Locally cartesian closed categories without chosen constructions

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Abstract

We show how to formulate locally cartesian closed category without chosen pullbacks, by the use of Makkai's theory of anafunctors. *MSC classification 2000: 18A35, 18A40.*

1 Introduction

The standard formulation of a locally cartesian closed category (LCCC) depends on the assumption of chosen pullbacks. One may not always assume that such pullbacks can be chosen, working inside a topos, or in a meta-theory which lacks the full axiom of choice. Such theories are, for instance, Zermelo-Fraenkel set theory, ZF, or constructive theories of types and sets. Makkai [2, 3] developed a theory of generalised functors, *anafunctors*, which can handle non-chosen limit constructions in a functorial way. We shall here apply this theory to the example of LCCC. We thus give a formulation of LCCCs without chosen constructions. In the course of this we also note that some basic results about adjoints carry over to the anafunctor setting (Theorems 2.3 and 3.2). The results in Sections 4 and 5 indicate that categorical logic may be developed smoothly without chosen constructions.

2 Anafunctors

We choose to use the "span formulation" of anafunctors given in [2]. Let $\mathcal{X}$ and $\mathcal{A}$ be categories. An *anafunctor* F from $\mathcal{X}$ to $\mathcal{A}$ is a category $|F|$ and pair of functors $F_0 : |F| \longrightarrow \mathcal{X}$ and $F_1 : |F| \longrightarrow \mathcal{A}$ such that F_0 satisfies the conditions

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(A1) F_0 is surjective on objects,

(A2) for any $s, t \in |F|$ and $g : F_0(s) \longrightarrow F_0(t)$ there is a unique $f : s \longrightarrow t$ with $g = F_0(f)$. Denote this f by $|g|_{s,t}$.

We write $F : \mathcal{X} \looparrowright \mathcal{A}$ for such an anafunctor. In short, it is thus a span of ordinary functors

$$\mathcal{X} \xleftarrow{\;F_0\;} |F| \xrightarrow{\;F_1\;} \mathcal{A}$$

where F_0 is full, faithful and surjective on objects.

Note that for $s, t \in |F|$ with $X = F_0(s) = F_0(t)$ the morphism $|\mathrm{id}_X|_{s,t} : s \longrightarrow t$ is an isomorphism. For $s = t$, $|\mathrm{id}_X|_{s,t}$ is id_s. Note, further, that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{X}$, then for any $r, s, t \in |F|$ with $F_0(r) = X$, $F_0(s) = Y$, $F_0(t) = Z$, we have $|g \circ f|_{r,t} = |g|_{s,t} \circ |f|_{r,s}$.

A standard functor $F : \mathcal{X} \longrightarrow \mathcal{A}$ becomes an anafunctor $\hat{F} : \mathcal{X} \looparrowright \mathcal{A}$ by letting $|\hat{F}| = \mathcal{X}$, $\hat{F}_1 = F$ and letting $\hat{F}_0$ be the identity functor on $\mathcal{X}$.

Remark 2.1 Using the axiom of choice, we may, given an anafunctor $G : \mathcal{X} \looparrowright \mathcal{A}$, construct a standard functor $\check{G} : \mathcal{X} \longrightarrow \mathcal{A}$ as follows. For any object X of $\mathcal{X}$ choose $H(X)$ in $|G|$ with $G_0(H(X)) = X$. Then put $\check{G}(X) = G_1(H(X))$ and, for $f : X \longrightarrow Y$, let $\check{G}(f) = G_1(|f|_{H(X),H(Y)})$.

Remark 2.2 The composition of anafunctors is a composition of spans using a pullback, and is associative merely up to canonical isomorphism. In [2] it is shown that categories, anafunctors and natural transformations form a (super large) bicategory.

An anafunctor $F : \mathcal{X} \looparrowright \mathcal{A}$ is said to *preserve colimits of type* I if, and only if, for every functor $H : I \longrightarrow |F|$ and $c \in |F|$, $\tau : H \overset{\cdot}{\longrightarrow} \Delta(c)$: whenever $F_0\tau : F_0H \overset{\cdot}{\longrightarrow} F_0\Delta(c)$ is a colimiting cone then so is $F_1\tau : F_1H \overset{\cdot}{\longrightarrow} F_1\Delta(c)$. Preservation of limits is defined dually. An anafunctor $F : \mathcal{X} \looparrowright \mathcal{A}$ is said to *preserve property* P of arrows, if for any $f : s \longrightarrow t$ in $|F|$, whenever $F_0(f)$ has property P, then so has $F_1(f)$. Let $F : \mathcal{X} \looparrowright \mathcal{A}$ and $G : \mathcal{A} \looparrowright \mathcal{X}$ be anafunctors. Then F is *left adjoint (or anaadjoint)* to G if for $t \in |F|$ and $v \in |G|$ there are bijections

$$\varphi_{t,v} : \mathcal{A}(F_1(t), G_0(v)) \longrightarrow \mathcal{X}(F_0(t), G_1(v)) \qquad (1)$$

satisfying the following naturality conditions

(N1) for $s, t \in |F|$, $v \in |G|$, $h : s \longrightarrow t$, $f \in \mathcal{A}(F_1(t), G_0(v))$

$$\varphi_{t,v}(f) \circ F_0(h) = \varphi_{s,v}(f \circ F_1(h))$$

(N2) for $t \in |F|$, $v, w \in |G|$, $k : v \longrightarrow w$, $g \in \mathcal{X}(F_0(t), G_1(v))$

$$G_0(k) \circ \varphi^{-1}_{t,v}(g) = \varphi^{-1}_{t,w}(G_1(k) \circ g).$$

Note a particular case of (N1) where $h : s \longrightarrow t$ is such that $F_0(h) = \mathrm{id}_X$ where $X = F_0(s) = F_0(t)$. Then $\varphi_{t,v}(f) = \varphi_{s,v}(f \circ F_1(h))$ and $F_1(h)$ is an isomorphism.

We have as for usual adjoints, and with a similar proof:

Theorem 2.3 *Let $F : \mathcal{X} \nrightarrow \mathcal{A}$ and $G : \mathcal{A} \nrightarrow \mathcal{X}$ be anafunctors such that F is left adjoint to G. Then:*

(i) F preserves colimits of any type,

(ii) F preserves epis,

(iii) G preserves limits of any type,

(iv) G preserves monos.

Example 2.4 Suppose that $F : \mathcal{X} \nrightarrow \mathcal{A}$ is left adjoint to $G : \mathcal{A} \nrightarrow \mathcal{X}$. For diagrams of finite type, i.e. where I is a finite category such as indicated by

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \qquad\qquad \bullet \rightrightarrows \bullet \tag{2}$$

The axiom of choice is not needed for a finite category and we can reason as follows. Suppose that

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\;q\;} Z$$

is a coequalizer diagram in $\mathcal{X}$. Pick some $r, s, t \in |F|$ with $F_0(r) = X$, $F_0(s) = Y$ and $F_0(t) = Z$ using (A1). Now, since F preserves colimits of the type to the right in (2), the following is a coequalizer diagram in $\mathcal{A}$

$$F_1(r) \underset{F_1(|g|_{r,s})}{\overset{F_1(|f|_{r,s})}{\rightrightarrows}} F_1(s) \xrightarrow{\;F_1(|q|_{s,t})\;} F_1(t)$$

This holds regardless of the choices of r, s, t satisfying the above equations.

For later reference we give details of the constructions involved in proofs (see [2]) of the equivalence between local and global existence conditions for anaadjoints. This is a generalisation of the corresponding results for ordinary functors [1]. An anafunctor $F : \mathcal{X} \looparrowright \mathcal{A}$ satisfies the *local existence condition for a right adjoint (LR)* if for any $A \in \mathcal{A}$ there are $s \in |F|$, $\varepsilon : F_1(s) \longrightarrow A$ such that

(*) for each $t \in |F|$ and each $h : F_1(t) \longrightarrow A$ there is a unique $\hat{h} : t \longrightarrow s$ with $\varepsilon \circ F_1(\hat{h}) = h$.

Lemma 2.5 *An anafunctor $F : \mathcal{X} \looparrowright \mathcal{A}$ satisfies (LR) if, and only if, there is a right adjoint $G : \mathcal{A} \looparrowright \mathcal{X}$ to F.*

Proof. ($\Leftarrow$) Suppose that $G : \mathcal{A} \looparrowright \mathcal{X}$ is right adjoint to F and that $\varphi_{t,v}$ is a family of bijections witnessing the adjunction as in (1). We verify (LR): Let $A \in \mathcal{A}$ be given. Then take $v \in |G|$ with $A = G_0(v)$ and then $s \in |F|$ with $F_0(s) = G_1(v)$. Put

$$\varepsilon = \varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)}) : F_1(s) \longrightarrow A. \tag{3}$$

Now consider any $t \in |F|$ and $h : F_1(t) \longrightarrow A$. Then $k = \varphi_{t,v}(h) : F_0(t) \longrightarrow G_1(v) = F_0(s)$. Let $\hat{h} = |k|_{t,s} : t \longrightarrow s$. Hence by (N1) and inverting $\varphi_{t,v}$

$$\begin{aligned}
\varepsilon \circ F_1(\hat{h}) &= \varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)} \circ F_1(\hat{h})) \\
&= \varphi^{-1}_{t,v}(\varphi_{s,v}(\varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)}) \circ F_0(\hat{h})) \\
&= \varphi^{-1}_{t,v}(\mathrm{id}_{G_1(v)} \circ F_0(\hat{h})) \\
&= \varphi^{-1}_{t,v}(F_0(\hat{h})) = \varphi^{-1}_{t,v}(k) = h.
\end{aligned}$$

Suppose that $h' : t \longrightarrow s$ satisfies $\varepsilon \circ F_1(h') = h$. As above $\varepsilon \circ F_1(\hat{h}) = \varphi^{-1}_{t,v}(F_0(h'))$. Thus $\varphi^{-1}_{t,v}(F_0(h')) = \varphi^{-1}_{t,v}(F_0(\hat{h}))$, and since $\varphi_{t,v}$ is a bijection $F_0(h') = F_0(\hat{h})$. As F_0 is faithful, we have in fact $h' = \hat{h}$.

For $h' : t \longrightarrow s$ we note a useful identity

$$\varepsilon \circ F_1(h') = \varphi^{-1}_{t,v}(F_0(h')). \tag{4}$$

($\Rightarrow$) We construct G as follows. Let $|G|$ be the category whose objects are triples (A, s, ε) where $A \in \mathcal{A}$, $s \in |F|$, $\varepsilon : F_1(s) \longrightarrow A$ satisfies universal property (*). In this category a morphism from (A, s, ε) to (A', s', ε') is a pair (f, g) where $f : s \longrightarrow s'$, $g : A$

$\longrightarrow A'$ are such that the square

$$\begin{array}{ccc} F_1(s) & \xrightarrow{\varepsilon} & A \\ {\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle g} \\ F_1(s') & \xrightarrow[\varepsilon']{} & A' \end{array}$$

commutes. According to the universal property of (A', s', ε') the morphism f is determined uniquely by g. Next, define $G_0 : |G| \longrightarrow \mathcal{A}$ by $G_0(A, s, \varepsilon) = A$ and $G_0(f, g) = g$, which is seen to be a functor that satisfies (A2). By (LR) it follows that (A1) holds. Then define $G_1 : |G| \longrightarrow \mathcal{X}$ by $G_1(A, s, \varepsilon) = F_0(s)$ and $G_1(f, g) = F_0(f)$. Thus $G : \mathcal{A} \nrightarrow \mathcal{X}$ is an anafunctor. To prove that F is left adjoint to G we construct, for $t \in |F|$ and $v = (A, p, \varepsilon) \in |G|$, the bijection

$$\varphi_{t,v} : \mathcal{A}(F_1(t), G_0(v)) \longrightarrow \mathcal{X}(F_0(t), G_1(v))$$

as follows. We have $\varepsilon : F_1(p) \longrightarrow A$ and $A = G_0(p)$ since $v \in |G|$. For any $h \in \mathcal{A}(F_1(t), G_0(v))$ there is a unique $\hat{h} : t \longrightarrow p$ with $\varepsilon \circ F_1(\hat{h}) = h$. Let $\varphi_{t,v}(h) = F_0(\hat{h})$. Now if $\varphi_{t,v}(h_2) = \varphi_{t,v}(h)$, then by faithfulness of F_0 we get $\hat{h} = \widehat{h_2}$. Therefore $h = h_2$ as well. For a given $k \in \mathcal{X}(F_0(t), G_1(v))$, we have for some $h : t \longrightarrow p$ that $F_0(h) = k$, as $G_1(v) = F_0(p)$. Thus trivially $\varphi_{t,u}(\varepsilon \circ F_1(h)) = k$.

To verify the naturality conditions (N1) and (N2) is straightforward. □

Dually we have the following notion. An anafunctor $G : \mathcal{A} \nrightarrow \mathcal{X}$ satisfies the *local existence condition for a left adjoint* (LL) if for any $X \in \mathcal{X}$ there is $s \in |G|$ and $\eta : X \longrightarrow G_1(s)$ such that

(**) for each $t \in |G|$ and each $f : X \longrightarrow G_1(t)$ there is a unique $\hat{f} : s \longrightarrow t$ with $G_1(\hat{f}) \circ \eta = f$.

Lemma 2.6 *An anafunctor $G : \mathcal{A} \nrightarrow \mathcal{X}$ satisfies (LL) if, and only if, there is a left adjoint $F : \mathcal{X} \nrightarrow \mathcal{A}$ to G.*

Proof. The anafunctor F is constructed as follows. It is analogous to that of Lemma 2.5, but we spell it out for completeness. The proof of its properties is omitted., being dual to that of the mentioned lemma.

The category $|F|$ consists of triples (X, s, η) satisfying property (**). A morphism $(f, g) : (X, s, \eta) \longrightarrow (X', s', \eta')$ consists of $f : X \longrightarrow X'$ and $g : s \longrightarrow s'$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\eta} & G_1(s) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G_1(g)} \\ X' & \xrightarrow[\eta']{} & G_1(s') \end{array}$$

commutes. According to the universal property, g is determined uniquely by f.

Define $F_0 : |F| \longrightarrow \mathcal{X}$ and $F_1 : |F| \longrightarrow \mathcal{A}$ by $F_0(X, s, \eta) = X$, $F_0(f, g) = f$, $F_1(X, s, \eta) = G_0(s)$ and $F_1(f, g) = g$. By the (LL) property it follows that F is an anafunctor. □

3 Natural transformations

We recall the definition from [2]. A *natural transformation* $h : F \longrightarrow G$ between two anafunctors $F, G : \mathcal{X} \looparrowright \mathcal{A}$ is a family $h_{s,t} : F_1(s) \longrightarrow G_1(t)$ ($s \in |F|, t \in |G|, F_0(s) = G_0(t)$) of morphisms in $\mathcal{A}$ such that for all $f : s \longrightarrow u$ and $g : t \longrightarrow v$, with $F_0(s) = G_0(t)$, $F_0(u) = G_0(v)$ and $F_0(f) = G_0(g)$, the diagram

$$\begin{array}{ccc} F_1(s) & \xrightarrow{F_1(f)} & F_1(u) \\ \downarrow{\scriptstyle h_{s,t}} & & \downarrow{\scriptstyle h_{u,v}} \\ G_1(t) & \xrightarrow[G_1(g)]{} & G_1(v) \end{array}$$

commutes. In case F and G are ordinary functors, i.e. $F_0 = G_0 = \mathrm{Id}_{\mathcal{X}}$, this reduces to the standard notion of natural transformation.

We now prove a little coherence result. Suppose that $k : G \longrightarrow H$ is another natural transformation, where $G, H : \mathcal{X} \looparrowright \mathcal{A}$ are anafunctors. Then we claim that the diagram

$$\begin{array}{ccc} F_1(s) & \xrightarrow{h_{s,t'}} & G_1(t') \\ \downarrow{\scriptstyle h_{s,t}} & & \downarrow{\scriptstyle k_{t',r}} \\ G_1(t) & \xrightarrow[k_{t,r}]{} & H_1(r) \end{array}$$

commutes for any $s \in |F|$, $t, t' \in |G|$, $r \in |H|$ with $X = F_0(s) = G_0(t) = G_0(t') = H_0(r)$. There is a unique $g : t \longrightarrow t'$ with $G_0(g) = \mathrm{id}_X$. Then $F_0(\mathrm{id}_s) = H_0(\mathrm{id}_t) = G_0(g)$, so by naturality we have

$$\begin{aligned} k_{t',r} \circ h_{s,t'} &= k_{t',r} \circ h_{s,t'} \circ F_1(\mathrm{id}_s) \\ &= k_{t',r} \circ G_1(g) \circ h_{s,t} \\ &= H_1(\mathrm{id}_r) \circ k_{t,r} \circ h_{s,t} \\ &= k_{t,r} \circ h_{s,t} \end{aligned}$$

Thus define the composition of k and h by

$$(k \cdot h)_{s,r} = k_{t,r} \circ h_{s,t}$$

where t is any element of $|G|$ with $F_0(s) = G_0(t) = H_0(r)$. Such t exists since G is surjective on objects. It follows that $(k \cdot h)_{s,r}$ is well-defined. Naturality is clear by the naturality of h and k.

For an anafunctor $F : \mathcal{X} \nrightarrow \mathcal{A}$ the identity natural transformation $1_F : F \longrightarrow F$ is defined by $(1_F)_{s,t} = F_1(|\mathrm{id}_X|_{s,t})$ for $s, t \in |F|$ with $X = F_0(s) = F_0(t)$. A natural transformation $h : F \longrightarrow G$ between two anafunctors $F, G : \mathcal{X} \nrightarrow \mathcal{A}$ is a *natural isomorphism* if there is a natural transformation $k : G \longrightarrow F$ such that $k \cdot h = 1_F$ and $h \cdot k = 1_G$. We omit the straightforward verification of the following lemma.

Lemma 3.1 *Let $F, G : \mathcal{X} \nrightarrow \mathcal{A}$ be anafunctors, and let $h : F \longrightarrow G$ be a natural transformation. Then h is a natural isomorphism if, and only if, $h_{s,t} : F_1(s) \longrightarrow G_1(t)$ is an isomorphism for all $s \in |F|$, $t \in |G|$ with $F_0(s) = G_0(t)$.*

It is now possible to generalise the uniqueness results for adjoints to the anafunctor case.

Theorem 3.2 *Left and right adjoints (if they exist) of an anafunctor $F : \mathcal{X} \nrightarrow \mathcal{A}$ are unique up to natural isomorphism.*

Proof. We prove this for right adjoints. Suppose that $G, G' : \mathcal{A} \nrightarrow \mathcal{X}$ are both right adjoints of F. Thus there are families of bijections

$$\varphi_{t,v} : \mathcal{A}(F_1(t), G_0(v)) \longrightarrow \mathcal{X}(F_0(t), G_1(v)) \qquad (t \in |F|, v \in |G|)$$

and

$$\varphi'_{t,v'} : \mathcal{A}(F_1(t), G'_0(v')) \longrightarrow \mathcal{X}(F_0(t), G'_1(v)) \qquad (t \in |F|, v' \in |G'|)$$

satisfying (N1) and (N2).

We construct $h_{v,v'} : G_1(v) \longrightarrow G'_1(v')$ for $v \in |G|$ and $v' \in |G'|$ with $G_0(v) = G'_0(v')$. Take $s, s' \in |F|$ with $F_0(s) = G_1(v)$ and $F_0(s') = G'_1(v')$ and consider the counits

$$\begin{aligned} \varepsilon_{s,v} &= \varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)}) : F_1(s) \longrightarrow G_0(v) \\ \varepsilon'_{s',v'} &= \varphi'^{-1}_{s',v'}(\mathrm{id}_{G'_1(v')}) : F_1(s') \longrightarrow G'_0(v'). \end{aligned}$$

Since $G_0(v) = G'_0(v')$, there is thus a unique $f : s \longrightarrow s'$ with $\varepsilon'_{s',v'} \circ F_1(f) = \varepsilon_{s,v}$ and a unique $g : s' \longrightarrow s$ with $\varepsilon_{s,v} \circ F_1(g) = \varepsilon'_{s',v'}$. It follows by the universal properties that g is the inverse to f. Write $\theta_{s,s',v,v'} = f$. Define

$$h_{v,v'} = F_0(\theta_{s,s',v,v'}).$$

This is an iso. We need to show that this definition does not depend on s and s'. Suppose $t, t' \in |F|$ with $F_0(t) = G_1(v)$ and $F_0(t') = G'_1(v')$. There are unique $k : s \longrightarrow t$ and $k' : s' \longrightarrow t'$ with $F_0(k) = \mathrm{id}_{G_1(v)}$ and $F_0(k') = \mathrm{id}_{G'_1(v')}$. Thus to show $F_0(\theta_{s,s',v,v'}) = F_0(\theta_{t,t',v,v'})$ it suffices to prove

$$k' \circ \theta_{s,s',v,v'} = \theta_{t,t',v,v'} \circ k.$$

This is done by verifying that

$$\varepsilon'_{t',v'} \circ F_1(k' \circ \theta_{s,s',v,v'}) = \varepsilon'_{t',v'} \circ F_1(\theta_{t,t',v,v'} \circ k)$$

from which the identity follows by the uniqueness. Indeed, we have using (N1) in the second step and properties of the units of the adjunction

$$\begin{aligned}
\varepsilon'_{t',v'} \circ F_1(k' \circ \theta_{s,s',v,v'}) &= {\varphi'}^{-1}_{t',v'}(\mathrm{id}_{G'_1(v')}) \circ F_1(k') \circ F_1(\theta_{s,s',v,v'}) \\
&= {\varphi'}^{-1}_{s',v'}({\varphi'}_{t',v'}({\varphi'}^{-1}_{t',v'}(\mathrm{id}_{G'_1(v')})) \circ F_0(k')) \circ F_1(\theta_{s,s',v,v'}) \\
&= {\varphi'}^{-1}_{s',v'}(\mathrm{id}_{G'_1(v')} \circ \mathrm{id}_{G'_1(v')}) \circ F_1(\theta_{s,s',v,v'}) \\
&= {\varepsilon'}_{s',v'} \circ F_1(\theta_{s,s',v,v'}) \\
&= \varepsilon_{s,v} \\
&= \varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)}) \\
&= \varphi^{-1}_{s,v}(\varphi_{t,v}(\varphi^{-1}_{t,v}(\mathrm{id}_{G_1(v)})) \circ F_0(k)) \\
&= \varphi^{-1}_{t,v}(\mathrm{id}_{G_1(v)}) \circ F_1(k) \\
&= \varepsilon_{t,v} \circ F_1(k) \\
&= \varepsilon'_{t',v'} \circ F_1(\theta_{t,t',v,v'}) \circ F_1(k) \\
&= \varepsilon'_{t',v'} \circ F_1(\theta_{t,t',v,v'} \circ k).
\end{aligned}$$

Thus $h_{v,w}$ is well-defined and iso. We finally verify that these morphisms form a natural transformation. Consider $\alpha : v \longrightarrow w$ and $\alpha' : v' \longrightarrow w'$ with $G_0(v) = G'_0(v')$, $G_0(w) = G'_0(w')$ and $G_0(\alpha) = G'_0(\alpha')$. We show that

$$\begin{array}{ccc}
G_1(v) & \xrightarrow{h_{v,v'}} & G'_1(v') \\
{\scriptstyle G_1(\alpha)}\downarrow & & \downarrow{\scriptstyle G'_1(\alpha')} \\
G_1(w) & \xrightarrow[h_{w,w'}]{} & G'_1(w')
\end{array}$$

commutes. We consider some s, s', t, t' with $F_0(s) = G_1(v)$, $F_0(s') = G'_1(v')$, $F_0(t) = G_1(w)$, $F_0(t') = G'_1(w')$. Then

$$h_{v,v'} = F_0(\theta_{s,s',v,v'}) \qquad h_{w,w'} = F_0(\theta_{t,t',w,w'}).$$

There are unique $a : s \longrightarrow t$ and $a' : s' \longrightarrow t'$ with $F_0(a) = G_1(\alpha)$ and $F_0(a') = G_1'(\alpha')$. It is sufficient to check

$$\theta_{t,t',w,w'} \circ a = a' \circ \theta_{s,s',v,v'} : s \longrightarrow t'.$$

As above we get the first step of

$$\begin{aligned}
\varepsilon'_{t',w'} \circ F_1(a' \circ \theta_{s,s',v,v'}) &= \varphi'^{-1}_{s',w'}(\varphi'_{t',w'}(\varphi'^{-1}_{t',w'}(\mathrm{id}_{G_1'(w')}) \circ F_0(a')) \circ F_1(\theta_{s,s',v,v'}) \\
&= \varphi'^{-1}_{s',w'}(F_0(a')) \circ F_1(\theta_{s,s',v,v'}) \\
&= \varphi'^{-1}_{s',w'}(G_1'(\alpha')) \circ F_1(\theta_{s,s',v,v'}) \\
&= G_0'(\alpha') \circ \varphi'^{-1}_{s',v'}(\mathrm{id}_{G_1'(v')}) \circ F_1(\theta_{s,s',v,v'}) \qquad \text{(using N2)} \\
&= G_0'(\alpha') \circ \varepsilon'_{s',v'} \circ F_1(\theta_{s,s',v,v'}) \\
&= G_0'(\alpha') \circ \varepsilon_{s,v} \\
&= G_0(\alpha) \circ \varepsilon_{s,v} \\
&= G_0(\alpha) \circ \varphi^{-1}_{s,v}(\mathrm{id}_{G_1(v)}) \\
&= \varphi^{-1}_{s,w}(F_0(\alpha)) \qquad \text{(using N2)} \\
&= \varepsilon_{t,w} \circ F_1(a) \qquad \text{(by (4))} \\
&= \varepsilon'_{t',w'} \circ F_1(\theta_{t,t',w,w'}) \circ F_1(a) \\
&= \varepsilon'_{t',w'} \circ F_1(\theta_{t,t',w,w'} \circ a)
\end{aligned}$$

This proves that h forms a natural transformation. □

4 Locally cartesian closed categories

Convention. The objects of a slice category $\mathcal{C}/X$ are morphisms $a : A \longrightarrow X$, and will be written (A, a). As a further abbreviation we write $\alpha = (A, a), \beta = (B, b), \gamma = (C, c)$ etc.

For any morphism $f : X \longrightarrow Y$ the functor $\Sigma_f : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ is given by composition with f on objects, $\Sigma_f(\alpha) = (A, f \circ a)$, and defined as the identity on morphisms $\Sigma_f(h) = h$. We regard this functor as an anafunctor $S = S_f : \mathcal{C}/X \looparrowright \mathcal{C}/Y$, so $|S| = \mathcal{C}/X$, $S_0 = \mathrm{Id}_{|S|}$ and $S_1 = \Sigma_f$. The (LR) condition for S now says: for any $\alpha \in \mathcal{C}/Y$ there is some $\pi \in \mathcal{C}/X$ and $e : (P, f \circ p) \longrightarrow \alpha$ such that for each $\pi' \in \mathcal{C}/X$ and each $h : (P', f \circ p') \longrightarrow \alpha$ there is a unique $\hat{h} : \pi' \longrightarrow \pi$ with $e \circ \hat{h} = h$. This says, in other words, that for any $\alpha \in \mathcal{C}/Y$ there are morphisms p and e such that the following diagram is a pullback

$$\begin{array}{ccc} P & \xrightarrow{e} & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Suppose that $\mathcal{C}$ is a category where pullbacks exists (but are not necessarily chosen). The following can now be obtained from the construction in Lemma 2.5. Define for $f : X \longrightarrow Y$ an anafunctor $F_f = F : \mathcal{C}/Y \looparrowright \mathcal{C}/X$ that expresses pullback along f. The category $|F|$ consists of objects (β, π, q) where $\beta \in \mathcal{C}/Y$, $\pi \in \mathcal{C}/X$ and where

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback square. A morphism $(h, k) : (\beta, \pi, q) \longrightarrow (\beta', \pi', q')$ consists of morphisms $h : \pi \longrightarrow \pi'$ and $k : \beta \longrightarrow \beta'$ in $\mathcal{C}/X$ and $\mathcal{C}/Y$ respectively, such that

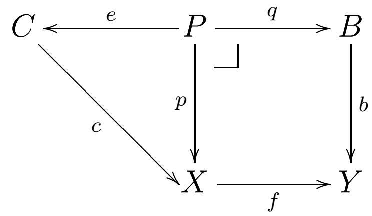
$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ P' & \xrightarrow[q']{} & B' \end{array}$$

commutes. Define functors $F_0 : |F| \longrightarrow \mathcal{C}/Y$ by $F_0(\beta, \pi, q) = \beta$ and $F_0(h, k) = k$ and $F_1 : |F| \longrightarrow \mathcal{C}/X$ by $F_1(\beta, \pi, q) = \pi$ and $F_1(h, k) = h$. The functor F_0 is surjective on objects since pullbacks exists. As for (A2) suppose $k : F_0(\beta, \pi, q) \longrightarrow F_0(\beta', \pi', q')$, i.e. $k : \beta \longrightarrow \beta'$. By the pullback property, there is a unique map $h : P \longrightarrow P'$ with $p'h = p$ and $kq = q'h$, i.e. such that $(h, k) : (\beta, \pi, q) \longrightarrow (\beta', \pi', q')$ is a morphism. This shows (A2). Hence we have shown:

Lemma 4.1 *Let $\mathcal{C}$ be a category with pullbacks. For any $f : X \longrightarrow Y$ the anafunctor $S_f : \mathcal{C}/X \looparrowright \mathcal{C}/Y$ is left adjoint to $F_f : \mathcal{C}/Y \looparrowright \mathcal{C}/X$.* $\Box$

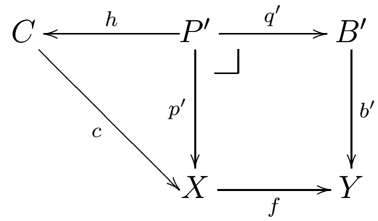
We shall employ the usual notations Σ_f and f^* for anafunctors S_f and F_f respectively. The next step is to spell out the condition for f^* to have a right anaadjoint.

This gives a functorial definition of LCCCs without chosen constructions. The (LR) condition for $F = f^* : \mathcal{C}/Y \looparrowright \mathcal{C}/X$ becomes explicitly: for every $\gamma \in \mathcal{C}/X$ there are (***) $s = (\beta, \pi, q) \in |F|$ and $e : F_1(s) = \pi \longrightarrow \gamma$ in $\mathcal{C}/X$, i.e. there is a commutative diagram



(5)

which is such that, if there is any other commutative diagram



(6)

i.e. $t = (\beta', \pi', q') \in |F|$ and $h : F_1(t) = \pi' \longrightarrow \gamma$ then there is a unique $(m, n) : t \longrightarrow s$ such that

$$e \circ m = h. \tag{7}$$

Note that m is determined by $n : \beta' \longrightarrow \beta$ because of the pullback property.

For any category $\mathcal{C}$, let $\mathrm{Mon}_{\mathcal{C}}(X)$ be the full subcategory of $\mathcal{C}/X$ determined by objects that are monomorphisms going into X.

Lemma 4.2 *Let $\mathcal{C}$ be a category with pullbacks. Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{C}$. If $f^* : \mathcal{C}/Y \looparrowright \mathcal{C}/X$ satisfies the (LR) condition, then the anafunctor $\Pi_f : \mathcal{C}/X \looparrowright \mathcal{C}/Y$, constructed as follows, is a right adjoint to Π_f. The category $|\Pi_f|$ consists of triples (γ, s, e) such that (***) above is satisfied. Moreover, for $s = (\beta, \pi, q)$ and $((h, k), \ell) : (\gamma, s, e) \longrightarrow (\gamma', s', e')$,*

$$\begin{aligned}
(\Pi_f)_0(\gamma, s, e) &= \gamma \\
(\Pi_f)_0((h, k), \ell) &= \ell \\
(\Pi_f)_1(\gamma, s, e) &= (f^*)_0(s) = \beta \\
(\Pi_f)_0((h, k), \ell) &= (f^*)_0(h, k) = k.
\end{aligned}$$

Moreover, the functor Π_f restricts to an anafunctor $\mathrm{Mon}_{\mathcal{C}}(X) \looparrowright \mathrm{Mon}_{\mathcal{C}}(Y)$.

Proof. The first part follows directly from the general construction of a right adjoint in Lemma 2.5.

As for the second part, let $(\gamma, s, e) \in |\Pi_f|$ and $s = (\beta, \pi, q)$ and suppose that $\gamma \in \mathrm{Mon}_{\mathcal{C}}(X)$, i.e. $c : C \longrightarrow X$ is mono. We show that $b : B \longrightarrow Y$ is mono. Let $r_1, r_2 : B' \longrightarrow B$ be so that $br_1 = br_2$. Let $b' = br_1$. Form the pullback

$$\begin{array}{ccc} P' & \xrightarrow{q'} & B' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle b'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

As $b' = br_1 = br_2$ there is, for each $k = 1, 2$, a unique $u_k : P' \longrightarrow P$ with $qu_k = r_k q'$ and $pu_k = p'$. By the equality $pu_1 = pu_2$ we get $ceu_1 = ceu_2$. Thus, since c is mono, $eu_1 = eu_2$. Let $h = eu_1$. Hence $ch = ceu_1 = pu_1 = p'$. Thus we have a diagram just as (6), with $t = (\pi', \beta', q')$. Let $(m, n) : t \longrightarrow s$ be the unique morphism such that $e \circ m = h$. By the above $(m, n) = (u_k, r_k)$, $k = 1, 2$, also satisfies these conditions. Hence $r_1 = r_2$, which proves b to be mono. □

5 Images and order reflection

Though images of morphisms may be formulated straightforwardly without any chosen construct, we show how they arise by a left anaadjoint of the inclusion functor.

For any category $\mathcal{C}$ and any object X of the category, let $\mathrm{Inc}_X : \mathrm{Mon}_{\mathcal{C}}(X) \longrightarrow \mathcal{C}/X$ be the inclusion functor, which we also regard as an anafunctor $\mathrm{Jnc}_X : \mathrm{Mon}_{\mathcal{C}}(X) \nrightarrow \mathcal{C}/X$. The (LL) condition for Jnc_X now reads as follows: for any $\alpha \in \mathcal{C}/X$ there is $\iota \in \mathrm{Mon}_{\mathcal{C}}(X)$ and $h : \alpha \longrightarrow \iota$ such that

(†) for any $\kappa \in \mathrm{Mon}_{\mathcal{C}}(X)$ and any $f : \alpha \longrightarrow \kappa$ there is a unique $\hat{f} : \iota \longrightarrow \kappa$ with $\hat{f} \circ h = f$.

Actually, the last constraint is unnecessary, since it follows from $k \circ f = a = i \circ h = k \circ \hat{f} \circ h$ and that k is mono. Consequently, the (LL) condition for Jnc_X is equivalent to the existence of images in $\mathcal{C}$. The left adjoint anafunctor $H = \mathrm{Jm}_X : \mathcal{C}/X \longrightarrow \mathrm{Mon}_{\mathcal{C}}(X)$ to Jnc_X is then, according to Lemma 2.6, given by the following. The category $|H|$ consists of as objects, triples (α, ι, h) such that $h : \alpha \longrightarrow \iota$ and $\alpha \in \mathcal{C}/X$ and $\iota \in \mathrm{Mon}_{\mathcal{C}}(X)$ which satisfies (†). In other words, $A \xrightarrow{h} I \xrightarrow{i} X$ is an image factorisation of $a : A \longrightarrow X$. A morphism $(f, g) : (\alpha, \iota, h) \longrightarrow (\alpha', \iota', h')$ then consists of $f : \alpha \longrightarrow \alpha'$ and $g : \iota \longrightarrow \iota'$ such that $g \circ h = h' \circ f$. Further $H_0(\alpha, \iota, \eta) = \alpha$, $H_0(f, g) = f$ and $H_1(\alpha, \iota, \eta) = \iota$, $H_1(f, g) = g$.

Each anafunctor between partial orders turns out to be naturally isomorphic to an ordinary functor, and may thus be regarded simply as a monotone map. In fact, we have a slightly stronger result.

Proposition 5.1 *If $F : (A, \leq) \longrightarrow (B, \leq)$ is some anafunctor from a preorder to a partial order, then it is naturally isomorphic to an ordinary functor $G : (A, \leq) \longrightarrow (B, \leq)$.*

Proof. When regarding a preorder $(P, \leq)$ as a category, we write $o_{a,b} : a \longrightarrow b$ for the unique arrow that exists if, and only if, $a \leq b$ holds. The functor G is given by $G(a) = F_1(s)$ where $s \in |F|$ is some object with $F_0(s) = a$. This is a good definition, since if $a = F_0(s) = F_0(t)$, there is an isomorphism $f : s \longrightarrow t$ with $F_0(f) = 1_a$. Thus also $F_1(f) : F_1(s) \longrightarrow F_1(t)$ is an isomorphism, and hence $F_1(s) = F_1(t)$, as B is a partial order. One shows using a similar lifting argument that G is monotone: if $a \leq a'$ then there are s and s' with $a = F_0(s)$, $a' = F_0(s')$, and hence there is some $f : s \longrightarrow s'$ with $F_0(f) = o_{a,a'} : a \longrightarrow a'$. Thereby $F_1(f) : F_1(s) \longrightarrow F_1(s')$, that is $G(s) = F_1(s) \leq F_1(s') = G(s')$. The natural isomorphism $f : F \looparrowright \check{G}$, where $\check{G} : (A, \leq) \longrightarrow (B, \leq)$ is the anafunctor version of G, is given by

$$f_{s,t} = \mathrm{id}_{G(t)} = o_{G(t),G(t)} : F_1(s) \longrightarrow \check{G}(t)$$

for $s \in |F|$, $t \in A$ with $F_0(s) = (\check{G})_0(t) = t$. $\square$

As an application, an anafunctor $\mathrm{Mon}_{\mathcal{C}}(X) \looparrowright \mathrm{Mon}_{\mathcal{C}}(Y)$ thus gives rise to an equivalent monotone map $\mathrm{Sub}_{\mathcal{C}}(X) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(Y)$ in an obvious way using the proposition.

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