Locally compact spaces and formal topology

Erik Palmgren
Uppsala University

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General topology and Bishop constructivism

In early work of the Bishop school (BISH) *metric space* or *uniform* topologies were the only ones seriously used. They do not encompass

- (without coding) manifolds
- quotient spaces, identification spaces
- "logical" and "algebraic" spaces e.g. Scott domains, T_0-spaces, locales
- ...

Constructive approaches to general topology

- Neighbourhood spaces — adequate with addition of FAN axiom. (INT = BISH + FAN here)
- Locale theory
- Formal topology — locale theory on predicative basis
- Apartness spaces (Bridges,...)

Some wishes for a category of topological spaces

R1. A continuous function $[0,1] \longrightarrow \mathbb{R}$ should be uniformly continuous functions ... even though it can be obtained as a factorisation through an arbitrary space $[0,1] \longrightarrow X \longrightarrow \mathbb{R}$.

- In metric spaces this may fail for $X = (0, \infty)$ when the map $X \longrightarrow \mathbb{R}$ is the reciprocal. (Julian, Richman)
- Neighbourhood spaces fails this req., unless the Fan Theorem is assumed.)

R2. The spaces should be closed under product, sum, glueing and quotient constructions. Categorically it suffices that finite limits and colimits exists.

- − Metric spaces fails to have quotients
- + Neighbourhood spaces have quotients (H. Ishihara and P. APAL 2006), but a predicative proof is non-trivial.

R3. The spaces should should have enough subspaces (e.g. definable by equations and inequalities).

- $+$ Metric spaces
- $+$ Neighbourhood spaces

R4. The spaces should include locally compact complete metric spaces. (Such as $\mathbb{R}$, $[a, b]$.)

- $-$ Neighbourhood spaces do not support local uniform continuity.

Wishes R1-R4 are fulfilled by formal topology.

R5. The spaces should support homotopy theory — for instance by satisfying axioms of some abstract homotopy theory.

- ? Unknown (?) for constructive topology

Locally compact spaces in BISH

Def An inhabited metric space X is *locally compact*, if each bounded subset can be included in some compact subset of X.

Prop Any locally compact space is *complete and separable*.
Rem $\mathbb{Q}$ and $(0, 1)$ are not locally compact in the above sense.

Def A function $f : X \longrightarrow Y$ between a locally compact metric space X and a metric space Y is *continuous*, if f is uniformly continuous on each compact subset of X.

Constructivity of the Fundamental Adjunction ?

By the adjunction

$$\Omega : \textbf{Top} \longrightarrow \textbf{Loc} \qquad \dashv \qquad \mathrm{Pt} : \textbf{Loc} \longrightarrow \textbf{Top}$$

the functor Ω gives a full and faithful embedding of all Hausdorff spaces into locales (in fact of all sober spaces).

However, the functor Ω yields only point-wise covers in **Loc**, so $\Omega([0,1])$ is not compact, (unless e.g. the Fan Theorem is assumed). The localic version of the interval is compact.

To embed the locally compact metric spaces used in BISH we need a more refined functor.

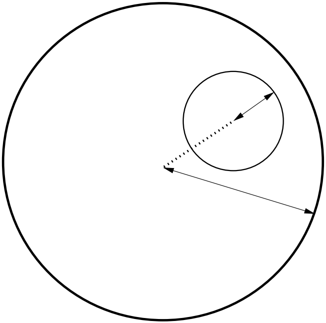
Localic completion (S. Vickers)

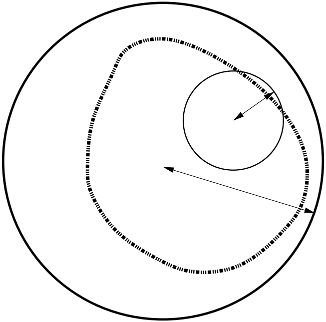
For any metric space (X, d) its **localic completion** $\mathcal{M} = \mathcal{M}_X$ is a formal topology $(M, \leq_{\mathcal{M}}, \lhd_{\mathcal{M}})$ where M is the set of *formal ball symbols*

$$\{\mathrm{b}(x, \delta) : x \in X, \delta \in \mathbb{Q}_+\}.$$

These symbols are ordered by formal inclusion

- $\mathrm{b}(x, \delta) \leq_{\mathcal{M}} \mathrm{b}(y, \varepsilon) \iff d(x, y) + \delta \leq \varepsilon$
- $\mathrm{b}(x, \delta) <_{\mathcal{M}} \mathrm{b}(y, \varepsilon) \iff d(x, y) + \delta < \varepsilon.$





The cover is generated by

(M1) $p \lhd \{q \in M : q < p\}$, for any $p \in M$,

(M2) $M \lhd \{\mathrm{b}(x, \delta) : x \in X\}$ for any $\delta \in \mathbb{Q}_+$,

The points of $\mathcal{M}_X$ form a metric completion of X. For $X = \mathbb{Q}$, we get the formal real numbers.

The covering relation is in the case of the Baire space given by an infinite wellfounded tree. Must allow generalised inductive definitions in the meta theory.

Recall that in BISH a non-void complete metric space is *locally compact* if every bounded set is included in a compact set.

Theorem (Vickers) X is compact iff $\mathcal{M}_X$ is compact.

Theorem 2 (P.) If $X = (X, d)$ is locally compact, then $\mathcal{M}_X$ is locally compact as a formal topology, i.e. $p \lhd \{q \in M : q \ll p\}$ for any p.

Here the waybelow relation is

$q \ll p$ iff for every U: $p \lhd U$ implies there is some subfinite (finitely enumerable) $U_0 \subseteq U$ with $q \lhd U_0$.

Elementary characterisation of the cover of $\mathcal{M}_X$

Refined cover relations

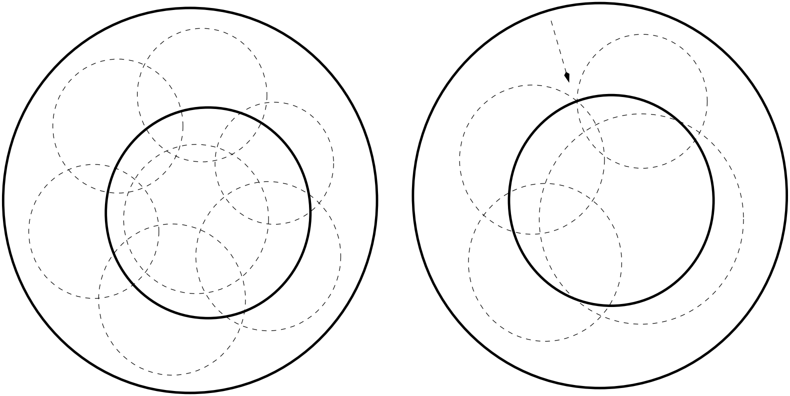
$$U \leq V \Longleftrightarrow_{\mathrm{df}} (\forall p \in U)(\exists q \in V) p \leq q$$

Note: If $U \leq V$, then $U \lhd V$. (Similar extension for $<$)

$$p \sqsubseteq_{\varepsilon} U \Leftrightarrow_{\mathrm{df}} (\forall q \leq p)[\mathrm{radius}(q) \leq \varepsilon \Rightarrow \{q\} \leq U]$$

Write $p \sqsubseteq U$ iff $p \sqsubseteq_{\varepsilon} U$ for some $\varepsilon \in \mathbb{Q}_+$.

Note: If $p \sqsubseteq V$, then $p \lhd V$.



Any sufficiently small disc included in the inner left disc “slips” into one of the dashed discs.

Elementary characterisation of the cover of $\mathcal{M}_X$

Lemma 4 Let X be locally compact. Let $\delta \in \mathbb{Q}_+$. For any formal balls $p < q$ there is a subfinite C such that

$$p \sqsubseteq C < q$$

where all balls in C have radius $< \delta$.

Remark: In fact, this is a sufficient condition for local compactness.

Let $A(p, q) = \{C \in \mathcal{P}_{\mathrm{subf}}(M_X) : p \sqsubseteq C < q\}$.

Elementary characterisation of the cover of $\mathcal{M}_X$

The following cover relation is generalising the one introduced by Vermeulen and Coquand for $\mathbb{R}$:

$$a \lessdot U \Leftrightarrow_{\mathrm{df}} (\forall b < c < a)\,(\exists U_0 \in A(b,c))\,U_0 < U.$$

Theorem 3 If X is a locally compact metric space, then

$$a \lhd U \Longleftrightarrow a \lessdot U$$

($\Leftarrow$ holds for any metric X space.)
Proof of $\Rightarrow$ goes by verifying that $\lessdot$ is a cover relation satisfying (M1) and (M2). Lemma 3 is employed several times.

Continuous maps relate the covers

Recall:

Let $\mathcal{X} = (X, \leq, \lhd)$ and $\mathcal{Y} = (Y, \leq', \lhd')$ be formal topologies. A relation $F \subseteq X \times Y$ is a *continuous mapping* $\mathcal{X} \longrightarrow \mathcal{Y}$ if

- $a\, F\, b,\ b \lhd' V \implies a \lhd F^{-1} V$,
- $a \lhd U,\ x\, F\, b$ for all $x \in U \implies a\, F\, b$,
- $X \lhd F^{-1} Y$,
- $a \lhd F^{-1} V,\ a \lhd F^{-1} W \implies a \lhd F^{-1}(V \wedge W)$.

Each such induces a continuous point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) R(a, b)\} : \mathrm{Pt}(\mathcal{X}) \longrightarrow \mathrm{Pt}(\mathcal{Y})$$

and that satisfies: $a\, F\, b \Rightarrow f[a^*] \subseteq b^*$.

Let $f : X \longrightarrow Y$ be a function between complete metric spaces. Define the relation $A_f \subseteq M_X \times M_Y$ by

$$a\, A_f\, b \Longleftrightarrow_{\mathrm{df}} a \lhd \{p : (\exists q < b) f[p_*] \subseteq q_*\}$$

Here $\mathrm{b}(x, \delta)_* = B(x, \delta)$.

Lemma 5 $A_f : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ is a continuous morphism between formal topologies, whenever X is locally compact and f is continuous.

Define thus $\mathcal{M}(f) : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ as A_f

Remark Classically, $a\, A_f\, b$ is equivalent to $f[a_*] \subseteq b_*$. However, this definition does not work constructively.

Lemma 6 Let X be locally compact and let Y be complete. For any continuous morphism $F : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$, the induced point-map

$$j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X : X \longrightarrow Y$$

is continuous. Here $j_Z : Z \longrightarrow \mathrm{Pt}(\mathcal{M}_Z)$ is the canonical metric isomorphism for a complete Z.

For X locally compact metric, Y complete metric spaces as above

$$f \mapsto A_f : \mathrm{Cont}(X, Y) \longrightarrow \mathrm{Cont}(\mathcal{M}_X, \mathcal{M}_Y)$$

has the explicit inverse

$$F \mapsto j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X$$

Embedding Theorem

Finally, to verify the functoriality of $\mathcal{M}$ we use the following property.

Lemma 7 Let X and Y be locally compact metric spaces, and suppose $f : X \longrightarrow Y$ is continuous. If $a < p$, $f[p_*] \subseteq q_*$, $q < b \sqsubseteq U$, then there is $S \in A(a, p)$ so that for any $d \in S$ there some $e \in U$ and $r < e$ with $f[d_*] \subseteq r_*$.

Embedding Theorem:
$\mathcal{M}$: **LCMet** $\longrightarrow$ **FTop** is a full and faithful functor from the category of locally compact metric spaces to the category of formal topologies.

Transfer of identities

Thm The functor $\mathcal{M}$: **LCMet** $\longrightarrow$ **FTop** preserves finite products.

By the Embedding Theorem, identities can be transferred between the point based metric spaces and their point-free version.
For $F_i : \mathcal{R}^n \longrightarrow \mathcal{R}$ and $f_i = \mathrm{Pt}(\mathrm{F}_i) : \mathbb{R}^n \longrightarrow \mathbb{R}$ we have

$$F = G \iff (\forall \bar{x}) f(\bar{x}) = g(\bar{x}).$$

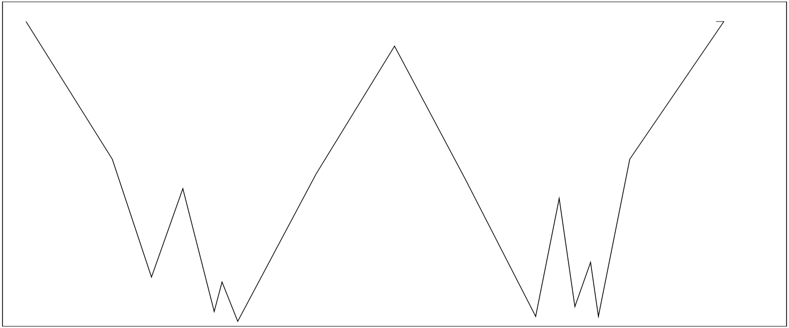
Establishing basic identities for formal real numbers is thus made easier.

The infimum problem on compact sets

Problem: In BISH find a positive uniform lower bound of a uniformly continuous function $f : [0,1] \longrightarrow (0,\infty)$.

Impossible by results of Specker, Julian & Richman using a recursive (computable) interpretation.

However in INT, a point where the minimum is attained can be found using the FAN theorem.



$\mathcal{X}^G$ — the open subspace of $\mathcal{X}$ given by a set of formal nbds G.
$E^G : \mathcal{X}^G \longrightarrow \mathcal{X}$ — the associated embedding.

Lemma Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a continuous morphism between formal topologies. Let $G \subseteq Y$ be a set of nbds. TFAE:

- F factorises through $E^G : \mathcal{Y}^G \longrightarrow \mathcal{Y}$
- $X \lhd F^{-1}[G]$
- $F : \mathcal{X} \longrightarrow \mathcal{Y}^G$ is continuous.

Def $F : \mathcal{X} \longrightarrow \mathcal{R}$ is **positive** if it factorises through $\mathcal{R}^P$, where P is the set of $\mathrm{b}(x, \delta)$ with $x > \delta$.

There is a corresponding characterisation of maps into closed subspaces. For $W \subseteq Y$ let $\mathcal{Y} \dot{-} W$ denote the corresponding closed subspace.

Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a continuous morphism. TFAE:

- F factors through E_W.
- $F^{-1}W \sim_{\mathcal{X}} \emptyset$.
- $F : \mathcal{X} \longrightarrow (\mathcal{Y} \dot{-} W)$ is continuous.

Def $F : \mathcal{X} \longrightarrow \mathcal{R}$ is **non-negative** if it factorises through $\mathcal{R} \dot{-} N$, where N is the set of $\mathrm{b}(x, \delta)$ with $x < \delta$.

Locally uniformly positive maps

Let X be a metric space. A function $f : X \longrightarrow \mathbb{R}$ is *locally uniformly positive* (l.u.p) if for every $x \in X$ and every $\delta > 0$ there is some $\varepsilon > 0$ so that for all $y \in Y$

$$d(y, x) < \delta \Longrightarrow f(y) > \varepsilon.$$

Theorem Let X be a locally compact metric space, and suppose $f : X \longrightarrow \mathbb{R}$ is continuous. Then f is l.u.p. if, and only if, $\mathcal{M}(f) : \mathcal{M}(X) \longrightarrow \mathcal{R}$ is positive.

Proof of ($\Leftarrow$): Suppose $\mathcal{M}(f)$ is positive. Using Theorem 3 we obtain

$$M(X) \lessdot \mathcal{M}(f)^{-1}[G].$$

Let $x \in X$ and $\delta > 0$. Then consider neighbourhoods $s = \mathrm{b}(x, \delta) < s' < p$. Thus we have some $p_1, \ldots, p_n < s'$ with

$$s \sqsubseteq \{p_1, \ldots, p_n\} < \mathcal{M}(f)^{-1}[G]$$

Thus there are $q_i > p_i$ and $r_i \in G$, $i = 1, \ldots, n$, satisfying $f[(q_i)^*] \subseteq (r_i)^*$. Thereby

$$f[s^*] = f[\mathrm{b}(x, \delta)^*] \subseteq f[p_1^*] \cup \cdots \cup f[p_n^*] \subseteq (r_1)^* \cup \cdots \cup (r_n)^*.$$

From which follows

$$f[B(x, \delta)] = f[\mathrm{b}(x, \delta)^*] \subseteq (\varepsilon, \infty),$$

so f is indeed l.u.p. $\square$

By this and previous results we have for locally compact X and continuous $f, g : X \longrightarrow \mathbb{R}$:

- $\mathcal{M}(f) < \mathcal{M}(g) \quad \Longleftrightarrow \quad g - f$ is l.u.p.
- $\mathcal{M}(f) \leq \mathcal{M}(g) \quad \Longleftrightarrow \quad f \leq g$.

To establish nonstrict inequality, pointwise proofs are sufficient in the second case. This is not so for strict in equalities.