From intuitionistic topology to point-free topology

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1. Constructive continuity

When computing the integral, calculating supremum or approximating zeroes of a continuous function

$$f : [0,1] \longrightarrow \mathbb{R}$$

we use its uniform continuity.

Using non-constructive axioms, e.g. the Fan Theorem, one can show uniform continuity from pointwise continuity.

In constructive mathematics following Bishop (BISH), or in recursive mathematics, one takes *uniform continuity as the fundamental notion.* Then one defines a function $\mathbb{R} \longrightarrow \mathbb{R}$ to be *continuous* if it is uniformly continuous on every $[a,b]$.

This makes it however difficult extend the continuity notion to general spaces, beyond metric spaces.

Consider a composition of continuous functions

$$\mathbb{R} \xrightarrow{g} X \xrightarrow{h} \mathbb{R} \qquad\qquad f = h \circ g.$$

INT (Brouwer): by the Fan Theorem f is uniformly continuous on every interval $[a,b]$. X may be a general space.

BISH: Unless the space X has some structure that can "transfer" local uniformity, the composed function f need not be Bishop-continuous. This works for metric spaces X.

The problem of reciprocals in BISH

The reciprocal $(\cdot)^{-1} : (0, \infty) \longrightarrow \mathbb{R}$ is continuous. Consider an arbitrary continuous $f : [0,1] \longrightarrow (0,\infty)$. *Suppose* that their composition is continuous on the compact set $[0,1]$. Then $\frac{1}{f(x)}$ is bounded above by (say) $M > 0$ for all $x \in [1,0]$. Thus $f(x)$ is bounded below by $1/M$ on the same interval.

We have shown that each continuous $f : [0,1] \longrightarrow (0,\infty)$ has a positive uniform lower bound. By a theorem of Richman (1987) this result implies the Fan Theorem.

Conclusion: Either the notion of continuity suggested in BISH is not appropriate for the reciprocal function, or the continuous functions are not closed under composition. This was a big concern to E. Bishop. A discussion and counterexamples can be found in the PhD thesis of Frank Waaldijk 1996 (Nijmegen).

Several extensions of constructive topology to solve this problem have been put forward (D.S. Bridges and others).

Point-free topology solves this problem and moreover allows for non-metric spaces.

Why general topology?

"Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size." (Bishop 1967, *Foundations of Constructive Analysis*.)

While adequate for most of analysis, metric space topologies favoured in Bishop (1967) do not encompass

- (without coding) manifolds, combinatorial topologies

- quotient spaces, identifi cation spaces

- "logical" and "algebraic" spaces e.g. Scott domains, T_0-spaces, locales

2. Point-free topology

Have you ever seen a point?

A.Pultr, *Frames*,

In: Handbook of Algebra (ed. Hazewinkel), vol. 3, 2003.

Points and continuous functions are derived notions. Instead basic open sets and their **covering and approximations relations** are taken as fundamental.

U, V, V_i

$$U \subseteq \cup\{V_i : i \in I\}$$

$$f[U] \subseteq V$$

Origins

I. Generalised spaces (sites, toposes) introduced in algebraic geometry by Grothendieck. $\Rightarrow$ locale theory.

II. The problem of constructive covering compactness (Heine-Borel principle)

a) Brouwer: the Fan Theorem and standard open covers.

b) Martin-Löf (1968): **inductively generate open covers** (consistent with recursive mathematics). Formal topology (1985-): Martin-Löf, Sambin, Stoltenberg, Sigstam, Coquand, Aczel, Negri, Curi,...

Def. A *formal topology* consists of a pre-order $X = (X, \leq)$ of *basic open neighbourhoods* and $\lhd \subseteq X \times \mathcal{P}(X)$, the *covering relation*, satisfying the four conditions

(Ref) $a \in U$ implies $a \lhd U$,

(Tra) $a \lhd U, U \lhd V$ implies $a \lhd V$,

(Loc) $a \lhd U, a \lhd V$ implies $a \lhd U \wedge V$,

(Ext) $a \leq b$ implies $a \lhd \{b\}$.

Here $U \lhd V \Leftrightarrow_{\text{def}} (\forall a \in U)\, a \lhd V$.

The topology is *set-presented* if there is a family $C(a,i)$ $(i \in I(a))$ of subsets of X so that

$$a \lhd U \Longleftrightarrow_{\text{def}} (\exists i \in I(a)) C(a,i) \subseteq U.$$

A *point* of X is a non-void subset $\alpha \subseteq S$ which is

- $\leq$-filtering, i.e. for $a, b \in \alpha$, there is $c \in \alpha$ with $c \leq a$ and $c \leq b$,
- such that α contains a neighbourhood from U, whenever $a \triangleleft U$ and $a \in \alpha$.

Thus any two approximations of a point has a common improvement, and how the approximations may improve is determined by the cover relation.

Formal reals $\mathcal{R}$

The basic neighbourhoods of $\mathcal{R}$ are $\{(a,b) \in \mathbb{Q}^2 : a < b\}$ given the inclusion order (as intervals), denoted by $\leq$. The cover $\lhd$ is generated by

$$(G1) \quad (a,b) \lhd \{(a',b') : a < a' < b' < b\} \text{ for all } a < b,$$

$$(G2) \quad (a,b) \lhd \{(a,c),(d,b)\} \text{ for all } a < d < c < b.$$

This means that $\lhd$ is the smallest covering relation satisfying (G1) and (G2). The set of points $\mathrm{Pt}(\mathcal{R})$ of $\mathcal{R}$ form a structure isomorphic to the Cauchy reals $\mathbb{R}$.

For a point α with $(a,b) \in \alpha$ we have by (G2) e.g.

$$(a,(a+2b)/3) \in \alpha \text{ or } ((2a+b)/3,b) \in \alpha.$$

(____)

⋮

(____)

⋮

(____)

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Continuous maps between formal topologies relate the covers so that composition preserves relevant "uniformities".

Let $\mathbf{S} = (S, \leq, \lhd)$ and $\mathbf{T} = (T, \leq', \lhd')$ be formal topologies. A relation $F \subseteq S \times T$ is a *continuous mapping* $\mathbf{S} \longrightarrow \mathbf{T}$ if

(A1) $a F b, b \lhd' V \Longrightarrow a \lhd F^{-1} V$,

(A2) $a \lhd U, x F b$ for all $x \in U \Longrightarrow a F b$,

(A3) $S \lhd F^{-1} T$,

(A4) $a \lhd F^{-1} V, a \lhd F^{-1} W \Longrightarrow a \lhd F^{-1} (V \wedge W)$.

Each such induces a point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) R(a, b)\} : \mathrm{Pt}(\mathbf{S}) \longrightarrow \mathrm{Pt}(\mathbf{T})$$

and that satisfies: $a F b \Rightarrow f[a^*] \subseteq b^*$.

Example. Consider a continuous mapping from a compact space into the formal reals $F : \mathbf{S} \longrightarrow \mathcal{R}$. For each rational $\varepsilon > 0$,

$$C_\varepsilon =_{\text{def}} \{(\frac{n\varepsilon}{2}, \frac{n\varepsilon}{2} + \varepsilon) : n \in \mathbb{Z}\}$$

covers $\mathcal{R}$, Using (A1) and (A3) we have

$$S \lhd F^{-1} C_\varepsilon.$$

By compactness, there is some finite sequence of basic neighbourhoods $s_1, \ldots, s_m$, and numbers $n_1, \ldots, n_m$ so that

$$s_i \, F \, (\frac{n_i\varepsilon}{2}, \frac{n_i\varepsilon}{2} + \varepsilon)$$

and

$$S \lhd \{s_1, \ldots, s_m\}.$$

Thm: A Bishop-continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is represented by the continuous mapping $A_f : \mathcal{R} \longrightarrow \mathcal{R}$ given by

$$(a,b)\, A_f\, (b,c) \iff f(a,b) \subseteq (c,d).$$

so that $f = \mathrm{Pt}(A_f)$.

For the converse we use the **Heine-Borel theorem** in the following form (Cederquist and Negri (1997):

If $[\alpha,\beta] \lhd' V$, *then* $[\alpha,\beta] \lhd' V_0$, *for some finite* $V_0 \subseteq V$ *such that* $\cup V_0$ *is an interval.*

The **Lebesgue lemma** yields: there is $\delta > 0$ so that if $x,y \in \cup V_0$ have distance $< \delta$, then $x,y \in I \in V_0$ for some I. Then we get:

Thm: If $G : [\alpha,\beta] \longrightarrow \mathcal{R}$ is continuous, then

$$g = \mathrm{Pt}(G) : \mathrm{Pt}([\alpha,\beta]) \longrightarrow \mathrm{Pt}(\mathcal{R})$$

uniformly continuous.

3. The category of formal topologies.

Categorical topology. As the category of topological spaces has limits and colimits, many spaces of interest can be built up using these universal constructions, starting from the real line and closed intervals.

The circle

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

is an equaliser of the constant 1 map and $(x,y) \mapsto x^2 + y^2$.

It can also be constructed as a coequaliser of $s,t : \{*\} \longrightarrow [0,2\pi]$ where $s(*) = 0, t(*) = 2\pi$. (Identifying ends.)

The categorical properties of the category **FTop** of set-presented formal topologies ought to be same as that of the category of locales **Loc**.

Theorem. Loc has small limits and small colimits.

However, since we are working under the restraint of predicativity (as when the meta-theory is Martin-Löf type theory) this is far from obvious. (Locales are *complete* lattices with an infinite distributive law.)

FTop has ...

- Products (ind. gen. covers) Tychonov theorem (Coquand JSL 1992).
- Equalisers (straightforward, ind. gen. covers)
- Sums (easy)
- Coequalisers (P. 2004).

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{p}{\dashrightarrow} Q$$

Further constructions using limits and colimits

1. The torus may be constructed as the coequaliser of the followings maps $\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$

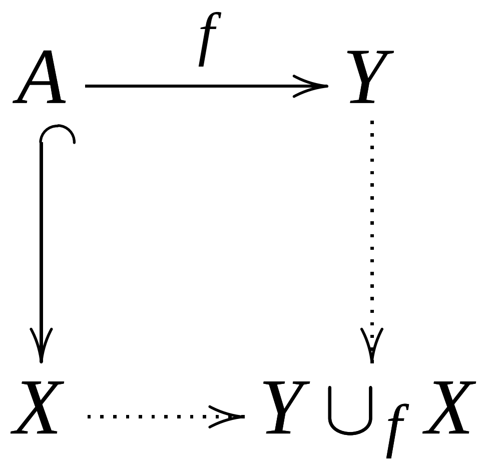
$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} \qquad (\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} + \mathbf{n}.$$

2. The real projective space $\mathbb{R}\mathrm{P}^n$ may be constructed as coequaliser of two maps

$$\mathbb{R}^{n+1} \times \mathbb{R}_{\neq 0} \longrightarrow \mathbb{R}^{n+1}$$

$$(\mathbf{x}, \lambda) \mapsto \mathbf{x} \qquad (\mathbf{x}, \lambda) \mapsto \lambda\mathbf{x}.$$

3. For $A \hookrightarrow X$ and $f : A \longrightarrow Y$, the pushout gives the attaching map construction:



4. The special case of 3, where $Y = 1$ is the one point space, gives the space X/A where A in X is collapsed to a point.

5. For concatenating homotopies we observe that

$$\begin{array}{ccc} X & \xrightarrow{\langle 1, b\circ !_X\rangle} & X \times [b,c] \\ {\scriptstyle \langle 1, b\circ !_X\rangle}\downarrow & & \downarrow{\scriptstyle 1\times i} \\ X \times [a,b] & \xrightarrow[1\times i]{} & X \times [a,c] \end{array}$$

is a pushout.

(Category theory has the necessary tools for point-free reasoning about spaces.)

4. Rejecting power sets by principles of constructive type theory

- Propositions-as-types.
- Types should be inductively defined and admit a recursion principle.

Let $\mathcal{U}$ be a type-universe which interprets propositions as well. The corresponding power set is $\mathcal{P}(S) = S \longrightarrow \mathcal{U}$.

A reducibility rule:

$$\frac{\psi : \mathcal{P}(S) \longrightarrow \mathcal{U}}{(\forall X \in \mathcal{P}(S))\psi(X) : \mathcal{U}}$$

Recursion on $\mathcal{U}$ gives, e.g, a fixed point of the negation operator $T \mapsto \neg T$ in $\mathcal{U}$. *Contradiction.*

(How can we know what a proposition means without allowing it to be decomposed completely?)

Predicitivity problems in point-free topology

Let S denote the collection of small types, Type the collection of types, and assume S:Type. Sets live among small types.

Type-theoretically $\mathcal{P}(X) = X \longrightarrow \mathsf{S}$ is a type, but not a small type.

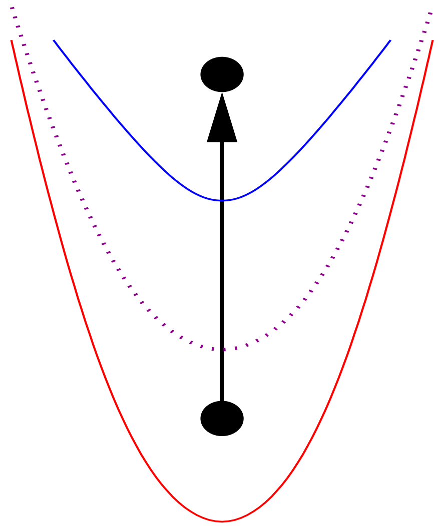
A cover relation may be regarded as a (closure) operator

$$U \mapsto \{a \in X : \mathrm{Cov}(a, U)\} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

A point α is an element of $\mathcal{P}(X)$.

Thus: neither the collection of topologies on a set, nor the points of a particular topology need to form a small type.

Point α is *partial* if it is properly contained in some point β.



For any proposition Q, define a new point

$$\alpha_Q = \{x \in \beta : x \in \alpha \text{ or } Q\}$$

Then $\alpha_Q = \beta$ iff Q is true. Thus the class of points is *not isomorphic to a set.*

X has only maximal points iff for all $\alpha, \beta \in \mathrm{Pt}(X)$

$$\alpha \subseteq \beta \Longrightarrow \alpha = \beta.$$

Reading $u \in \alpha$ as α is a point of the formal open u, this becomes nothing but the T_1-*separation axiom.* (Every formal topology is T_0 accordingly: $\subseteq$ is antisymmetric.)

Theorem (P. 2002) If X is a set-presented T_1-topology, then $\mathrm{Pt}(X)$ is a set.

Remark. Classically, every space which is not T_1 has a partial point.

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Function spaces need not be sets

For fixed formal topologies $\mathbf{X}$ and $\mathbf{Y}$, an approximable mapping $R : \mathbf{X} \longrightarrow \mathbf{Y}$ is given by a subset of $X \times Y$. Thus the collection of such mappings

$$\mathrm{Cont}(\mathbf{X}, \mathbf{Y})$$

is *prima facie* merely a type.

Indeed, it maybe a proper type since $\mathrm{Cont}(\mathbf{1}, \mathbf{Y}) \cong \mathrm{Pt}(\mathbf{Y})$ and the latter is a proper type, for $\mathbf{Y}$ with partial points.

However, for some important classes of spaces they turn out to be sets, just as for point-based topology.

Regular topologies

In a regular topology a basic neighbourhood is covered by the basic neighbourhoods “well inside” the neighbourhood. E.g. for the real formal topology $\mathcal{R}$

$$(a,b) \lhd \{(c,d) : a < c < d < b\}$$

It has the basic property that for points α, β and a neighbourhood p

$$p \in \alpha \Longrightarrow p \in \beta \quad \text{or} \quad \alpha \text{ is apart from } \beta.$$

Thm (Sambin) The points of a regular formal topology are maximal.

This result may be generalised to certain $\mathbf{Cont}(\mathbf{X},\mathbf{Y})$. Introduce the familiar graph order $\subseteq$ between approximable mappings.

Thm 1 If $\mathbf{Y}$ is regular, then the mappings in $\mathbf{Cont}(\mathbf{X},\mathbf{Y})$ are all maximal with respect to $\subseteq$.

Thm 2 If $\mathbf{X}$ and $\mathbf{Y}$ are set-presented and $\mathbf{Y}$ is regular, then $\mathbf{Cont}(\mathbf{X},\mathbf{Y})$ is a set.

Rem. Curi proved Thm 2 in the case where $\mathbf{X}$ is locally compact.

Coequalisers

In **Loc** coequalisers may be constructed as equalisers in the opposite category of frames, which is straightforward. A direct translation into **FTop** yields the following suggestion for a construction:

$$\{U \in \mathcal{P}(Y) : \widetilde{F^{-1}U} = \widetilde{G^{-1}U}\}$$

for a pair of continuous mappings $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$ between formal topologies. Here $\widetilde{W} = \{a \in X : a \lhd W\}$. (*Problem:* the new basic neighbourhoods U do not form a set.)

Any family of types (presets) $\mathsf{T}(t)$ $t : \mathsf{U}$ [write $\mathcal{U} = (\mathsf{U}, \mathsf{T})$] gives rise to a subcategory of sets. We say that a set S is a $\mathcal{U}$*-set*, if it is isomorphic to a set of the form

$$(\mathsf{T}(t_0), =_e)$$

where $t_0 \in \mathsf{U}$ and $=_e$ is an equivalence relation given by

$$x =_e y \Longleftrightarrow \mathsf{T}(e(x,y)) \text{ inhabited,}$$

and some $e : \mathsf{T}(t_0) \times \mathsf{T}(t_0) \longrightarrow \mathsf{U}$.

Then all the $\mathcal{U}$*-subsets* of a given set X can be represented by a set $\mathcal{R}_{\mathcal{U}}(X)$.

The power set $\mathcal{R}(X) = \mathcal{R}_{\mathcal{U}}(X)$ is restricted and its allowed set-theoretic operations depend strongly on the family $\mathcal{U}$.

If $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$ is a pair of continuous mappings, then we say that $\mathcal{R}(Y)$ is *adequate* for F and G if

(H1,H2) $\mathcal{R}(Y)$ contains Y and is closed under $\wedge_{\mathcal{Y}}$

(H3) For any subset U of Y with $b \in U$ such that U satisfi es the equivalence

$$\widetilde{F^{-1}U} = \widetilde{G^{-1}U}$$

there is already some $V \in \mathcal{R}(Y)$ with $b \in V \subseteq U$ satisfying the equation.

Lemma 1 If $\mathcal{R}(Y)$ is adequate for $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$, then the formal topology whose basic neighbourhoods are

$$Q = \{U \in \mathcal{R}(Y) : \widetilde{F^{-1}U} = \widetilde{G^{-1}U}\},$$

and where

$U \lhd_{\mathcal{Q}} \mathcal{U}$ iff $U \lhd_{\mathcal{Y}} \cup \mathcal{U}$

for $\mathcal{U} \subseteq \mathcal{R}(Y)$, defi nes a coequaliser. Moreover the coequalising morphism $P : \mathcal{Y} \longrightarrow \mathcal{Q}$ is given by:

aPU iff $a \lhd_{\mathcal{Y}} U$.

Lemma 2 Let $\mathcal{U}$ be a universe closed under certain standard type-theoretic operations[a] and containing certain data for the maps $F, G : \mathbf{X} \longrightarrow \mathbf{Y}$, including the set presentation of $\mathbf{X}$. Suppose $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$. Take a subset U of Y such that

$$\widetilde{F^{-1}U} \subseteq \widetilde{G^{-1}U}. \tag{1}$$

Then for any $V \in \mathcal{R}(Y)$ with $V \subseteq U$, there is $W \in \mathcal{R}(Y)$ with $W \subseteq U$ and

$$\widetilde{F^{-1}V} \subseteq \widetilde{G^{-1}W}. \tag{2}$$

Moreover, the above holds with F and G interchanged in both (1) and (2).

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[a] Π, Σ, W

Lemma 3 For any pair of continuous maps F and G there is a type-theoretic universe $\mathcal{U}$ so that $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$ is adequate for the maps.

Proof. The proof goes by iteration of Lemma 2 along a wellfounded-tree. □

Theorem. (P. 2004) Under the assumption of universe forming operators, coequalisers exists in the category **FTop**.

Remark Aczel (2005) abstracted from the proof a new predicative set existence principle *-REA which is implies the theorem in CZF (without using choice principles).