Continuity on the real line and in formal spaces

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1 Introduction

As is well-known, Brouwer introduced his axioms for intuitionism in order to regain central results about continuity. A notable example is the classical theorem that every real-valued continuous function on a finite, closed interval is uniformly continuous. His proof relied on the covering compactness of such intervals, which in turn was derived from the axioms using only constructive principles. In fact, the axiom known as the Fan Theorem (classically equivalent to König's lemma) suffices. The special axioms were avoided altogether in Bishop's development of constructive analysis, a development which is consistent with classical mathematics, as well as, recursive mathematics. Bishop simply modified the definition of continuous function on the real numbers to mean: *uniformly continuous* on each finite and closed interval. This was a very successful step. However, it may also lead to difficulties, when going beyond metric spaces. If X is a general space with a topology given by a neighbourhood basis, the composition of two continuous functions

$$\mathbb{R} \to X \to \mathbb{R},$$

need not be a continuous function. That a class of topological spaces form a category seems to be a minimal requirement, for instance in the theory of manifolds, or in algebraic topology. Though little emphasised, the continuous functions of the category of locales, or formal spaces, agrees with Bishop's definition of continuous function on real numbers. Proving this within the framework of (Bishop) constructive mathematics is the purpose of the present paper (Theorem 4.1 and 4.6). In addition, we show that the reciprocal map

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is included in the category (Section 5). With formal spaces it is thus not necessary to adopt the Fan Theorem as an axiom, for instance, to get a good category. For a discussion of this proposal, and some earlier constructions of categories of spaces addressing these problems, we refer to Waaldijk (1998) and Schuster (2003). For a recursive version of formal topology and related results there we refer to Sigstam (1995).

Locale theory developed from the wealth of work in topos theory (see Johnstone 1982), without regard for being a constructive alternative to Brouwer's solution. On the other hand, this was the explicit purpose in Martin-Löf's doctoral thesis (1970), and he showed how to regain covering compactness of the Cantor space by inductively generating the covers in a constructive manner. This was the precursor to the theory of formal spaces as developed by Martin-Löf and Sambin.

2 Point-free topology

The fundamental principle of formal spaces is to work primarily with the basic neighbourhoods, and their relation with respect to covering. For instance in the case of real numbers, these can be open intervals (a, b) with rational endpoints (see Section 3). It is often essential that covers can be generated inductively, rather than merely being defined in terms of points. We refer to Sambin (1987) and Negri and Soravia (1999) for a general background on formal topology, and briefly recall some basic definitions.

Definition 2.1 Let S be a set, and let $\lhd$ be a relation between elements of S and subsets of S, i.e. $\lhd \subseteq S \times \mathcal{P}(S)$. Extend $\lhd$ to a relation between subsets by letting $U \lhd V$ if and only if $a \lhd V$ for all $a \in U$. For a preorder $(X, \leq)$ and a subset $U \subseteq X$, its *downwards closure* $U_{\leq}$ consists of those $x \in X$ such that $x \leq y$ for some $y \in U$. Write $a_{\leq}$ for $\{a\}_{\leq}$.

Definition 2.2 A *formal topology* $\mathcal{S}$ is a pre-ordered set $S = (S, \leq)$ (of so-called *basic neighbourhoods*) together with a relation $\lhd \subseteq S \times \mathcal{P}(S)$, the *covering relation,* satisfying the four conditions

(R) $a \in U$ implies $a \lhd U$,	(L) $a \lhd U$, $a \lhd V$ implies $a \lhd U_{\leq} \cap V_{\leq}$,
(T) $a \lhd U$, $U \lhd V$ implies $a \lhd V$,	(E) $a \leq b$ implies $a \lhd \{b\}$.

A *point* is an inhabited subset $\alpha \subseteq S$ which is filtering with respect to $\leq$, and such that $U \cap \alpha$ is non-void, whenever $a \lhd U$ for some $a \in \alpha$. The collection of points is denoted $\mathrm{Pt}(\mathcal{S})$.

We use the term *formal space* interchangeably with *formal topology.*

3 Continuous mappings

A continuous mapping between formal topologies is a certain relation between their basic neighbourhoods. The fundamental example of such a mapping is the one associated with a continuous function $f : \mathbb{R} \to \mathbb{R}$:

$$(a, b)\, A_f\, (c, d) \Longleftrightarrow f(a, b) \subseteq (c, d).$$

See Theorem 4.1 below.

To define the general concept we introduce some notation. For a relation $R \subseteq S \times T$ the *inverse image of* $V \subseteq T$ *under the relation* R is, as usual,

$$R^{-1}[V] =_{\mathrm{def}} \{a \in S : (\exists b \in V)\, a\, R\, b\}$$

Thus for instance

$$A_f^{-1}\{(c, d)\} = \{(a, b) : f(a, b) \subseteq (c, d)\}.$$

Notice that, in general, $R^{-1}[U] \subseteq R^{-1}[V]$ whenever $U \subseteq V$, and

$$R^{-1}[\cup_{i\in I} U_i] = \cup_{i\in I} R^{-1}[U_i].$$

The relation R is naturally extended to subsets as follows. For $U \subseteq S$, let $U\, R\, b$ mean $(\forall u \in U)\, uRb$, and for $V \subseteq T$, we let $a\, R\, V$ mean $a \lhd R^{-1}[V]$.

Definition 3.1 Let $\mathcal{S} = (S, \leq, \lhd\,)$ and $\mathcal{T} = (T, \leq', \lhd\,')$ be formal topologies. A relation $R \subseteq S \times T$ is a *continuous mapping* from $\mathcal{S}$ to $\mathcal{T}$ (and we write $R : \mathcal{S} \to \mathcal{T}$) if

(A1) $a\, R\, b$, $b \lhd' V$ implies $a\, R\, V$,

(A2) $a \lhd U$, $U\, R\, b$, implies $a\, R\, b$,

(A3) $a\, R\, T$, for all $a \in S$,

(A4) $a\, R\, V$, $a\, R\, W$ implies $a\, R\, (V_{\leq'} \cap W_{\leq'})$.

Remark 3.2 Note that by $b \lhd' \{b\}$, (A1) and (A2)

$$\{a\}\, R\, b \Longleftrightarrow a\, R\, b \Longleftrightarrow a \lhd R^{-1}\{b\} \Longleftrightarrow a\, R\, \{b\}.$$

A continuous mapping R thus satisfies

$$a\, R\, b, b \lhd' \{b'\} \Longrightarrow a\, R\, b'.$$

Moreover (A4) may be replaced by the condition

(A4') $a\,R\,b, a\,R\,c \Longrightarrow a\,R\,(b_{\leq'} \cap c_{\leq'})$.

The next properties are useful for checking closure under composition. Denote by $\tilde{U} = \{a : a \lhd U\}$ — the saturation of U in the topology.

Proposition 3.3 *Let $R : \mathcal{S} \to \mathcal{T}$ be a continuous mapping. Then:*

(i) $U \lhd V$ *implies* $R^{-1}[U] \lhd R^{-1}[V]$,

(ii) $b\,R\,U$ *iff* $b\,R\,\tilde{U}$,

(iii) $R^{-1}[U]^{\sim} = R^{-1}[\tilde{U}]^{\sim}$. $\Box$

Let **FTop** be the following category of formal topologies and continuous mappings. For a formal topology $\mathcal{S} = (S, \leq, \lhd)$ we define a continuous mapping $I : \mathcal{S} \to \mathcal{S}$ (the identity) by

$$aIb \Longleftrightarrow a \lhd \{b\}.$$

For continuous mappings, $R_1 : \mathcal{S}_1 \to \mathcal{S}_2$ and $R_2 : \mathcal{S}_2 \to \mathcal{S}_3$, between formal spaces, define the composition

$$a(R_2 \circ R_1)b \Longleftrightarrow a \lhd R_1^{-1}[R_2^{-1}\{b\}].$$

This is continuous mapping $(R_2 \circ R_1) : \mathcal{S}_1 \to \mathcal{S}_3$. The category is not locally small, within any known predicative meta-theory.

Let **Nbhd** be the category of (large) neighbourhood spaces and point-wise continuous functions. Only set-based spaces are considered in (Bishop and Bridges 1985). Then we have a functor $\mathrm{Pt} : \mathbf{FTop} \to \mathbf{Nbhd}$ given by

$$\mathrm{Pt}(R)(\alpha) = \{y \in T : (\exists x \in \alpha)\, x\,R\,y\}$$

for $R : \mathcal{S} \to \mathcal{T}$. For a basic neighbourhood a of the formal topology $\mathcal{S}$,

$$a^* =_{\mathrm{def}} \{\alpha \in \mathrm{Pt}(\mathcal{S}) : a \in \alpha\}$$

is a basic open of the neighbourhood space $\mathrm{Pt}(\mathcal{S})$. Moreover,

$$a\,R\,b \Longrightarrow \mathrm{Pt}(R)[a^*] \subseteq b^*. \tag{1}$$

Remark 3.4 Sambin (1987) defines continuous mappings in a slightly different way. We can most easily explain it by removing axiom (A2) and introducing an equivalence $\sim$ of mappings $\mathcal{S} \to \mathcal{T}$ as follows

$$F \sim G \Longleftrightarrow \overline{F} = \overline{G}.$$

Here $\overline{H}$ is the relation given by $a \, \overline{H} \, b$ iff $a \lhd H^{-1}b$. Composition is ordinary composition of relations. Moreover, relations on $S \times T$ are written as functions $T \to \mathcal{P}(S)$. The resulting category $\mathbf{FTop'}$ is categorically equivalent to $\mathbf{FTop}$, via the forgetful functor $\mathbf{FTop} \to \mathbf{FTop'}$ and the reverse functor given by $F \mapsto \overline{F}$. An advantage of $\mathbf{FTop'}$ is the simple definition of composition, which does not involve the cover relation.

Remark 3.5 Instead of *continuous mapping* the term *approximable mapping* is often used to emphasise the generalisation of the corresponding notion in Scott's domain theory.

4 Functions on real numbers

We recall a standard construction of the formal space $\mathcal{R}$ of real numbers. Here we follow Cederquist and Negri (1997), but omit the positivity predicate and use a smaller set of neighbourhoods. The basic neighbourhoods of $\mathcal{R}$ are $\{(a, b) \in \mathbb{Q}^2 : a < b\}$ given the inclusion order (as intervals), denoted by $\leq$. The cover $\lhd$ is generated by

(G1) $(a, b) \lhd \{(a', b') : a < a' < b' < b\}$ for all $a < b$,

(G2) $(a, b) \lhd \{(a, c), (d, b)\}$ for all $a < d < c < b$.

Recall that this means that $\lhd$ is the smallest covering relation satisfying (G1) and (G2). The points $\mathrm{Pt}(\mathcal{R})$ of $\mathcal{R}$ form a structure isomorphic to the Cauchy reals $\mathbb{R}$ (see e.g. Negri and Soravia 1999), via $^{-} : \mathbb{R} \to \mathrm{Pt}(\mathcal{R})$ given by

$$\overline{x} = \{(a, b) \in \mathbb{Q}^2 : a < x < b\}.$$

The points are ordered as follows

$$\alpha < \beta \Longleftrightarrow_{\text{def}} \exists (a, b) \in \alpha \; \exists (c, d) \in \beta \; b < c,$$

$$\alpha \leq \beta \Longleftrightarrow_{\text{def}} \neg \beta < \alpha.$$

In Bishop's constructive analysis a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be continuous iff it is uniformly continuous on each compact interval (closed and finite interval). Each such function gives rise to a continuous mapping $A_f : \mathcal{R} \to \mathcal{R}$ given by

$$(a,b)\, A_f\, (c,d) \Longleftrightarrow f(a,b) \subseteq (c,d).$$

Theorem 4.1 *A continuous function $f : \mathbb{R} \to \mathbb{R}$ is represented by the continuous mapping A_f in the sense that $g = \mathrm{Pt}(A_f) : \mathrm{Pt}(\mathcal{R}) \to \mathrm{Pt}(\mathcal{R})$ satisfies $g(\overline{x}) = \overline{f(x)}$ for all $x \in \mathbb{R}$.*

Proof. The proof that A_f is continuous is straightforward for (A2-A4), noting that for (A3) we use that the image of a compact interval under f is bounded. Property (A1) is equivalent to

$$I \lhd U \Longrightarrow (\forall J \in \mathcal{R})(J\, A_f\, I \Rightarrow J \lhd A_f^{-1}[U]). \tag{2}$$

This is proved by "induction on $\lhd$". Denoting the right hand side of (2) by $I\, K\, U$, this amounts to proving that K satisfies the axioms of a cover relation and the generators of $\mathcal{R}$. Since $\lhd$ is the least such relation, the statement (2) then follows. All these axioms are essentially straightforward to check, but let us note how to verify (G2), since this uses uniform continuity. We have to show

$$(u,v)\, K\, \{(u,c),(d,v)\}, \tag{3}$$

for $u < d < c < v$. Suppose $f(a,b) \subseteq (u,v)$. Let $\varepsilon = c - d$, and let $(a_1,b_1),\ldots,(a_n,b_n)$ be a cover of (a,b), with $a \le a_k < b_k \le b$, so finely meshed that for all k

$$x, y \in (a_k,b_k) \Longrightarrow |f(x) - f(y)| < \varepsilon/3. \tag{4}$$

Put $d' = d + \varepsilon/3$ and $c' = c - \varepsilon/3$. Let k be arbitrary and take some $x \in (a_k,b_k)$. Since $d' < c'$ we have by contransitivity of the order that $d' < f(x)$ or $f(x) < c'$. If $d' < f(x)$, then $f(a_k,b_k) \subseteq (d,v)$ by (4). On the other hand, if $f(x) < c'$, then $f(a_k,b_k) \subseteq (u,c)$ again by (4). This means that

$$\{(a_1,b_1),\ldots,(a_n,b_n)\} \subseteq A_f^{-1}\{(u,c),(d,v)\}.$$

The set on the left hand side was assumed to cover (a,b), so we are done proving (3).

For $x \in \mathbb{R}$ we have by definition of g

$$g(\bar{x}) = \{(c,d) : (\exists (a,b) \in \bar{x})\, f(a,b) \subseteq (c,d)\}.$$

Thus $(c,d) \in g(\bar{x})$ iff there are a, b with $a < x < b$ and $f(a,b) \subseteq (c,d)$. By the continuity of f the latter is equivalent to $c < f(x) < d$, i.e. $(c,d) \in \overline{f(x)}$. □

Next we show that every continuous mapping $\mathcal{R} \to \mathcal{R}$ gives rise to a continuous function in the sense of Bishop. We use the Heine-Borel theorem of Cederquist and Negri (1995).

For a formal topology $S = (S, \leq, \lhd)$ and a subset U of S define the *closed subspace topology* $S_U = (S, \leq, \lhd_U)$ (intuitively on the complement of the union of U) by defining a new cover relation

$$a \lhd_U V \Longleftrightarrow a \lhd U \cup V.$$

(For a general discussion of closed and open sublocales we refer to Johnstone 1982.) Then $E : S_U \to S$ given by

$$a\, E\, b \Longleftrightarrow a \lhd_U \{b\}$$

is a continuous monomorphism in the category of formal spaces. In fact

$$a \lhd_U E^{-1}[V] \Leftrightarrow a \lhd_U V$$

Let $i = \mathrm{Pt}(E)$. Then $i(\alpha) = \alpha$, for any $\alpha \in \mathrm{Pt}(S_U)$. Moreover, for $\beta \in \mathrm{Pt}(S)$,

$$\beta \in \mathrm{Pt}(S_U) \Leftrightarrow \beta \cap U = \emptyset.$$

We will be interested in the special case when $S = \mathcal{R}$ and when U is the set of basic intervals bounded away from $[\alpha, \beta]$, more precisely,

$$C(\alpha, \beta) = \{(a,b) : \; \bar{b} < \alpha \text{ or } \beta < \bar{a}\},$$

where $\alpha < \beta$ are some given points of $\mathcal{R}$. Any $\gamma \in \mathrm{Pt}(\mathcal{R})$ satisfies

$$\gamma \in \mathrm{Pt}(\mathcal{R}_{C(\alpha,\beta)}) \Leftrightarrow \gamma \cap C(\alpha,\beta) = \emptyset \Leftrightarrow \alpha \leq \gamma \leq \beta.$$

Thus we take $\mathcal{R}_{C(\alpha,\beta)}$ to be the formal space for the closed interval $[\alpha, \beta]$. Denote it by $\mathcal{I}(\alpha, \beta)$.

The Heine-Borel theorem of Cederquist and Negri (1995) goes through, without any important changes, for the version of formal spaces we have used here. They used an auxiliary relation $\lhd_{\mathrm{fin}}$ (suggested by Thierry Coquand) to show the result

$$(a,b) \lhd V \Leftrightarrow (\forall u, v)(a < u < v < b \Rightarrow (u,v) \lhd_{\mathrm{fin}} V).$$

Here $\lhd_{\mathrm{fin}}$ is the cover relation generated by (G2) only. It satisfies the following important property

Lemma 4.2 *If* $(a,b) \lhd_{\text{fin}} U$ *then there is a finite* $U_0 \subseteq U$ *such that*

$$(a,b) \subseteq \cup U_0 \tag{5}$$

as intervals.

Proof. Define the relation $(a,b) \lhd_{\text{extfin}} U$ to hold iff there is some finite $U_0 \subseteq U$ satisfying (5). Note that since the endpoints of intervals in U_0 are rational it does not matter whether the intervals are considered as subsets of rational or real numbers. In fact (5) is decidable. The relation $\lhd_{\text{extfin}}$ clearly satisfies (G2), (R) and (E). (L) can easily be checked for (5) over the rational numbers. As for (T) there is only a finite choice principle involved. We conclude that $\lhd_{\text{fin}}$ is smaller than $\lhd_{\text{extfin}}$, thereby proving the lemma. □

For a finite set V_0, the relation $(a,b) \lhd_{\text{fin}} V_0$ is thus equivalent to (a,b) being covered by $\cup V_0$ (as intervals). The coherence property of $\lhd_{\text{fin}}$ then follows.

Theorem 4.3 *(Cederqvist and Negri 1995) For* $\alpha < \beta$, *suppose* $\mathcal{I}(\alpha,\beta) \lhd_{C(\alpha,\beta)} V$. *Then*

(i) $\mathcal{I}(\alpha,\beta) \lhd_{C(\alpha,\beta)} V_0$ *for some finite* $V_0 \subseteq V$.

(ii) $(r,s) \lhd_{\text{fin}} C(\alpha,\beta) \cup V$, *where* $\bar{r} < \alpha < \beta < \bar{s}$.

To prove Theorem 4.6 below we need furthermore a simple version of Lebesgue's lemma.

Lemma 4.4 *If* $I_1, \ldots, I_n$ *are open intervals with rational end points whose union* S *is an interval, then there is rational* $\delta > 0$ *such that for every pair of real numbers* $x, y \in S$, *where* $|x-y| < \delta$, *there is some* k *with* $x, y \in I_k$. □

The δ is called the *Lebesgue number* of the covering.

Here then is a strengthening of the conclusion of Theorem 4.3(i).

Corollary 4.5 *For* $\alpha < \beta$, *suppose* $\mathcal{I}(\alpha,\beta) \lhd_{C(\alpha,\beta)} V$. *Then* $\mathcal{I}(\alpha,\beta) \lhd_{C(\alpha,\beta)} V_0$ *for some finite* $V_0 \subseteq V$ *and* $\cup V_0$ *is an interval.*

Proof. Write $C = C(\alpha, \beta)$. Theorem 4.3(ii) and the coherence of $\lhd_{\text{fin}}$ gives a finite

$$W_0 = \{I_1, \ldots, I_n\} \subseteq C \cup V, \tag{6}$$

and $\bar{r} < \alpha < \beta < \bar{s}$ such that

$$(r, s) \lhd_{\text{fin}} W_0.$$

We may assume that $\cup W_0$ is an interval (otherwise we may remove certain intervals outside (r, s)). By the Lebesgue lemma there is a positive rational δ so that for all real $x, y \in \cup W_0$,

$$|x - y| < \delta \Longrightarrow (\exists k)\, x, y \in I_k.$$

Now $\alpha \in \cup W_0$. Pick r' with $r < r' < \alpha$ and $\alpha - r < \delta$. Hence by the lemma there is some k_α such that $r', \alpha \in I_{k_\alpha}$. Similarly there is $\beta < s' < s$ and k_β with $\beta, s' \in I_{k_\beta}$. By (6) we find a finite choice function $f : \{1, \ldots, n\} \to \{1, 2\}$ so that

(i) $f(k) = 1 \Rightarrow I_k \in C$,

(ii) $f(k) = 2 \Rightarrow I_k \in V$.

Now form $V_0 = \{I_k : f(k) = 2\} \subseteq V$. The equality between basic neighbourhood's is decidable so the subset V_0 is finite as well. Since $\alpha \in I_{k_\alpha}$, $f(k_\alpha) = 2$. Similarly $f(k_\beta) = 2$. We now claim that

$$(r', s') \lhd_{\text{fin}} V_0.$$

By density, pick rational numbers $u \in I_{k_\alpha}$, $v \in I_{k_\beta}$ with $\alpha < u$ and $v < \beta$. If $v < u$ the claim is clear. If $u < v$, then for any rational $q \in (u, v)$ there is some k with $q \in I_k$. For such k, $f(k) = 1$ is impossible by the definition of C. Thus $f(k) = 2$ and the claim follows. Again we may assume that $\cup V_0$ is an interval. Finally, since C covers what is outside the interval $[\alpha, \beta]$ we have

$$\mathcal{I}(\alpha, \beta) \lhd_C V_0. \quad \Box$$

Now the following converse of Theorem 4.1 is fairly easy

Theorem 4.6 *Let $\alpha < \beta \in \mathrm{Pt}(\mathcal{R})$. If $G : \mathcal{I}(\alpha, \beta) \to \mathcal{R}$ is a continuous mapping, then*

$$g = \mathrm{Pt}(G) : \mathrm{Pt}(\mathcal{I}(\alpha, \beta)) \to \mathrm{Pt}(\mathcal{R})$$

is uniformly continuous.

Proof. Write $C = C(\alpha, \beta)$ and $\mathcal{I} = \mathcal{I}(\alpha, \beta)$. Let ε be a positive rational number. The intervals $I_n = (n\varepsilon/2, n\varepsilon/2 + \varepsilon)$, $n \in \mathbb{Z}$, form a cover S_ε of $\mathcal{R}$. Then since $\mathcal{I} \lhd_C G^{-1}[\mathcal{R}]$, we get

$$\mathcal{I} \lhd_C G^{-1}[S_\varepsilon].$$

By Corollary 4.5 there is a finite

$$V_0 = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq G^{-1}[S_\varepsilon]$$

with $\mathcal{I} \lhd_C V_0$. For each $k = 1, \ldots, n$ there is thus some $(c_k, d_k) \in S_\varepsilon$ with

$$(a_k, b_k)\, G\, (c_k, d_k). \tag{7}$$

Now $\cup V_0$ is an interval, so let $\delta > 0$ be the Lebesgue number of V_0. Suppose that $\gamma, \gamma' \in \mathrm{Pt}(\mathcal{I})$ and $|\gamma - \gamma'| < \delta$. Take any $(a, b) \in \gamma$. Then $(a, b) \lhd_C V_0$, so there is some $(a', b') \in (C \cup V_0) \cap \gamma$. Since $C \cap \gamma = \emptyset$, we have in fact $(a', b') \in V_0$. Hence $\gamma \in \cup V_0$. Similarly $\gamma' \in \cup V_0$. Thus by the Lebesgue lemma, $\gamma, \gamma' \in (a_k, b_k)$ for some k. By (7) we have

$$(c_k, d_k) \in g(\gamma) \cap g(\gamma'),$$

and thus $|g(\gamma) - g(\gamma')| < \varepsilon$. This proves that g is uniformly continuous. □

By an analogous argument as above: suppose that $G : \mathcal{I}(\alpha, \beta) \to \mathcal{X}$ is continuous, and U is a covering of the space $\mathcal{X}$. Then there is a finite cover $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ of the interval, and a finite list of neighbourhoods $z_1, \ldots, z_n$ in U, with the property that for each point α in the interval (a_i, b_i), the function value $g(\alpha)$ is in the neighbourhood z_i. This is an often used device in homotopy theory.

Since the formal real topology $\mathcal{R}$ is regular, we have as a special case of Theorem 4.3 in (Palmgren 2003)

Lemma 4.7 *Any two continuous mappings $F, G : \mathcal{A} \to \mathcal{R}$ with $F \subseteq G$ are equal.* □

We identify $\mathbb{R}$ and $\mathrm{Pt}(\mathcal{R})$ and establish the following bijective correspondence of Bishop continuous functions on $\mathbb{R}$ and continuous mappings on $\mathcal{R}$ in the formal sense. Consider a continuous mapping $F : \mathcal{R} \to \mathcal{R}$. Then the composition with $E_{\alpha,\beta} : \mathcal{I}(\alpha, \beta) \to \mathcal{R}$ is continuous, and so

$$\mathrm{Pt}(F \circ E_{\alpha,\beta}) = \mathrm{Pt}(F) \circ \mathrm{Pt}(E_{\alpha,\beta}),$$

is uniformly continuous by Theorem 4.6. Hence $f = \mathrm{Pt}(F)$ is uniformly continuous on every compact interval, i.e. continuous in the sense of Bishop. Moreover, $A_f \subseteq F$ by (1). Thus $A_f = F$, by Lemma 4.7 and Theorem 4.1, and we are back to the continuous mapping. Conversely, if $f : \mathbb{R} \to \mathbb{R}$ is Bishop continuous, then by Theorem 4.1, $A_f : \mathcal{R} \to \mathcal{R}$ is a continuous mapping, and morever $\mathrm{Pt}(A_f) = f$.

5 Open subspaces and the reciprocal map

We show that the reciprocal map is a continuous mapping, so that any composition with it is again continuous.

Consider an arbitrary formal topology $X = (X, \leq, \lhd)$ and a set of neighbourhoods $G \subseteq X$ which is *downwards closed,* i.e. $G_{\leq} = G$. Define the *open subspace topology* $X^G = (X, \leq', \lhd')$ by letting

$$a \lhd' U \Longleftrightarrow_{\mathrm{def}} a_{\leq} \cap G \lhd U$$

and $a \leq' b$ iff $a \lhd' \{b\}$. These relations can be seen to extend $\lhd$ and $\leq$ respectively.

Theorem 5.1 *For a formal topology X and a downwards closed set of basic neighbourhoods G,*

(a) X^G *is a formal topology,*

(b) $\alpha \in \mathrm{Pt}(X^G)$ *iff* $\alpha \cap G$ *is inhabited and* $\alpha \in \mathrm{Pt}(X)$. □

Consider now the formal topology of real numbers $\mathcal{R} = (\mathcal{R}, \leq, \lhd)$. The set of neighbourhoods bounded away from 0

$$G = \{(a, b) \in \mathcal{R} : 0 < a \text{ or } b < 0\}$$

is downwards closed. $\mathcal{R}^G$ is the formal topology of *off-zero real numbers.* For $u = (a, b) \in G$ define the *reciprocal interval*

$$u^{-1} = (b^{-1}, a^{-1}).$$

Note that the operation is monotone with respect to $\leq$. The *reciprocal* is a continuous mapping I from $\mathcal{R}^G$ to $\mathcal{R}$. It is defined by

$$u\, I\, v \Longleftrightarrow u \in G \;\&\; u^{-1} \subseteq v.$$

A key lemma in the proof that I is indeed continuous is the following:

Lemma 5.2 *If $u \lhd U$, then $v \lhd' I^{-1}U$, for all $v \in G$ with $v\, I\, u$.*

Proof. By induction on $\lhd$. □

Then it is straightforward to check that for all $x \in \mathbb{R}$ apart from 0,

$$\mathrm{Pt}(I)(\overline{x}) = \overline{x^{-1}},$$

which proves I to represent the reciprocal function.

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References

E. Bishop and D.S. Bridges (1985). *Constructive Analysis.* Springer.

J. Cederquist and S. Negri (1995). A constructive proof of the Heine-Borel covering theorem for formal reals. *Types for proofs and programs (Torino, 1995),* pp. 62 – 75, Lecture Notes in Comput. Sci., 1158, Springer, Berlin, 1996.

T. Coquand (1996). Formal topology with posets. Preprint.

M.P. Fourman and R.J. Grayson (1982). Formal spaces. In: A.S. Troelstra and D. van Dalen (eds.) *The L.E.J. Brouwer Centenary Symposium*, pp. 107 – 122. North-Holland.

P.T. Johnstone (1982). *Stone Spaces.* Cambridge University Press.

P. Martin-Löf (1970). *Notes on Constructive Mathematics.* Almqvist and Wiksell, Stockholm.

S. Negri and D. Soravia (1999). The continuum as a formal space. *Arch. Math. Logic* 38 (1999), no. 7, 423 – 447.

E. Palmgren (2002). Maximal and partial points in formal topology. *Annals of Pure and Applied Logic,* to appear.

E. Palmgren (2003). Predicativity problems in point-free topology. In: V. Stoltenberg (ed.), *Proceedings of Logic Colloquium, Helsinki 2003.* To appear.

E. Palmgren (2004). Regular universes and formal spaces. *Annals of Pure and Applied Logic,* to appear.

G. Sambin (1987). Intuitionistic formal spaces — a first communication. In: D. Skordev (ed.) *Mathematical logic and its applications,* Plenum Press, pp. 187 – 204.

G. Sambin (2003). Some points in formal topology. *Theoretical Computer Science,* 305(2003), 347 – 408.

P. Schuster (2003). *What is continuity, constructively?* Preprint.

I. Sigstam (1995). Formal spaces and their effective presentations. *Arch. Math. Logic* 34(1995), 211 – 246.

F. Waaldijk (1998). *Modern Intuitionistic Topology.* PhD Thesis, Nijmegen 1998.

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