rectified theorem is also necessary (see P. 2007):

Theorem 3 Let $f_n : \mathbb{R} \to \mathbb{R}$ $(n = 1, 2, 3, \ldots)$ be a sequence of continuous functions. Suppose that it converges point-wise to a continuous function $f : \mathbb{R} \to \mathbb{R}$. Then for every positive $\varepsilon \in \mathbb{R}$, and each convergent $x \in \mathbb{R}^\star$, there is n so that for all $m \geq n$,

$$|(f_m)^\star(x) - f^\star(x)| <^\star \varepsilon^\star.$$

Proof. Let $x \in \mathbb{R}^\star$ be convergent and let $\varepsilon \in \mathbb{R}$ be positive. Write x as a sum of a standard number and an infinitesimal $u^\star + \alpha$. By the point-wise convergence it follows that there is n so that for each $m \geq n$

$$|f_m(u) - f(u)| < \varepsilon/3.$$

Hence also $|f_m^\star(u^\star) - f^\star(u^\star)| < (\varepsilon/3)^\star$. But f, and all f_m, are continuous so

$$|f_m^\star(u^\star) - f_m^\star(x)| < (\varepsilon/3)^\star \qquad |f^\star(u^\star) - f^\star(x)| < (\varepsilon/3)^\star.$$

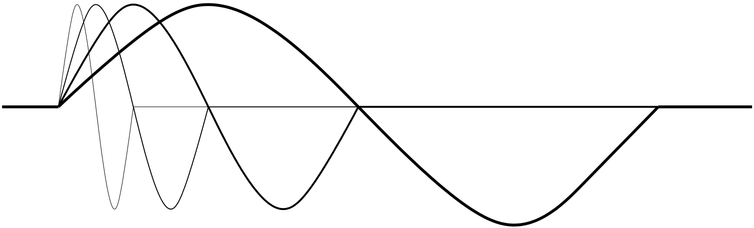
Summing we obtain for all $m \geq n$ that

$$|f_m^\star(x) - f^\star(x)| < \varepsilon^\star. \quad \square$$

Cauchy's rectified condition is not uniform convergence either. Define $h_n : \mathbb{R} \to \mathbb{R}$ to be $h_n(x) = \sin(2\pi nx)$ on the interval $[0, 1/n]$ (one full period) and 0 otherwise. Then h_n converges point-wise to the constant $h(t) = 0$ function. The convergence is not uniform even on the interval $[0, 1]$, since

$$\sup_{0 \le t \le 1} |h_n(t) - h(t)| = 1,$$

for all n.



It can be seen directly that this sequence (h_n) satisfies Cauchy's rectified condition: Let $x \simeq t^\star$, $t \in \mathbb{R}$, be a convergent number in $\mathbb{R}^\star$. If $t = 0$, then we have $h_n^\star(x) \simeq h_n^\star(t) = 0 = h^\star(x)$. For $t > 0$ it follows that $h_n^\star(x) \simeq h_n^\star(t) = 0$ for each sufficiently large n.

Using the reading of Cauchy that Laugwitz (1987a) makes and the non-standard model in the Ω-calculus, we see that Cauchy indeed had sharper condition than uniform convergence. The condition is however not so convenient to express in standard terms.

So maybe Cauchy had good reasons to stick to his condition.

— PART 2 —
Limitations and extensions of the Ω-calculus

Robinson's non-standard extensions

Robinson's classical nonstandard extension ${}^*\mathbb{R}$ of $\mathbb{R}$ is obtained by by changing the nonstandard relations.
We let ${}^*M = M^\star = \mathbb{N} \to M$ as before.
A non-principal ultra filter on $\mathbb{N}$ is $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ such that

- $\mathbb{N} \in \mathcal{U}$,
- $A, B \in \mathcal{U} \implies A \cap B \in \mathcal{U}$,
- $A \in \mathcal{U},\ A \subseteq B \implies B \in \mathcal{U}$,
- $\{n\} \notin \mathcal{U}$ for all $n \in \mathbb{N}$,
- for any $A \subseteq \mathbb{N}$: $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

For each relation $R \subseteq M \times N$ define for $\mathbf{x} \in {}^*M$ and $\mathbf{y} \in {}^*N$

$$ {}^*R(\mathbf{x}, \mathbf{y}) \Longleftrightarrow \{i \in \mathbb{N} : R(x(i), y(i)\} \in \mathcal{U}. $$

It follows easily that this relation extends the Ω-calculus relation $R^\star$:

$$ R^\star(\mathbf{x}, \mathbf{y}) \Rightarrow {}^*R(\mathbf{x}, \mathbf{y}). $$

Moreover, the last condition implies that for any $R \subseteq M \times N$ and all $\mathbf{x} \in {}^*M$ and $\mathbf{y} \in {}^*N$:

$$ {}^*R(\mathbf{x}, \mathbf{y}) \vee {}^*\overline{R}(\mathbf{x}, \mathbf{y}) $$

where $\overline{R} = M \times N \setminus R$.

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Ω-calculus vs. Robinson's nonstandard analysis

A characteristic property of Robinson's (1966) nonstandard extension ${}^*\mathbb{R}$ of the structure of real numbers is that it satisfies the first-order transfer principle - sometimes called Leibniz' permanence principle:

$$(\text{Trans}) \qquad \mathbb{R} \models \varphi \Longleftrightarrow {}^*\mathbb{R} \models \varphi$$

Here φ is a first-order formula over the language of $\mathbb{R}$ (functions and predicates).

In particular, this means that ${}^*\mathbb{R}$ satisfies the same algebraic properties as $\mathbb{R}$. Thus ${}^*\mathbb{R}$ is an ordered field.

Another characteristic of ${}^*\mathbb{R}$: every finite number is convergent.

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Another characteristic of ${}^*\mathbb{R}$: every finite number is convergent.

Theorem (BISH) If there is an extension ${}^*\mathbb{N}$ of $\mathbb{N}$ satisfying (Trans) and which contains an infinite number, then LPO holds.

Proof. For any decidable predicate $P(n)$ of $\mathbb{N}$ we can show

$$(\forall m \in \mathbb{N})[(\forall n < m)P(n) \vee (\exists n < m)\neg P(n)].$$

By the transfer principle

$$(\forall m \in {}^*\mathbb{N})[(\forall n <^* m)^*P(n) \vee (\exists n <^* m)\neg^*P(n)].$$

So let m be an infinite number. Thus either
(1): for all $n \in \mathbb{N}$, ${}^*n < m$ and ${}^*P(n)$. By transfer back, we get $(\forall n \in \mathbb{N})P(n)$.
Or
(2): there is $(\exists n \in {}^*\mathbb{N})\neg^*P(n)$, in which transfer back gives $(\exists n \in \mathbb{N})\neg P(n)$.
This shows LPO. $\square$

In contrast to ${}^*\mathbb{R}$ the Ω-calculus extension $\mathbb{R}^\star$ is not a field. It has zero-divisors: let

$$\begin{aligned} \mathbf{x} &= (0,1,0,1,0,1,\ldots) \\ \mathbf{y} &= (1,0,1,0,1,0,\ldots) \end{aligned}$$

then $\mathbf{x}\cdot\mathbf{y} =^\star 0$ but neither $\mathbf{x} =^\star 0$ nor $\mathbf{y} =^\star 0$.

Also $\mathbb{R}^\star$ is not linearly ordered, since neither $\mathbf{x} \leq^\star \mathbf{y}$ nor $\mathbf{y} \leq^\star \mathbf{x}$ holds.

Moreover $\mathbf{x}$ and $\mathbf{y}$ are finite but not convergent.

The Ω-calculus extension satisfies a limited first-order transfer principle

$$\mathbb{R} \models \varphi \Longleftrightarrow \mathbb{R}^{\star} \models \varphi$$

in BISH for first-order formulas φ built up from atomic formulas using only $\exists$ and $\wedge$ (existential conjuctive). For the direction $\Rightarrow$ it holds in fact for wider class of constructive Horn-formulas.

Using classical logic and the axiom of choice one can extend this (Palyutin 1980s) to formulas φ built up from atomics using $\exists$, $\forall$, $\wedge$ and the bounded universal quantifier construction $(\exists x \varphi') \wedge (\forall x)(\varphi' \rightarrow \varphi'')$.

Question of reverse math: Does any of the classical valid theorems for the Ω-calculus extension imply unacceptable principles in BISH?
Yes: Palyutin's general transfer theorem implies BD-$\mathbb{N}$ (Nordvall-Forsberg 2009 - MSc thesis). It was also shown in this paper that ω_1-saturation for all disjunctive formulas implies König's Lemma.

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In fact we have:

Theorem (Nordvall-Forsberg 2009) Suppose that for each universal atomic formula $\varphi(x_1, \ldots, x_n)$ in the signature L and for parameters $f_1, \ldots, f_n \in M^\star$:

$$M^\star \models \varphi(f_1, \ldots, f_n) \iff (\exists k)(\forall i \geq k) M \models \varphi(f_1(i), \ldots, f_n(i)).$$

Then BD-$\mathbb{N}$ holds.

Proof. Suppose that $A \subseteq \mathbb{N}$ is an inhabited countable pseudobounded set. According to Richman's characterization pseudbounded means that for each sequence $f : \mathbb{N} \to A$, there is $k \in \mathbb{N}$ so that $f(n) \leq n$ for all $n \geq k$. Consider the structure M with sorts A and $\mathbb{N}$ and relation $\leq$ restricted to $A \times \mathbb{N}$.

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Let $h(i) = i$. Then for each $\mathbf{x} : \mathbb{N} \to A$ it holds that $M^\star \models \mathbf{x} \leq h$. Thus $M^\star \models (\forall x \in A)(x \leq h)$. The assumption gives that there is k such that for all $i \geq k$

$$M \models (\forall x \in A) x \leq i.$$

Thus $k = i$ is a bound for A.

Suppose that $B \subseteq \mathbb{N}$ is a countable pseudobounded set. Then $A = \{0\} \cup \{x + 1 : x \in B\}$ is inhabited, countable and pseudobounded. By the above A is bounded, and hence B is bounded. $\square$

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Theorem 6 (BISH, Martin-Löf 1989) For each existential conjunctive formula $\varphi(x_1, \ldots, x_n)$ in the signature L and for parameters $f_1, \ldots, f_n \in M^\star$:

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The proof goes by induction on φ and uses countable choice for the $\exists$-case in the direction $(\Leftarrow)$. $\square$

Somewhat surprisingly it is possible to strengthen the result using a general continuity principle:

The *weak continuity principle for a set S* says that for any predicate $P \subseteq (S^{\mathbb{N}}) \times \mathbb{N}$

$$(\forall \alpha)(\exists n) P(\alpha, n) \Longrightarrow (\forall \alpha)(\exists n)(\exists k)(\forall \beta)[(\forall i < k)\beta(i) = \alpha(i) \Rightarrow P(\beta, n)].$$

Here α and β varies over $S^{\mathbb{N}}$. The case when $S = \mathbb{N}$ is the familiar WC-N (Troelstra and van Dalen 1988).

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A quantified conjunctive formula is any formula built up from atomic formulas using $\forall$, $\exists$ and $\wedge$.

Theorem (BISH + WC) Assume that weak continuity principle holds for each sort of the structure M. For each quantified conjunctive formula $\varphi(x_1, \ldots, x_n)$ in the signature L and for parameters $f_1, \ldots, f_n \in M^\star$:

$$M^\star \models \varphi(f_1, \ldots, f_n) \iff (\exists k)(\forall i \geq k) M \models \varphi(f_1(i), \ldots, f_n(i)).$$

Proof. In view of the inductive proof of Theorem 6 we need only check the $\forall$-case.

($\Leftarrow$) Suppose that $(\exists k)(\forall i \geq k) M \models (\forall y \in S)\varphi(f_1(i), \ldots, f_n(i), y)$, that is $(\exists k)(\forall i \geq k)(\forall y \in S)\ M \models \varphi(f_1(i), \ldots, f_n(i), y)$. Hence for any $g \in S^\star$,

$$(\exists k)(\forall i \geq k)\ M \models \varphi(f_1(i), \ldots, f_n(i), g(i)).$$

The inductive hypothesis gives $M^\star \models (\forall y \in S)\varphi(f_1(i), \ldots, f_n(i), y)$.

($\Rightarrow$) Suppose $M^\star \models (\forall y \in S)\varphi(f_1(i), \ldots, f_n(i), y)$. This implies by the inductive hypothesis that for each $g : \mathbb{N} \to S$ there exists $k \in \mathbb{N}$ such that

$$(\forall i \geq k)\ M \models \varphi(f_1(i), \ldots, f_n(i), g(i)).$$

Hence by the weak continuity principle for S, there is $\ell \in \mathbb{N}$ so that for each $h : \mathbb{N} \to S$ with $(\forall j < \ell)h(j) = g(j)$ it holds that

$$(\forall i \geq k)\ M \models \varphi(f_1(i), \ldots, f_n(i), h(i)).$$

Let $g(i) = m$ for all i, where m is a fixed element in S. Then we get k and ℓ so that for each $h : \mathbb{N} \to S$ with $(\forall j < \ell)h(j) = g(j)$ it holds that

$$(\forall i \geq k)\ M \models \varphi(f_1(i), \ldots, f_n(i), h(i)). \tag{1}$$

We may assume that $k \geq \ell$. Let y be an arbitrary element in S. Define $h : \mathbb{N} \to S$ by letting $h(i) = g(i)$ for $i < \ell$ and $h(i) = y$ for $i \geq \ell$. Then since $k \geq \ell$, the statement (1) gives

$$(\forall i \geq k)\ M \models \varphi(f_1(i), \ldots, f_n(i), y).$$

But y was arbitrary, so

$$(\forall i \geq k)\ M \models (\forall y \in S)\varphi(f_1(i), \ldots, f_n(i), y).$$

This proves the equivalence. $\square$

We note that the direction ($\Leftarrow$) does not use the continuity principle. The ($\Rightarrow$) direction can also be proved under the assumption of classical logic.

Constructive extensions of the Ω-calculus

The Ω-calculus is very dependent on "sequential" formulations of notions like uniform continuity to obtain good nonstandard characterizations. These formulations are often to weak in BISH.

For instance the *classical* characterization of a sequence $a : \mathbb{N} \to \mathbb{R}$ to be Cauchy is in Ω-calculus that for all infinite $\mathbf{m}, \mathbf{n} \in \mathbb{N}^\star$,

$$a^\star(\mathbf{m}) \simeq a^\star(\mathbf{n}).$$

Reading out the nonstandard statement: for any two sequences (m_k) and (n_k) that goes to infinity, the sequence $a(m_k) - a(n_k)$ goes to 0.

But this is probably not equivalent to **a** being Cauchy in BISH. (Does BD-$\mathbb{N}$ suffice?)

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But this is probably not equivalent to **a** being Cauchy in BISH. (Does BD-$\mathbb{N}$ suffice?)

A solution to obtain natural characterizations, is to index sequences not only by $\mathbb{N}$ but by arbitrary sets and filters.

A filter (base) $\mathcal{F}$ consists of a set S together with a family of F_i $(i \in I)$ of subsets of S such that

$$(\forall i, j \in I)(\exists k \in I) F_k \subseteq F_i \cap F_j.$$

For any set M we have a nonstandard extension $M^\star(\mathcal{F})$ consisting of all functions $\alpha : S \to M$.
Define for $\alpha, \beta \in \mathbb{R}^\star(\mathcal{F})$,

$$\alpha =^\star_{\mathcal{F}} \beta \Longleftrightarrow (\exists i \in I)(\forall x \in F_i)\alpha(x) = \beta(x).$$

$$\alpha <^\star_{\mathcal{F}} \beta \Longleftrightarrow (\exists i \in I)(\forall x \in F_i)\alpha(x) < \beta(x).$$

Example1: $F_i = \{x \in \mathbb{N} : i \leq x\}$ gives the Ω-calculus extension.

Example 2: $F_{(i,j)} = \{(x,y) \in \mathbb{N}^2 : i \leq x, j \leq y\}$ gives an extension $\mathbb{N}^\star(\mathcal{F})$ in which the first and second projection π_1 and π_2 are infinite. We have then

$$
\begin{aligned}
&|a^\star(\pi_1) - a^\star(\pi_2)| <^\star \varepsilon^\star \\
&\quad \Leftrightarrow \quad (\exists (i,j) \in \mathbb{N}^2)(\forall (x,y) \in F_{(i,j)})|a(x) - a(y)| < \varepsilon \\
&\quad \Leftrightarrow \quad (\exists i,j \in \mathbb{N})(\forall x \geq i)(\forall y \geq j)|a(x) - a(y)| < \varepsilon \\
&\quad \Leftrightarrow \quad (\exists i \in \mathbb{N})(\forall x,y \geq i)|a(x) - a(y)| < \varepsilon
\end{aligned}
$$

From which the proper characterization follows.

Drawing on ideas from schemes in algebraic geometry Moerdijk (1995) introduced a novel kind of constructive non-standard model of arithmetic. (It was later generalized to analysis in (P. 1997,1998).)

The idea is to consider a whole category $\mathbb{F}$ of filter bases, and a suitable notion of continuous map between filters. Furthermore a suitable notion of Grothendieck topology is defined on $\mathbb{F}$ (finite joint surjective cover) which allows, among other things, to split a filter in two parts according to any property. This allows one to dispense with ultrafilters.

The non-standard extensions are the representable presheaves $M^\star = \mathrm{Hom}(-, M)$ in the resulting category of sheaves $\mathrm{Sh}(\mathbb{F})$. The semantics in this model is non-standard (forcing), and one can prove a full transfer principle of the form, for φ internal:

$$\mathcal{F} \Vdash \varphi(\alpha, \beta, \ldots) \Longleftrightarrow (\exists i \in \mathbb{N})(\forall x \in F_i)\varphi(\alpha(x), \beta(x), \ldots).$$

For this model $\mathrm{Sh}(\mathbb{F})$

- the real numbers $\mathbb{R}^{\star}$ is in fact an constructive ordered field (unlike in Ω-calculus).
- good nonstandard characterizations are possible
- saturation properties are valid
- validates LLPO for standard quantifiers inside the model - ok, since semantics is nonstandard.

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