An intuitionistic axiomatisation of real closed fields

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Abstract

We give an intuitionistic axiomatisation of real closed fields, which has the constructive reals as a model. The main result is that this axiomatisation together with just the decidability of the order relation gives the classical theory of real closed fields. To establish this we rely on the quantifier elimination theorem for real closed fields due to Tarski, and a conservation theorem of classical logic over intuitionistic logic for geometric theories.

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1 Intuitionistic real closed fields

The axioms (1) – (7) below constitute the theory of ordered fields OF in the language $L_{\mathrm{RCF}} = \{+, \cdot, 0, 1, P\}$. P is an unary predicate intended to express positivity.

1. The axioms for a commutative ring with 1.
2. $P(1)$.
3. If $P(-x)$ or $P(x)$, then x is invertible.
4. $\neg P(0)$.
5. If $\neg P(-x)$ and $\neg P(x)$, then $x = 0$.
6. If $P(x)$ and $P(y)$, then $P(x + y)$ and $P(xy)$.
7. If $P(x + y)$ then $P(x)$ or $P(y)$.

Write $x > y$ for $P(x - y)$, and $x \leq y$ for $\neg(x > y)$.

Remark. OF proves that $=$ is stable, i.e. if $\neg\neg x = y$ then $x = y$. Axiom (7) is classically superfluous. Axiom (5) is classically equivalent to the *trichotomy axiom*

$$P(-x) \vee x = 0 \vee P(x).$$

However, OF proves the constructive substitute for this axiom: if $x < y$ then $x < z$ or $z < y$.

We take the intermediate value theorem for polynomials $f(x) = y_0x^n + y_1x^{n-1} + \cdots + y_n$ as an axiom scheme:

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8. If $f(u) < 0 < f(v)$ and $u < v$, then for some x, $u < x < v$ and $f(x) = 0$.

The axioms (1) – (8) define the (intuitionistic) theory RCF of real closed fields. The constructive real numbers provide a model for RCF: to verify (8) use Exercise 2.17 in Bishop and Bridges (1985). The theory DRCF is RCF extended with the axiom that positivity is decidable

9. $P(x)$ or $\neg P(x)$.

A constructive model of DRCF is given by the real algebraic numbers, cf. pp. 8–9 in Beeson (1985).

2 Relation to the classical axiomatisation

Below $\vdash$ and $\vdash_c$ denote derivability in intuitionistic and classical logic, respectively. If T is a first order theory, then T^c denotes the extension of this theory with the principle of excluded middle for all formulae. The main result is

Theorem 2.1 DRCF *is intuitionistically equivalent to* RCF^c*.*

We postpone the proof. A formula is *geometric* if it is constructed from atomic formulae (including $\top$ and $\bot$) using only $\exists$, $\wedge$ and $\vee$. Clearly any geometric formula can be written as $\exists \overline{x}\,(\phi_1 \vee \cdots \vee \phi_n)$, where each ϕ_i is a conjunction of atomic formulae. A *geometric implication* is a formula $\forall \overline{x}\,(\phi \longrightarrow \psi)$, where both ϕ and ψ are geometric formulae. Any geometric formula can trivially be regarded as a geometric implication. A theory is *geometric* if all its axioms are geometric implications. All axioms of RCF except (5) are given by geometric implications. The theory resulting when replacing (5) by the trichotomy axiom is called GRCF. This is a geometric theory which is intuitionistically equivalent to DRCF.

The following result is well-known from topos theory, for a proof using Kripke models see Exercise 2.6.14 in Troelstra and van Dalen (1988). We give a purely proof-theoretic argument.

Lemma 2.2 *Let* Γ *be a geometric theory, and let* θ *be a geometric implication. If* $\Gamma^c \vdash \theta$*, then* $\Gamma \vdash \theta$*.*

Proof. Let $\theta \equiv \forall \overline{x}\,(\phi \longrightarrow \psi)$, and suppose $\Gamma^c \vdash \theta$, i.e. $\Gamma \vdash_c \theta$. By replacing the variables $\overline{x}$ with fresh constants, and putting ϕ among the axioms Γ, the problem reduces to the case where θ is geometric. Furthermore any geometric theory is intuitionistically equivalent to a theory where all the axioms have the form

$$\forall \overline{x}\,(\phi \longrightarrow \exists \overline{y}\,(\phi_1 \vee \cdots \vee \phi_n)), \tag{1}$$

where $\phi, \phi_1, \ldots, \phi_n$ are conjunctions of atomic formulae. We assume that the axioms of Γ has this form and that $\theta = \exists \overline{z}\,(\psi_1 \vee \cdots \vee \psi_n)$, where the ψ_i are conjunctions of atomic formulae.

The rest of the proof is an application of Gödel's negative translation and the Dragalin–Friedman syntactic translation (cf. Troelstra and van Dalen 1988). By the negative translation, $\Gamma^g \vdash \theta^g$. Using intuitionistic logic we find that $\theta^g \leftrightarrow \neg\neg\theta$. Moreover using the tautology $(A \longrightarrow B) \longrightarrow (\neg\neg A \longrightarrow \neg\neg B)$ one easily verifies that $\Gamma \vdash \Gamma^g$. Hence $\Gamma \vdash \neg\neg\theta$. Now using the Dragalin-Friedman syntactic translation with the formula $A = \theta$, we obtain $\Gamma^A \vdash (\neg\neg\theta)^A$. We have $\Gamma \vdash \Gamma^A$, so $\Gamma \vdash (\neg\neg\theta)^A$. But $(\neg\neg\theta)^A = (\theta^A \longrightarrow A) \longrightarrow A$ and $\theta^A \leftrightarrow \exists\overline{z}(\psi_1 \vee \cdots \vee \psi_n \vee A)$. Hence $\theta^A \longrightarrow A$ is a tautology, so $\Gamma \vdash A$. ∎

A *polynomial relation* is a boolean combination of atomic formulae from L_{RCF}.

Theorem 2.3 (Tarski) *For any polynomial relation $A(x_1, \ldots, x_n, y)$ we can find, effectively, a polynomial relation $B(x_1, \ldots, x_n)$ such that*

$$\mathrm{RCF}^c \vdash B(x_1, \ldots, x_n) \leftrightarrow (\exists y)\, A(x_1, \ldots, x_n, y). \blacksquare$$

Proof of Theorem 2.1. Clearly $\mathrm{RCF}^c \vdash \mathrm{DRCF}$. To prove the converse, note first that any polynomial relation is equivalent to a (quantifier free) geometric formula in virtue of the trichotomy axiom. The equivalence formula in Tarski's theorem is thus equivalent to a conjunction of two geometric implications. Using this observation and Lemma 2.2 we see that Tarski's theorem holds also for DCRF (or GRCF) in place of RCF^c. We can eliminate quantifiers inductively and conclude that DRCF derives the principle of excluded middle. ∎

Remark. The simplest proofs of Tarski's theorem use classical model theory. Delzell (1996) however states that Tarski's original proof of quantifier elimination is already constructive. For a different constructive proof see Lombardi and Roy (1991).

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