First-order many sorted logic and its interpretation in categories

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Notes, November 2, 2006

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We recall a standard formulation of first-order many sorted logic especially suitable for interpretation using adjoints. The main reference is P.T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, vol. 1-2. Oxford 2002.

Let Σ be a many-sorted first-order signature. There is in addition a binary predicate $=_A$ on each sort (or type) A of the signature.

Example 1.1 A signature for the first order theory of modules over a ring: $\Sigma_{\text{Mod}} = \{\mathsf{R}, \mathsf{M}, 0 : \mathsf{R}, 1 : \mathsf{R}, \mathsf{a} : \mathsf{R}\,\mathsf{R} \longrightarrow \mathsf{R}, \mathsf{m} : \mathsf{R}\,\mathsf{R} \longrightarrow \mathsf{R}, \overline{0} : \mathsf{M}, \overline{\mathsf{a}} : \mathsf{M}\,\mathsf{M} \longrightarrow \mathsf{M}, \mathsf{s} : \mathsf{R}\,\mathsf{M} \longrightarrow \mathsf{M}\}$ There are two sorts R (for the ring) and M (for the module). Constants such as 1 are regarded as function symbols with zero arguments.

Example 1.2 A signature for the first order theory of groups has only one sort $\Sigma_{\text{Grp}} = \{\mathsf{G}, 1 : \mathsf{G}, (-) \cdot (-) : \mathsf{G}\,\mathsf{G} \longrightarrow \mathsf{G}, (-)^{-1} : \mathsf{G} \longrightarrow \mathsf{G}\}$.

A *context* is a finite list of distinct typed variables $\vec{x} = x_1, \ldots, x_n$, which may be empty, and then denoted (). A formula φ is *suitable* for the context $\vec{x}$ if each free variable of φ is in the context. Similarly, a term t is *suitable* for $\vec{x}$ if the variables of t are in the context. A *formula-in-context* is an expression $\vec{x}.\varphi$, where φ is a formula suitable for the context $\vec{x}$. A *term-in-context* is an expression $\vec{x}.t$, where t is a term suitable for the context $\vec{x}$.

A *sequent* over the signature Σ is an expression of the form

$$\varphi \vdash_{\vec{x}} \psi \tag{1}$$

where φ and ψ are formulae over Σ which are suitable for the context $\vec{x}$.

That a term t has sort A is indicated by $t : A$. Each term t over the signature Σ has a unique sort which is denoted $\sigma(t)$. For a sequence of terms $\vec{t} = t_1, \ldots, t_n$ define $\sigma(\vec{t}) = \sigma(t_1), \ldots, \sigma(t_n)$. Two sequences of terms $\vec{s}$ and $\vec{t}$ are *sort compatible* if $\sigma(\vec{s}) = \sigma(\vec{t})$.

Definition 1.3 The basic rules and axioms of *first order logic* are the following, grouped into (a) structural rules, (b) equality rules and (c) conjunctive rules, (d) disjunctive rules, (e) implicative rules, (f) rules for existential quantification, (g) rules for universal quantification.

(a1) *Identity axiom:* for a formula φ suitable for $\vec{x}$:

$$\varphi \vdash_{\vec{x}} \varphi$$

(a2) *Cut rule:* for formulae φ, ψ, θ suitable for $\vec{x}$

$$\frac{\varphi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \theta}{\varphi \vdash_{\vec{x}} \theta}$$

(a3) *Substitution rule:* For $\vec{t} = t_1, \ldots, t_n$ a sequence of terms, sort compatible with the context $\vec{x}$, and whose free variables are among $\vec{y}$, the rule

$$\frac{\varphi \vdash_{\vec{x}} \psi}{\varphi(\vec{t}/\vec{x}) \vdash_{\vec{y}} \psi(\vec{t}/\vec{x})}$$

is applicable.

The rules for equality are the following

(b1) *Reflexivity axiom:*

$$\top \vdash_{x} x = x.$$

(b2) *Equality axiom:*

$$\vec{x} = \vec{y} \wedge \varphi \vdash_{\vec{z}} \varphi(\vec{y}/\vec{x}).$$

Here $\vec{x} = x_1, \ldots, x_n$ are distinct variables, and the variables $\vec{y} = y_1, \ldots, y_n$ are distinct, and sort compatible with $\vec{x}$. The expression $\vec{x} = \vec{y}$ is short for $x_1 = y_1 \wedge \cdots \wedge x_n = y_n$.

The conjunctive rules are, for formulae φ, ψ, θ suitable for $\vec{x}$:

(c1-3) $$\varphi \vdash^{\vec{x}} \top \qquad \varphi \wedge \psi \vdash^{\vec{x}} \varphi \qquad \varphi \wedge \psi \vdash^{\vec{x}} \psi$$

(c4)
$$\frac{\varphi \vdash^{\vec{x}} \psi \qquad \varphi \vdash^{\vec{x}} \theta}{\varphi \vdash^{\vec{x}} \psi \wedge \theta}.$$

The disjunctive rules are, for formulae φ, ψ, θ suitable for $\vec{x}$:

(d1-3) $$\bot \vdash^{\vec{x}} \varphi \qquad \varphi \vdash^{\vec{x}} \varphi \vee \psi \qquad \psi \vdash^{\vec{x}} \varphi \vee \psi$$

(d4)
$$\frac{\psi \vdash^{\vec{x}} \varphi \qquad \theta \vdash^{\vec{x}} \varphi}{\psi \vee \theta \vdash^{\vec{x}} \varphi}.$$

The implication rules are

(e1)
$$\frac{\varphi \wedge \psi \vdash^{\vec{x}} \theta}{\varphi \vdash^{\vec{x}} \psi \Rightarrow \theta}$$

(e2)
$$\frac{\varphi \vdash^{\vec{x}} \psi \Rightarrow \theta}{\varphi \wedge \psi \vdash^{\vec{x}} \theta}$$

The rules for existential quantification are

(f1)
$$\frac{\varphi \vdash^{\vec{x,y}} \psi}{(\exists y)\varphi \vdash^{\vec{x}} \psi}$$

(f2)
$$\frac{(\exists y)\varphi \vdash^{\vec{x}} \psi}{\varphi \vdash^{\vec{x,y}} \psi}$$

Here y is not free in ψ.

The rules for universal quantification are

(g1)

$$\frac{\varphi \vdash^{\vec{x},y} \psi}{\varphi \vdash^{\vec{x}} (\forall y)\psi}$$

(g2)

$$\frac{\varphi \vdash^{\vec{x}} (\forall y)\psi}{\varphi \vdash^{\vec{x},y} \psi}$$

Here y is not free in φ.

So-called *coherent logic* is obtained by forbidding universal quantification and implication in formulas, and replacing the rules (e1-2), (g1-2) by the axioms

(i1) Distributivity:

$$\varphi \wedge (\psi \vee \theta) \vdash^{\vec{x}} (\varphi \wedge \psi) \vee (\varphi \wedge \theta)$$

(i2) Frobenius axiom: (y not in $\vec{x}$)

$$\varphi \wedge (\exists y)\psi \vdash^{\vec{x}} (\exists y)(\varphi \wedge \psi).$$

Exercise 1:

(a) Show that (i1) and (i2) are derivable in full first-order logic as presented above.

(b) Consider a standard formulation of first-order logic using natural deduction. Now take away rules for $\forall$ and $\Rightarrow$. Show that (i1) and (i2) are still derivable.

(c) Try to do the same as in (b) with the first-order logic presented above. Can you explain why the logics behave so differently? (Restrict to the propositional part of the logic if this is easier.)

Regular logic is obtained by omitting from coherent logic the rules (d1-4) and (i1) and forbidding as well disjunction and absurdity in formulas.

Horn logic is obtained by omitting from regular logic the rules (f1-2) and (i2) and forbidding as well existential quantifiers in formulas. The formulas that can appear are formed by conjunction from relations and $\top$. We call such *conjunctive formulas.*

2

In any category with finite products the signature of a first-order many sorted logic can be interpreted in a way that generalises the standard[1] model-theoretic interpretation in set theory.

Let $\mathcal{C}$ be a category with finite products and let Σ be a signature. A Σ-*structure* in $\mathcal{C}$ is a function M that assigns an object $M\,A$ to each sort (type) A of Σ. For any sequence $A_1 \cdots A_n$ of sorts $M(A_1, \ldots, A_n)$ is defined as the product $M\,A_1 \times \cdots \times M\,A_n$, which is the terminal object 1 of $\mathcal{C}$ in case $n = 0$. Moreover

- To each function symbol $f : A_1 \cdots A_n \longrightarrow B$ of Σ, a morphism $M\,f : M(A_1, \ldots, A_n) \longrightarrow M\,B$ in $\mathcal{C}$ is assigned.
- To each relation symbol $R \rightarrowtail A_1 \cdots A_n$ of Σ a subobject $M\,R \rightarrowtail M(A_1, \ldots, A_n)$ in $\mathcal{C}$ is assigned.

This interpretation is now extended to terms over Σ by recursion. We write $[\![\vec{x}.t]\!]_M$ for interpretation of the term t in context $\vec{x}$. If $\vec{x} = x_1, \ldots, x_n$ where x_i has sort A_i, and t has sort B, then

$$[\![\vec{x}.t]\!]_M : M(A_1, \ldots, A_n) \longrightarrow M\,B,$$

which is defined by recursion on terms:

- $[\![\vec{x}.x_k]\!]_M = \pi_k$ which is kth projection
- $[\![\vec{x}.f(t_1, \ldots, t_m)]\!]_M = (M\,f) \circ \langle [\![\vec{x}.t_1]\!]_M, \ldots, [\![\vec{x}.t_m]\!]_M \rangle$. Note that $\langle\rangle = !_{M(A_1,\ldots,A_n)}$, i.e. the unique map from $M(A_1, \ldots, A_n)$ to the terminal object 1.

Example 2.1 Consider the signature Σ_{Mod} for modules. The term $t = \mathsf{a}(\overline{0}, \mathsf{s}(r, x))$ in the context r, x, y, where $r : \mathsf{R}$, $x : \mathsf{M}$, $y : \mathsf{M}$ would get the following interpretation in a model N

$$[\![r, x, y.t]\!]_N = N(\mathsf{a}) \circ \langle N(\overline{0}) \circ !_U, N(\mathsf{s}) \circ \langle \pi_1^U, \pi_2^U \rangle \rangle : U \longrightarrow N(\mathsf{M}).$$

Here $U = N(\mathsf{R}, \mathsf{M}, \mathsf{M}) = N(\mathsf{R}) \times N(\mathsf{M}) \times N(\mathsf{M})$.

Example 2.2 *Groups in various categories.* Let $\mathcal{C}$ be a category with finite products. A *group in* $\mathcal{C}$ is then a Σ_{Grp}-structure M in $\mathcal{C}$ such that the following equations are valid,

[1] See for instance D. van Dalen: *Logic and Structure,* Springer 1997.

writing $[\![-]\!]$ for $[\![-]\!]_M$:

$$\begin{array}{rcl}
[\![x.\quad 1\cdot x]\!] & = & [\![x.\quad x]\!] : M(\mathsf{G}) \longrightarrow M(\mathsf{G}) \\
[\![x.\quad x\cdot 1]\!] & = & [\![x.\quad x]\!] : M(\mathsf{G}) \longrightarrow M(\mathsf{G}) \\
[\![x.\quad x\cdot x^{-1}]\!] & = & [\![x.\quad 1]\!] : M(\mathsf{G}) \longrightarrow M(\mathsf{G}) \\
[\![x.\quad x^{-1}\cdot x]\!] & = & [\![x.\quad 1]\!] : M(\mathsf{G}) \longrightarrow M(\mathsf{G}) \\
[\![x,y,z.\quad x\cdot(y\cdot z)]\!] & = & [\![x,y,z.\quad (x\cdot y)\cdot z]\!] : M(\mathsf{G})\times M(\mathsf{G})\times M(\mathsf{G}) \longrightarrow M(\mathsf{G})
\end{array}$$

Exercise 2. Investigate what are the groups in each of the following categories:

(a) $\hat{\mathbb{C}}$, presheaves over $\mathbb{C}$.

(b) M-sets for a monoid M. (What happens when M is a group?)

(c) an inf-semilattice.

(d) **Cat**, the category of small categories.

(e) **Rec**, the category recursive subsets of $\mathbb{N}$ and (total) recursive functions.

Let $\vec{t} = t_1, \ldots, t_m$ be a sequence of terms suitable to the context $\vec{x} = x_1, \ldots, x_n$, where x_i has sort A_i and t_j has sort B_j. Define

$$[\![\vec{x}.\vec{t}]\!]_M = \langle [\![\vec{x}.t_1]\!]_M, \ldots, [\![\vec{x}.t_m]\!]_M \rangle : M(A_1, \ldots, A_n) \longrightarrow M(B_1, \ldots, B_m).$$

Lemma 2.3 *Let $\vec{t}$ and M be as above. Then, for a term r suitable to the context $\vec{y}$, and where $\vec{y}$ is sort compatible with $\vec{t}$,*

$$[\![\vec{x}.r(\vec{t}/\vec{y})]\!]_M = [\![\vec{y}.r]\!]_M \circ [\![\vec{x}.\vec{t}]\!]_M.$$

2.1

A category is *cartesian* (or *left exact*) iff it has all finite limits.

Lemma 2.4 *The subobjects* $(\mathrm{Sub}_{\mathcal{C}}(X), \leq)$ *in a cartesian category* $\mathcal{C}$ *forms an* inf-semilattice.

(i) $C \leq X$ *as subobjects of* X*,*

(ii) for all subobjects A, B *of* X *there is a subobject* $A \wedge B$ *of* X*, so that for any subobject* C*:*

$$C \leq A \text{ and } C \leq B \text{ iff } C \leq A \wedge B.$$

Let $\mathcal{C}$ be a cartesian category. For any of its morphism $f : X \longrightarrow Y$ there is a functor

$$f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$$

given by pullbacks along f. It induces in turn a functor between subobjects

$$f^* : \mathrm{Sub}_{\mathcal{C}}(Y) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X).$$

Sometimes f^* is then denoted f^{-1} since it is essentially an inverse image operation.

Let Σ be a signature.

Let $R \rightarrowtail B_1 \cdots B_n$ be a relation symbol in the signature Σ. Let $\vec{t} = t_1, \ldots, t_n$ be a sequence of terms suitable to the context $\vec{x} = x_1, \ldots, x_m$, where x_i has sort A_i and t_j has sort B_j. Then define

$$[\![\vec{x}.R(\vec{t})]\!]_M = [\![\vec{x}.\vec{t}]\!]^*_M(M(R)).$$

The interpretation of the equality $=_A$ on the sort A is always the diagonal subobject on $M(A)$:

$$M(=_A) = [(M(A), \Delta_{M(A)})]$$

where $\Delta_X = \langle \mathrm{id}_X, \mathrm{id}_X \rangle : X \rightarrowtail X \times X$.

The interpretation of the true constant $\top$ is

$$[\![\vec{x}.\top]\!]_M = [(M(\sigma(\vec{x})), \mathrm{id})].$$

The interpretation of conjunction is given by

$$[\![\vec{x}.\varphi \wedge \psi]\!]_M = [\![\vec{x}.\varphi]\!]_M \wedge [\![\vec{x}.\psi]\!]_M.$$

This defines the interpretation of all conjunctive formulas of a signature Σ. A sequent $\varphi \overset{\vec{x}}{\vdash\!\!-\!\!-} \psi$ *is valid under the interpretation* M if

$$[\![\vec{x}.\varphi]\!]_M \leq [\![\vec{x}.\psi]\!]_M$$

Sometimes we use $M \models (\varphi \overset{\vec{x}}{\longmapsto} \psi)$ for this judgement. We wish to show that all sequents that can be derived using Horn logic are valid, i.e. that the interpretation is sound for Horn logic. For this it suffices to show that the axioms are valid and that rules are sound, i.e. preserves validity.

Note that the rules (a1), (a2) are sound since each $\mathrm{Sub}(X)$ is a preorder. Since it is also an inf-lattice, the rules (c1-4) are sound too.

To verify the soundness of the substitution rule we need to check the following *semantic-syntactic substitution property* for each conjunctive formula φ suitable for the context $\vec{y}$, and each sequence of terms $\vec{t}$, which is sort compatible with this context

$$\text{(SP)} \qquad [\![\vec{x}.\vec{t}\,]\!]_M^*([\![\vec{y}.\varphi]\!]_M) = [\![\vec{x}.\varphi(\vec{t}/\vec{y})]\!]_M.$$

Once (SP) is known, the validity of the substitution rule (a3) follows from part (i) of Lemma 2.5 below.

Lemma 2.5 *Let $\mathcal{C}$ be a cartesian category. Let X, Y, Z be objects of the category, and suppose that $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are morphisms. Then*

(i) $A \leq B$ in $\mathrm{Sub}_{\mathcal{C}}(Y)$ implies $f^*(A) \leq f^*(B)$ in $\mathrm{Sub}_{\mathcal{C}}(X)$.

(ii) $f^*(Y) = X$ in $\mathrm{Sub}_{\mathcal{C}}(X)$.

(iii) $f^*(A \wedge B) = f^*(A) \wedge f^*(B)$

(iv) $f^*(g^*(C)) = (g \circ f)^*(C)$ for $C \in \mathrm{Sub}_{\mathcal{C}}(Z)$.

The (SP) follows from (ii) - (iv) of Lemma 2.5 and Lemma 2.3:

Lemma 2.6 *(SP) is true for all conjunctive formulas.*

For any parallel pair of maps $g, h : Y \longrightarrow Z$, let $\mathrm{Equ}(g, h)$ denote the subobject of Y given by the equaliser $e : E \longrightarrow Y$ of g and h. This equaliser may be gotten as a pullback of the diagonal Δ_Z along $\langle g, h\rangle : Y \longrightarrow Z \times Z$. From this follows that the interpretation of equality can be made in terms of equalisers:

$$[\![\vec{x}.\ s = t]\!]_M = \mathrm{Equ}([\![\vec{x}.s]\!]_M, [\![\vec{x}.t]\!]_M).$$

as subobjects. It is easy to check that a sequent $\top \overset{\vec{x}}{\longmapsto} s = t$ is valid under M if, and only if,

$$[\![\vec{x}.s]\!]_M = [\![\vec{x}.t]\!]_M.$$

This makes the validity of the reflexivity axiom (b1) evident. The equality axiom (b2) is a consequence of a slightly more general axiom

$$\vec{x} = \vec{y} \wedge \varphi(\vec{x}/\vec{u}) \overset{\vec{z}}{\longmapsto} \varphi(\vec{y}/\vec{u}),$$

where φ is suitable for $\vec{u}$. The validity of this axiom is left as an exercise to apply the following lemma:

Lemma 2.7 *Let $\mathcal{C}$ be a cartesian category.*

(i) For $\mathcal{C}$-morphisms $f_i : X \longrightarrow Y_i$, $g_i : X \longrightarrow Y_i$, $i = 1, \ldots, n$:

$$\mathrm{Equ}(f_1, g_1) \wedge \cdots \wedge \mathrm{Equ}(f_n, g_n) \leq \mathrm{Equ}(\langle f_1, \ldots, f_n \rangle, \langle g_1, \ldots, g_n \rangle).$$

(ii) Let $g, h : Y \longrightarrow Z$ be morphisms in $\mathcal{C}$. For any $A \in \mathrm{Sub}_{\mathcal{C}}(Z)$, it holds as subobjects of X:

$$\mathrm{Equ}(g, h) \wedge g^*(A) \leq h^*(A).$$

We have proved

Theorem 2.8 *Any interpretation in a cartesian category is sound for Horn logic.*

2.2

A category $\mathcal{C}$ *has images* if any morphism $g : A \longrightarrow B$ may be factorised as $e : A \longrightarrow I$ followed a monomorphism $m : I \rightarrowtail B$, so that if there is any other factorisation into $f : A \longrightarrow J$ and a mono $n : J \rightarrowtail B$, then there is a morphism $h : I \longrightarrow J$ (necessarily unique) with $n \circ h = m$.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, "f"'] & I \arrow[r, tail, "m"] \arrow[d, dotted, "h"] & B \\ & J \arrow[ur, tail, "n"'] & \end{tikzcd}$$

In other words, $m : I \rightarrowtail B$ is the smallest subobject through which g factorises. It follows from the universal property that it is unique up to isomorphism. We call the pair $(e : A \longrightarrow I, m : I \rightarrowtail B)$ an image factorisation of $g : A \longrightarrow B$.

For a category $\mathcal{C}$ let $\mathrm{Mon}(\mathcal{C}, X)$ denote the full subcategory of $\mathcal{C}/X$ determined objects that are monomorphisms. There is an inclusion functor $\mathrm{Inc}_X : \mathrm{Mon}(\mathcal{C}, X) \longrightarrow \mathcal{C}/X$.

Theorem 2.9 *The category $\mathcal{C}$ has images if, and only if, for each X the inclusion functor Inc_X has a left adjoint Im_X.*

Proof. ($\Rightarrow$) Choose for each object $\alpha : A \longrightarrow X$ of $\mathcal{C}$ an image factorisation $(e_\alpha : A \longrightarrow I_\alpha, m_\alpha : I_\alpha \longrightarrow X)$ of α. Then let $\mathrm{Im}_X)(\alpha : A \longrightarrow X) = (m_\alpha : I_\alpha \longrightarrow X)$ and $\eta_{\alpha : A \longrightarrow X} = e_\alpha$ defines the functor and unit of adjunction on objects. It is easily checked that this gives an adjunction.

($\Leftarrow$) Left to the reader. $\square$

A morphism $f : A \longrightarrow B$ is a *cover* (or *is surjective*) if whenever it factorises as $m \circ g$, where m is mono, then m must be iso. (In particular, f does not factorise through a proper subobject.)

Lemma 2.10 *Let $(e : A \longrightarrow I, m : I \rightarrowtail B)$ be an image factorisation of $g : A \longrightarrow B$. Then e is a cover.*

Proof. Suppose that $e = n \circ e'$ where n is mono. Thus $m \circ n$ is mono. So there is a unique h with $m \circ n \circ h = m$. But m is mono so $n \circ h = \mathrm{id}$. Hence $n \circ h \circ n = n$. Since n is mono, also $h \circ n = 1$. This shows n iso. $\square$

Lemma 2.11 *In a cartesian category any cover is epi.*

Proof. Suppose that $e : A \longrightarrow B$ is a cover. Suppose that $f, g : B \longrightarrow C$ satisfies $fe = ge$. Let $m : E \rightarrowtail B$ be the equaliser of f and g. Thus there is a unique morphism

$h : C \longrightarrow E$ with $mh = e$. Since e is a cover, h must be an iso. Hence $f = g$ as required. $\square$

A category $\mathcal{C}$ is *regular* iff it is cartesian, has images and all its cover morphisms are stable under pullbacks.

- That covers are stable under pullbacks means that the pullback of a cover along any map is again a cover.

A morphism which arises as a coequaliser is called a *regular epi*. Any regular epi is a cover. Indeed, the converse is also true, see Theorem 3.3.

Example 2.12 1. The category of sets is regular.

2. The category of groups is regular. The quotient map $q : \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is a regular epi (the coequaliser of $x \mapsto 0$ and $x \mapsto 2x$). It is not split epi, since any $f : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$ must be zero.

3. The category of torsion free abelian groups is regular, and the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is epi, but not regular epi.

4. The category of presheaves over any small category $\mathbb{C}$ is regular. Consider the case where $\mathbb{C}$ is the category

$$0 \rightrightarrows 1.$$

There are only two arrows from 0 to 1, say s and t. The presheaves $\widehat{\mathbb{C}}$ can be considered as the *category of directed multigraphs*. For $F \in \widehat{\mathbb{C}}$, $F(0)$ is the set of nodes, $F(1)$ is the set of arcs, and $F(s)$ and $F(t)$ assigns sources and targets, respectively, to the arcs. A natural transformation between such presheaves, is thus a graph morphism.

Let $\mathbf{1}$ be the terminal presheaf over $\mathbb{C}$: $\mathbf{1}(0) = \mathbf{1}(1) = \{*\}$, $\mathbf{1}(s) = \mathbf{1}(t) = \mathrm{id}$. Consider the presheaf given by $F(0) = F(1) = \{a, b\}$ and $F(s) = \mathrm{id}$ and

$$F(t)(a) = b \qquad F(t)(b) = a.$$

The unique natural transformation $\tau : F \overset{\cdot}{\longrightarrow} \mathbf{1}$ is regular epi, but not split epi. For suppose there is even one $\sigma : \mathbf{1} \overset{\cdot}{\longrightarrow} F$. Then we have by naturality

$$F(t)(\sigma_1(*)) = \sigma_0(\mathbf{1}(t)(*)) = \sigma_0(*)$$

and

$$\sigma_1(*) = F(s)(\sigma_1(*)) = \sigma_0(\mathbf{1}(s)(*)) = \sigma_0(*)$$

Hence $\sigma_1(*)$ is a fixed point of $F(t)$, contrary to the construction of $F(t)$. $\square$

Let $\mathcal{C}$ be a cartesian category. For any $f : X \longrightarrow Y$ define the functor $\Sigma_f : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ by $\Sigma_f(\alpha : A \longrightarrow X) = (f \circ \alpha : A \longrightarrow Y)$ on objects, and $\Sigma_f(h) = h$ on morphisms. Then Σ_f is left adjoint to f^*. Then define

$$\exists_f : \mathrm{Mon}(\mathcal{C}, X) \longrightarrow \mathrm{Mon}(\mathcal{C}, Y)$$

by

$$\exists_f = \mathrm{Im}_Y \circ \Sigma_f \circ \mathrm{Inc}_X.$$

This induces $\exists_f : \mathrm{Sub}_{\mathcal{C}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(Y)$. In fact, we have

$$\exists_f([(\alpha : A \rightarrowtail X)] = [(m : I \rightarrowtail X)]$$

where $(e : A \longrightarrow I, m : I \rightarrowtail X)$ is any image factorisation of $f \circ \alpha$.

Lemma 2.13 *In a regular category, $\exists_f$ is left adjoint to $f^* : \mathrm{Sub}_{\mathcal{C}}(Y) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$, for any $f : X \longrightarrow Y$*

Proof. We show

$$\exists_f(A) \leq B \Longleftrightarrow A \leq f^*(B).$$

Here A and B are represented by monos $\alpha : A \longrightarrow X$ and $\beta : B \longrightarrow Y$, respectively.

($\Rightarrow$) Suppose $\exists_f(A) \leq B$. Thus $f \circ \alpha$ has an image factorisation $(e : A \longrightarrow I, m : I \rightarrowtail Y)$ where $\beta \circ t = m$ for some $t : I \longrightarrow B$. Thus we have a pullback diagram as follows where $t' = te$

$$\begin{tikzcd}
A \arrow[dr, dotted, "u"] \arrow[drr, "t'"] \arrow[ddr, "\alpha"'] & & \\
& f^*B \arrow[r, "q"'] \arrow[d, tail] & B \arrow[d, "\beta"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{2}$$

The unique morphism $u : A \longrightarrow f^*B$ making the triangles above commute, thus gives $A \leq f^*B$ as subobjects of X.

($\Leftarrow$) If $A \leq f^*B$ as subobjects of X, then there is some u making the lower triangle of (2) commute. The upper triangle is obtained by letting $t' = q \circ u$. Consider any image factorisation $(e : A \longrightarrow I, m : I \rightarrowtail X)$ of $f \circ \alpha$ and thus of $\beta \circ t'$. Since β is mono, there is a unique $h : I \longrightarrow B$ with $\beta \circ h = m$. This shows $\exists_f(A) \leq B$. $\Box$

Lemma 2.14 *In a category with images: For $f : X \longrightarrow Y$ and $\alpha : A \rightarrowtail X$ we have $\exists_f(A) = Y$ (as subobjects of Y) iff $f \circ \alpha$ is a cover.*

Proof. ($\Rightarrow$) Let $(e : A \longrightarrow I, m : I \longrightarrow Y)$ be an image factorisation of $f \circ \alpha$. Since $[(Y, \mathrm{id}_Y] = [(I, m)]$ the mono m must indeed be an iso. But the composition of a cover with an iso is a cover, so $m \circ e = f \circ \alpha$ is a cover.

($\Leftarrow$) Generally, if $g : A \longrightarrow Y$ is a cover, then id_Y is an image of g. Thus $f \circ \alpha$ cover implies $\exists_f(A) = Y$. $\Box$

Supposing that $[\![\vec{x}, y.\varphi]\!]_M$ has been defined we let

$$[\![\vec{x}.(\exists y)\varphi]\!]_M = \exists_p([\![\vec{x}, y.\varphi]\!]_M)$$

where p is the projection $p : M(\sigma(\vec{x}, y)) \longrightarrow M(\vec{x})$. Now by Lemma 2.13 we have

$$[\![\vec{x}.(\exists y)\varphi]\!]_M \leq [\![\vec{x}.\psi]\!]_M \Longleftrightarrow [\![\vec{x}, y.\varphi]\!] \leq p^*([\![\vec{x}.\psi]\!]_M).$$

The projection p is given by $[\![\vec{x}, y.\vec{x}]\!]$. Thus assuming (SP) is valid for ψ we have

$$p^*([\![\vec{x}.\psi]\!]_M) = [\![\vec{x}, y.\psi]\!]_M.$$

Thus for ψ suitable for $\vec{x}$ with no occurrence of y, we have that $(\exists y)\varphi \overset{\vec{x}}{\vdash} \psi$ is valid under M if, and only if, $\varphi \overset{\vec{x},y}{\vdash} \psi$ is valid under M. This verifies the rules (f1) and (f2).

The next result is the Frobenius lemma.

Lemma 2.15 *In a regular category: for $f : X \longrightarrow Y$, $\alpha : A \rightarrowtail X$ and $\beta : B \rightarrowtail Y$,*

$$\exists_f(f^*(B) \wedge A) = B \wedge \exists_f(A)$$

as subobjects of Y.

Proof. Exercise, or see Johnstone 2002. $\Box$

Suppose that ψ is suitable for the context $\vec{x}, y$, and that φ is suitable for $\vec{x}$. If φ satisfies (SP), then from Frobenius it follows that

$$\begin{array}{rcl}
[\![\vec{x}.\varphi \wedge (\exists y)\psi]\!] & = & [\![\vec{x}.\varphi]\!] \wedge [\![\vec{x}.(\exists y)\psi]\!] \\
& = & [\![\vec{x}.\varphi]\!] \wedge \exists_{[\![\vec{x},y.\vec{x}]\!]}([\![\vec{x}, y.\psi]\!]) \\
& = & \exists_{[\![\vec{x},y.\vec{x}]\!]}([\![\vec{x}, y.\vec{x}]\!]^*([\![\vec{x}.\varphi]\!]) \wedge [\![\vec{x}, y.\psi]\!]) \\
& = & \exists_{[\![\vec{x},y.\vec{x}]\!]}([\![\vec{x}, y.\varphi]\!] \wedge [\![\vec{x}, y.\psi]\!]) \\
& = & [\![\vec{x}.(\exists y)(\varphi \wedge \psi)]\!].
\end{array}$$

This verifies axiom (i2) of regular logic.

It remains to prove (SP) for all regular formulas. This follows from the so-called Beck-Chevalley lemma

Lemma 2.16 *In a regular category: if*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback diagram then

$$\exists_{f'}(h^*(A)) = g^*(\exists_f(A))$$

as subobjects of V, for any subobject A of X.

Proof. Exercise. □

Indeed the following diagram is a pullback: for terms $\vec{t}$ suitable for $\vec{u}$, and sort compatible with $\vec{x}$ and where y is not in $\vec{u}$ or $\vec{x}$

$$\begin{tikzcd} M(\sigma(\vec{u}, y)) \arrow[r, "[\![\vec{u},y.\vec{t},y]\!]"] \arrow[d, "[\![\vec{u},y.\vec{u}]\!]"'] & M(\sigma(\vec{x}, y)) \arrow[d, "[\![\vec{x},y.\vec{x}]\!]"] \\ M(\sigma(\vec{u})) \arrow[r, "[\![\vec{u}.\vec{t}]\!]"'] & M(\sigma(\vec{x})) \end{tikzcd}$$

For φ suitable for $\vec{x}, y$ and for which (SP) holds, we get using the Beck-Chevalley Lemma

$$\begin{array}{rcl} [\![\vec{u}.\ ((\exists y)\varphi)(\vec{t}/\vec{x})]\!] & = & [\![\vec{u}.\ (\exists y)\varphi(\vec{t}, y/\vec{x}, y)]\!] \\ & = & \exists_{[\![\vec{u},y.\vec{u}]\!]}([\![\vec{u}, y.\ \varphi(\vec{t}, y/\vec{x}, y)]\!]) \\ & = & \exists_{[\![\vec{u},y.\vec{u}]\!]}([\![\vec{u}, y.\vec{t}, y]\!]^*([\![\vec{x}, y.\ \varphi]\!])) \\ & = & [\![\vec{u}.\vec{t}]\!]^*(\exists_{[\![\vec{x},y.\vec{x}]\!]}([\![\vec{x}, y.\varphi]\!])) \\ & = & [\![\vec{u}.\vec{t}]\!]^*([\![\vec{x}.(\exists y)\varphi]\!]). \end{array}$$

From this we can conclude:

Lemma 2.17 *(SP) hold for all regular formulas.*

Theorem 2.18 *Any interpretation in a regular category is sound for regular logic.*

2.3

A category $\mathcal{C}$ is *coherent* iff it is regular, each $\mathrm{Sub}_{\mathcal{C}}(X)$ is a sup-semilattice, and $f^* : \mathrm{Sub}_{\mathcal{C}}(Y) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$ preserves finite sups, for any $f : X \longrightarrow Y$.

Recall that a partial order $(L, \leq)$ is a *sup-semilattice* if there exists $\perp \in L$ and $\vee : L \times L \longrightarrow L$ such that

(i) $\perp \leq c$ for all c,

(ii) $a \leq c$ and $b \leq c$ iff $a \vee b \leq c$.

(This is the dual to an inf-semilattice.)

That f^* preserves finite sups is thus equivalent to

$$f^*(\perp_Y) = \perp_X \qquad\qquad f^*(A \vee B) = f^*(A) \vee f^*(B),$$

for any subobjects A, B of Y.

Lemma 2.19 *For a coherent category $\mathcal{C}$, the subobjects $\mathrm{Sub}_{\mathcal{C}}(X)$ for a distributive lattice.*

Proof. For any mono $\gamma : C \rightarrowtail X$ we have that the composition $\exists_\gamma \circ \gamma^* : \mathrm{Sub}_{\mathcal{C}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$ satisfies

$$(\exists_\gamma \gamma^*)(B) = \exists_\gamma(\gamma^*(B) \wedge C) = B \wedge \exists_\gamma(C) = C \wedge B, \tag{3}$$

for any subobject B. (This follows from the Frobenius lemma.) Since $\mathcal{C}$ is coherent γ^* commutes with finite sups. Also $\exists_\gamma$ commutes with finite sups, since it is a left adjoint. Thus

$$C \wedge (A \vee B) = \exists_\gamma \gamma^*(A \vee B) = \exists_\gamma \gamma^*(A) \vee \exists_\gamma \gamma^*(B) = (C \wedge A) \vee (C \vee B).$$

This proves the distributivity law. □

It is evident that in a coherent category the (SP) extendes to all coherent formulas. The distributive law for subobjects verifies (i1). Thus we can conclude:

Theorem 2.20 *Any interpretation in a coherent category is sound for coherent logic.*

2.4

A category $\mathcal{C}$ is *Heyting* iff it is coherent and for each $f : X \longrightarrow Y$ the functor $f^* : \mathrm{Sub}_{\mathcal{C}}(Y) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$ has a right adjoint $\forall_f$.

Supposing that $[\![\vec{x}, y.\varphi]\!]_M$ has been defined we let

$$[\![\vec{x}.(\forall y)\varphi]\!]_M = \forall_p([\![\vec{x}, y.\varphi]\!]_M)$$

where p is the projection $p : M(\sigma(\vec{x}, y)) \longrightarrow M(\vec{x})$.

It is now easy check that the validity of the rules (g1) and (g2) follows from adjunction property, analogously as for $\exists$, once (SP) is known for the formulas.

To define implication, we consider the composition $\forall_\beta \circ \beta^* : \mathrm{Sub}_{\mathcal{C}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(X)$, where $\beta : B \rightarrowtail X$. Then

$$A \leq (\forall_\beta \circ \beta^*)(C) \Leftrightarrow \beta^* A \leq \beta^* C \Leftrightarrow \exists_\beta \beta^* A \leq C.$$

But we know $\exists_\beta \beta^* A = A \wedge B$ from (3). Thus letting $(B \Rightarrow C) = (\forall_\beta \circ \beta^*)(C)$, we obtain

$$A \leq (B \Rightarrow C) \Longleftrightarrow A \wedge B \leq C.$$

Which is the defining equivalence for implication. The rules (f1) and (f2) are then straightforward to verify.

Theorem 2.21 *For any Heyting category $\mathcal{C}$ the subobjects $(\mathrm{Sub}_{\mathcal{C}}(X), \leq)$ form a Heyting algebra.*

The (SP) property has to be checked for all first-order formulas.

Lemma 2.22 *In a regular category: if*

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback diagram then

$$\forall_{f'}(h^*(A)) = g^*(\forall_f(A))$$

as subobjects of V, for any subobject A of X.

Proof. Use adjointness and Beck-Chevalley lemma for $\exists$. (Exercise.) $\Box$

Using this lemma one gets as for the $\exists$-case

$$[\![\vec{u}.\ ((\forall y)\varphi)(\vec{t}/\vec{x})]\!] = [\![\vec{u}.\vec{t}]\!]^*([\![\vec{x}.(\forall y)\varphi]\!]).$$

We leave as an exercise to show that

$$f^*(B \Rightarrow C) = f^*B \Rightarrow f^*C.$$

From this follows (SP) for all first-order formulas. We have then shown:

Theorem 2.23 *Any interpretation in a Heyting category is sound for intuitionistic first-order logic.*

Fact Any topos is a Heyting category.

3

Lemma 3.1 *Let $\mathcal{C}$ be a category.*

(i) Left or right composition of a cover morphism with isomorphism is a cover

(ii) A monic cover is an isomorphism.

(iii) In a cartesian category, the composition of two covers is a cover.

Proof. (i): Suppose $f : A \longrightarrow B$ is a cover, and that $g : X \longrightarrow A$ and $h : B \longrightarrow C$ are isos. If $fg = mk$ and m mono, then $f = mkg^{-1}$, so m is iso. If $hf = mk$ and m mono, then $f = h^{-1}mk$, so since $h^{-1}m$ is mono, it is iso, and so is m.

(ii): If $f : A \longrightarrow B$ is mono and a cover, then since $f \circ \mathrm{id}_A = f$, indeed f is an iso.

(iii): Suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are covers. Suppose that gf can be factorised as mh where $m : M \rightarrowtail C$ and $h : A \longrightarrow M$. Then form a pullback square

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, tail, "m'"'] & M \arrow[d, tail, "m"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Thus m' is mono. Since the square is a pullback and $gf = mh$, there is a unique map $t : A \longrightarrow P$ with $m't = f$ and $g't = h$. Now f is a cover, so m' is iso. By part (i), gm' is a cover. Since $gm' = mg'$, m must be iso as required. $\square$

Let $f : A \longrightarrow C$ be a morphism. A *kernel pair* for f is any pair of morphisms $p : P \longrightarrow A$ and $q : P \longrightarrow A$ such that

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is a pullback.

Lemma 3.2 *In a regular category, any cover f is the coequaliser of (any of) its kernel pair.*

Theorem 3.3 *In a regular category, the covers and regular epimorphisms coincide.*

4

Let $\mathcal{C}$ be a regular category. From an interpretation M in $\mathcal{C}$ of a signature we can obtain another useful form of semantics. The *Beth-Kripke-Joyal semantics* is obtained by defining for a formula φ, suitable for $\vec{x} = x_1, \ldots, x_n$, and for parameters $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle : U \longrightarrow M(\sigma(\vec{x}))$, the *forcing relation*

$$U \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha}] \tag{4}$$

to hold iff $\mathrm{Im}_{M(\sigma(\vec{x}))}(\vec{\alpha}) \leq [\![\vec{x}.\varphi]\!]_M$. We frequently omit M when it is clear from the context. The relation (4) is equivalent to: for any representing mono $\lambda : L \rightarrowtail M(\sigma(\vec{x}))$ of the subobject $[\![\vec{x}.\varphi]\!]_M$, there is some $t : U \longrightarrow L$ with $\lambda \circ t = \vec{\alpha}$. The *monotonicity* of the forcing relation is then evident: for any $f : V \longrightarrow U$

$$U \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha}] \Longrightarrow V \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha} \circ f]. \tag{5}$$

The *covering property* or *local character* of the forcing relation is the following converse: for any cover $f : V \longrightarrow U$,

$$V \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha} \circ f] \Longrightarrow U \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha}]. \tag{6}$$

This is seen as follows: Let $\lambda : L \rightarrowtail M(\sigma(\vec{x}))$ be any representing mono of the subobject $[\![\vec{x}.\varphi]\!]_M$. Suppose that $V \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha} \circ f]$, so that there is some $t : V \longrightarrow L$ with $\lambda \circ t = \vec{\alpha} \circ f$. Consider an image factorization $(g : U \longrightarrow I, m : I \rightarrowtail M(\sigma(\vec{x})))$ of $\vec{\alpha}$. By Lemma 3.1 (iii) $g \circ f$ is a cover, so there is a map $s : I \longrightarrow L$ with $\lambda \circ s = m$. Thus $\lambda \circ (s \circ g) = m \circ g = \vec{\alpha}$ as required, so $U \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha}]$.

Further, the validity of a sequent $\varphi \overset{\vec{x}}{\vdash} \psi$ under M, i.e. $[\![\vec{x}.\varphi]\!]_M \leq [\![\vec{x}.\psi]\!]_M$ is equivalent to: for all $U \in \mathcal{C}$ and for all $\vec{\alpha} : U \longrightarrow M(\sigma(\vec{x}))$,

$$U \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha}] \Longrightarrow U \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha}]. \tag{7}$$

($\Rightarrow$): Suppose the sequent is valid under M. Let $U \in \mathcal{C}$ and $\vec{\alpha} : U \longrightarrow M(\sigma(\vec{x}))$ satisfy $U \overset{\vec{x}}{\Vdash} \varphi[\vec{\alpha}]$, i.e. $\mathrm{Im}_{M(\sigma(\vec{x}))}(\vec{\alpha}) \leq [\![\vec{x}.\varphi]\!]_M$. Then by transitivity $\mathrm{Im}_{M(\sigma(\vec{x}))}(\vec{\alpha}) \leq [\![\vec{x}.\psi]\!]_M$, so $U \overset{\vec{x}}{\Vdash} \psi[\vec{\alpha}]$.

($\Leftarrow$): Suppose (7) for any U and $\vec{\alpha}$. Let $\lambda : L \rightarrowtail M(\sigma(\vec{x}))$ be a mono representing the subobject $[\![\vec{x}.\varphi]\!]_M$. Trivially, $\mathrm{Im}(\lambda) \leq [\![\vec{x}.\varphi]\!]_M$. Then for $U = L$ and $\vec{\alpha} = \lambda$, we obtain $U \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha}]$. Thus by (7), $U \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha}]$, so $\mathrm{Im}(\lambda) \leq [\![\vec{x}.\psi]\!]_M$. But $\mathrm{Im}(\lambda) = [(L, \lambda)] = [\![\vec{x}.\varphi]\!]_M$ and we are done.

Note the special case where $\varphi = \top$. Then since $U \overset{\vec{x}}{\Vdash}_M \top[\vec{\alpha}]$ is always true, we have that $\top \overset{\vec{x}}{\vdash} \psi$ is valid under M, iff for all $U \in \mathcal{C}$ and for all $\vec{\alpha} : U \longrightarrow M(\sigma(\vec{x}))$,

$$U \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha}]. \tag{8}$$

Specialising further, to the case where ψ has no free variables, we have that $\top \overset{()}{\vdash} \psi$ is valid under M, iff

$$1 \overset{()}{\Vdash}_M \top[\epsilon_1].$$

Here ϵ_U is the unique morphism $U \longrightarrow M(())$. (Recall $M(())$ is terminal.)

Lemma 4.1 *In a regular category the forcing relation satisfies*

(i) $U \overset{\vec{x}}{\Vdash}_M \top[\vec{\alpha}]$ *is true.*

(ii) $U \overset{\vec{x}}{\Vdash}_M (\varphi \wedge \psi)[\vec{\alpha}]$ *if, and only if,* $U \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha}]$ *and* $U \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha}]$.

(iii) $U \overset{\vec{x}}{\Vdash}_M (\exists y)\varphi[\vec{\alpha}]$ *if, and only if,* $V \overset{\vec{x},y}{\Vdash}_M \varphi[\langle \vec{\alpha} \circ f; \gamma \rangle]$ *for some cover* $f : V \longrightarrow U$ *and some* $\gamma : V \longrightarrow M(\sigma(y))$.

Proof. (i) and (ii) are straightforward since the subobjects of an object form an inf-semilattice in a cartesian category.

(iii) Let $\lambda : L \rightarrowtail M(\sigma(\vec{x}, y))$ be a representative for the subobject $[\![\vec{x}, y.\varphi]\!]_M$. Let $(e : L \longrightarrow I, m : I \longrightarrow M(\sigma(\vec{x}))$ be an image factorization of $p \circ \lambda$ where $p : M(\sigma(\vec{x}, y)) \longrightarrow M(\sigma(\vec{x}))$ is the canonical projection. Thus $[(I, m)] = [\![\vec{x}.(\exists y)\varphi]\!]_M$.

($\Rightarrow$): Suppose $U \overset{\vec{x}}{\Vdash}_M (\exists y)\varphi[\vec{\alpha}]$. Thus there is $t : U \longrightarrow I$ with $m \circ t = \vec{\alpha}$. Let $f : V \longrightarrow U$ be the pullback of e along t as in the pullback square

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "s"'] & U \arrow[d, "t"] \\ L \arrow[r, "e"'] & I \end{tikzcd}$$

By stability of covers under pullbacks, f is a cover. Let $q : M(\sigma(\vec{x}, y)) \longrightarrow M(\sigma(y))$ be the canonical leftmost projection. Put $\gamma = q \circ \lambda \circ s$. Form the canonical pairing $\langle \vec{\alpha} \circ f; \gamma \rangle : V \longrightarrow M(\sigma(\vec{x}, y))$. We claim that $\langle \vec{\alpha} \circ f; \gamma \rangle = \lambda \circ s$. Indeed, this follows since

$$p \circ \langle \vec{\alpha} \circ f; \gamma \rangle = \vec{\alpha} \circ f = mtf = mes = p\lambda s$$

and

$$q \circ \langle \vec{\alpha} \circ f; \gamma \rangle = \gamma = q\lambda s.$$

Thus

$$V \overset{\vec{x},y}{\Vdash}_M \varphi[\langle \vec{\alpha} \circ f; \gamma \rangle]. \tag{9}$$

($\Leftarrow$): Suppose now (9) where $f : V \longrightarrow U$ is a cover and $\gamma : V \longrightarrow M(\sigma(y))$. Thus there is some $s : V \longrightarrow L$ with $\langle \vec{\alpha} \circ f; \gamma \rangle = \lambda \circ s$. Now consider $(e : L \longrightarrow I, m : I \longrightarrow M(\sigma(\vec{x})))$, the mentioned image factorisation of $p \circ \lambda$. Thus we have a commutative square,

$$\begin{tikzcd} V \arrow[r, "es"] \arrow[d, "f"'] & I \arrow[d, "m"] \\ U \arrow[r, "\vec{\alpha}"'] & X \end{tikzcd}$$

Here $X = M(\sigma(\vec{x}))$. Considering an image factorisation $(h : L \longrightarrow J, n : J \rightarrowtail X)$ of $\vec{\alpha}$, and since f is a cover, we get by Lemma 3.1 that (hf, n) is an image factorisation of mes. Hence there is a morphism $t : J \longrightarrow I$ such that $mt = n$, and hence $mth = \vec{\alpha}$ as required. $\Box$

Lemma 4.2 *In a coherent category the forcing relation satisfies*

(i) $U \overset{\vec{x}}{\Vdash}_M \bot[\vec{\alpha}]$ *iff* $\mathrm{Im}_{M(\sigma(\vec{x}))}(\vec{\alpha}) = \emptyset$, *i.e. the image of* $\vec{\alpha}$ *is empty.*

(ii) $U \overset{\vec{x}}{\Vdash}_M (\varphi \vee \psi)[\vec{\alpha}]$ *iff there are monos* $m : V \rightarrowtail U$ *and* $n : W \rightarrowtail U$ *with*

$$V \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha} \circ m] \text{ and } W \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha} \circ n],$$

and where $[(V, m)] \vee [(W, n)] = [(U, \mathrm{id}_U)]$.

Proof. (i): This is immediate since $[\![\vec{x}.\bot]\!]$ is by definition the smallest element, i.e. the empty subobject, of $M(\sigma(\vec{x}))$.

(ii): Let $\kappa : K \rightarrowtail X$ and $\lambda : L \rightarrowtail X$ be representatives of the subobjects $[\![\vec{x}.\varphi]\!]_M$ and $[\![\vec{x}.\psi]\!]_M$ of $X = M(\sigma(\vec{x}))$. Let $\delta : D \rightarrowtail X$ be such that $[(D, \delta)] = [(K, \kappa)] \vee [(L, \lambda)]$, so $[(D, \delta)] = [\![\vec{x}.(\varphi \vee \psi)]\!]_M$.

($\Rightarrow$): Suppose $U \overset{\vec{x}}{\Vdash}_M (\varphi \vee \psi)(\vec{\alpha})$. Thus there is a $t : U \longrightarrow D$ such that $\delta \circ t = \vec{\alpha}$. This means that the following is a pullback diagram.

$$\begin{tikzcd} U \arrow[r, "t"] \arrow[d, tail, "\mathrm{id}_U"'] & D \arrow[d, tail, "\delta"] \\ U \arrow[r, "\vec{\alpha}"'] & X \end{tikzcd} \tag{10}$$

Now form pullbacks of κ and λ along $\vec{\alpha}$:

$$
\begin{tikzcd}
V \arrow[r, "r"] \arrow[d, "m"'] & K \arrow[d, "\kappa"] \\
U \arrow[r, "\vec{\alpha}"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "s"] \arrow[d, "n"'] & L \arrow[d, "\lambda"] \\
U \arrow[r, "\vec{\alpha}"'] & X
\end{tikzcd}
\tag{11}
$$

Thus m and n are monos. Since suprema are preserved by pullbacks along any morphism (in particular $\vec{\alpha}$) in a coherent category, we have indeed

$$[(V, m)] \vee [(W, n)] = [(U, \mathrm{id}_U)].$$

Now since (11) are commuting we get $V \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha} \circ m]$ and $W \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha} \circ n]$.

($\Rightarrow$): Assume the righthand side. Then there are r and s so that the diagrams in (11) commute. Thus $[(V, m)] \leq \vec{\alpha}^*([(K, \kappa)])$ and $[(W, n)] \leq \vec{\alpha}^*([(L, \lambda)])$. Therefore

$$[(U, \mathrm{id}_U)] = \vec{\alpha}^*([(K, \kappa)]) \vee \vec{\alpha}^*([(L, \lambda)]) = \vec{\alpha}^*([(K, \kappa)] \vee [(L, \lambda)]) = \vec{\alpha}^*([(D, \delta)]).$$

This implies the existence of a t such that (10) commutes, and hence $U \overset{\vec{x}}{\Vdash}_M (\varphi \vee \psi)[\vec{\alpha}]$.
$\square$

Lemma 4.3 *In a Heyting category the forcing relation satisfies*

(i) $U \overset{\vec{x}}{\Vdash}_M (\varphi \Rightarrow \psi)(\vec{\alpha})$ *iff for all* V *and all* $f : V \longrightarrow U$

$$V \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha} \circ f] \Longrightarrow V \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha} \circ f]$$

(ii) $U \overset{\vec{x}}{\Vdash}_M ((\forall y)\varphi)[\vec{\alpha}]$ *iff for all* V*, all* $f : V \longrightarrow U$ *and all* $\gamma : V \longrightarrow M(\sigma(y))$*,*

$$V \overset{\vec{x},y}{\Vdash}_M \varphi[\langle \vec{\alpha} \circ f; \gamma \rangle].$$

Proof. Let $X = M(\sigma(\vec{x}))$.

(i, $\Rightarrow$): Suppose $U \overset{\vec{x}}{\Vdash}_M (\varphi \Rightarrow \psi)[\vec{\alpha}]$. Thus $\mathrm{Im}_X(\vec{\alpha}) \leq [\![\vec{x}.\varphi \Rightarrow \psi]\!]_M$. Take any $f : V \longrightarrow U$, and suppose that $V \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha} \circ f]$. Thus $\mathrm{Im}_X(\vec{\alpha} \circ f) \leq [\![\vec{x}.\varphi]\!]_M$. Since $\mathrm{Im}_X(\vec{\alpha} \circ f) \leq \mathrm{Im}_X(\vec{\alpha})$, we have

$$\mathrm{Im}_X(\vec{\alpha} \circ f) \leq [\![\vec{x}.\varphi \Rightarrow \psi]\!]_M \wedge [\![\vec{x}.\varphi]\!]_M = ([\![\vec{x}.\varphi]\!]_M \Rightarrow [\![\vec{x}.\psi]\!]_M) \wedge [\![\vec{x}.\varphi]\!]_M \leq [\![\vec{x}.\psi]\!]_M.$$

Therefore $V \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha} \circ f]$.

(i, $\Leftarrow$): Suppose the condition on the righthand side of (i). It suffices to show

$$\mathrm{Im}_X(\vec{\alpha}) \wedge [\![\vec{x}.\varphi]\!]_M \leq [\![\vec{x}.\psi]\!]_M. \tag{12}$$

Let $\kappa : K \rightarrowtail X$ and $\lambda : L \rightarrowtail X$ be representatives of the subobjects $[\![\vec{x}.\varphi]\!]_M$ and $[\![\vec{x}.\psi]\!]_M$ of $X = M(\sigma(\vec{x}))$. Consider an image factorization $(e : U \longrightarrow I, m : I \rightarrowtail X)$ of $\vec{\alpha} : U \longrightarrow X$. Construct successive pullbacks (right to left)

$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ U \arrow[r, "e"'] & I \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & K \arrow[d, "\kappa"] \\ I \arrow[r, tail, "m"'] & X \end{tikzcd} \tag{13}$$

It follows that $V \overset{\vec{x}}{\Vdash}_M \varphi[\vec{\alpha} \circ f]$, so also $V \overset{\vec{x}}{\Vdash}_M \psi[\vec{\alpha} \circ f]$, which means that there is some $t : V \longrightarrow L$ with $\lambda \circ t = \vec{\alpha} \circ f$. But $\vec{\alpha} \circ f = (\kappa \circ q) \circ r$, where $\kappa \circ q$ is mono and r is a cover. Thus $(r, \kappa \circ q)$ is an image factorization of $\lambda \circ t$. Hence there is $s : P \longrightarrow L$ with $\lambda \circ s = \kappa \circ q$, which implies (12).

(ii): Let $Z = M(\sigma(\vec{x}, y))$ and suppose that $\lambda : L \rightarrowtail Z$ is a representative of the subobject $[\![\vec{x}, y.\varphi]\!]_M$. Let $p : Z \longrightarrow X$ be the canonical projection.

($\Rightarrow$): Assume $U \overset{\vec{x}}{\Vdash}_M ((\forall y)\varphi)[\vec{\alpha}]$ where $\vec{\alpha} : U \longrightarrow X$. Let $(e : U \longrightarrow I, m : I \rightarrowtail X)$ be an image factorization of $\vec{\alpha}$. By assumption, we have $[(I, m)] \leq \forall_p([(L, \lambda)])$, which is equivalent to $p^*([(I, m)]) \leq [(L, \lambda)]$. This is in turn the same as saying that for the pullback square

$$\begin{tikzcd} Q \arrow[r, "s"] \arrow[d, "n"'] & I \arrow[d, "m"] \\ Z \arrow[r, "p"'] & X \end{tikzcd} \tag{14}$$

there is some $t : Q \longrightarrow L$ with $\lambda \circ t = n$. For any $f : V \longrightarrow U$ and $\gamma : V \longrightarrow Y = M(\sigma(y))$, there is a canonical pairing $\langle \vec{\alpha} \circ f; \gamma \rangle : V \longrightarrow Z$. Thus $p \circ \langle \vec{\alpha} \circ f; \gamma \rangle = \vec{\alpha} \circ f = m \circ e \circ f$. By the pullback there is some $r : V \longrightarrow Q$ with $n \circ r = \langle \vec{\alpha} \circ f; \gamma \rangle$. Thus also $\lambda \circ (t \circ r) = \langle \vec{\alpha} \circ f; \gamma \rangle$. This shows

$$V \overset{\vec{x},y}{\Vdash}_M \varphi[\langle \vec{\alpha} \circ f; \gamma \rangle]. \tag{15}$$

($\Leftarrow$): Conversely, suppose that (15) holds for all $f : V \longrightarrow U$ and $\gamma : V \longrightarrow Y$. We have a pullback as in (14). Next form another pullback by pulling back e along s.

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "h"'] & U \arrow[d, "e"] \\ Q \arrow[r, "s"'] & I \end{tikzcd} \tag{16}$$

Note that h is a cover. Now put $f = g$ and let $\gamma = qnh$, where $q : Z \longrightarrow Y$ is the canonical projection. Thus by (15), $\langle \vec{\alpha} \circ f ; \gamma \rangle = \lambda \circ w$ for some $w : V \longrightarrow L$. We have $pnh = msh = meg = \vec{\alpha} f$ and so $\langle \vec{\alpha} \circ f ; \gamma \rangle = nh$ by the universal property of products. Since (h, n) is an image factorization and since λ is mono, there is $t : Q \longrightarrow L$ with $\lambda \circ t = n$ as required. $\square$

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