A constructive and functorial embedding of locally compact metric spaces into locales

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August 29, 2006

Abstract

The paper establishes, within constructive mathematics, a full and faithful functor $\mathcal{M}$ from the category of locally compact complete metric spaces and continuous functions into the category of formal topologies (or equivalently locales). The functor preserves finite products, and moreover satisfies $f \leq g$ if, and only if, $\mathcal{M}(f) \leq \mathcal{M}(g)$ for continuous $f, g : X \to \mathbb{R}$. This makes it possible to transfer results between Bishop's constructive theory of metric spaces and constructive locale theory.

MSC: 03F60, 06D22

Keywords: Locales, formal topologies, locally compact metric spaces

The main result of this paper is that the category of locally compact complete metric spaces may be embedded in the category of formal topologies via a full and faithful functor. The proof is constructive in the sense of Bishop's constructive mathematics (BISH) [2]. This makes it possible to directly use certain results from BISH in formal topology, or locale theory. The classical standard proof of this result would simply use the adjunction between topological spaces and locales [6], and the observation that metric spaces are sober. Aczel [1] gives a constructive version of this adjunction. However, this general adjunction approach does not take advantage of the ability of locales to represent (local) uniform continuity of functions on locally compact complete metric spaces, as the embedding gives a locale where the cover is defined in a point-wise fashion. Instead we use Vickers' notion of a localic completion of a metric space [13, 14]. In [14] it is proved that the localic completion $\mathcal{M}_X$ of a complete metric X space is compact as a locale if, and only if, X is totally bounded. In this paper we extend this construction to

*The author is supported by a grant from the Swedish Research Council (VR).

a full and faithful functor from the category of locally compact complete metric spaces to the category of formal topologies. (The latter category is equivalent to the category of locales in a topos.) This functor takes, in fact, a locally compact complete metric space to a locally compact formal topology. For $X = \mathbb{R}$, the localic completion $\mathcal{M}_X$ is the localic reals.

Curi [5] sketches another embedding, that of uniform spaces and uniform continuous functions, into uniform formal topologies. Some early work on the representation of metric spaces using power domains is Blanck [3].

In Section 1 we provide some preliminaries about formal topology. Section 2 recalls from [13] how localic completions may be regarded as completions of metric spaces. What we actually present is a reformulation of a special case of Vickers' more general result. In Section 3 we study some useful cover relations between formal balls, the basic neighbourhoods of $\mathcal{M}_X$. Section 4 gives a characterisation of the cover relation on $\mathcal{M}_X$ for a locally compact complete metric space X (Theorem 4.16) which is crucial for the functorial embedding. This characterisation may be regarded as a generalisation of the one given for real numbers by Mulvey and Coquand, see [8]. The functorial embedding is established in Section 5. We show that the functor preserves finitary products in Section 6. In Section 7 it is shown that $\mathrm{Cont}(X,\mathbb{R})$ and $\mathrm{Cont}(\mathcal{M}_X,\mathcal{R})$ are order isomorphic with respect to $\leq$ and via the functor $\mathcal{M}$.

1 Formal topologies

A representation of locales that is particularly convenient from the constructive and predicative point of view, and thus suitable for BISH, is formal topologies. We refer to [10, 11] for background.

1.1 Basic definitions and results

Definition 1.1 A *formal topology* consists of a pre-order $X = (X, \leq)$ of *basic open neighbourhoods* and $\lhd \subseteq X \times \mathcal{P}(X)$, the *(formal) covering relation,* which satisfies the four conditions

(Ref) $a \in U$ implies $a \lhd U$,

(Tra) $a \lhd U$, $U \lhd V$ implies $a \lhd V$,

(Loc) $a \lhd U$, $a \lhd V$ implies $a \lhd U \wedge V$,

(Ext) $a \leq b$ implies $a \lhd \{b\}$.

Here $U \lhd V \Leftrightarrow_{\text{def}} (\forall a \in U)\, a \lhd V$, and $U \wedge V = (U_{\leq}) \cap (V_{\leq})$, where $Z_{\leq} =_{\text{def}} \{x \in X : (\exists z \in Z)\, x \leq z\}$. The set $U \wedge V$ is called the *formal intersection* of U and V. Write $a \wedge b$ for $\{a\} \wedge \{b\}$.

Furthermore we require that the cover relation is *set-presented* in the sense that there is a family $C(a,i)$ $(i \in I(a))$ of subsets of X so that

$$a \lhd U \iff (\exists i \in I(a)) C(a,i) \subseteq U.$$

We write the components of a formal space $\mathcal{X}$ as $(X, \leq_{\mathcal{X}}, \lhd_{\mathcal{X}}, C_{\mathcal{X}})$, often omitting the set-presentation $C_{\mathcal{X}}$.

Define the mutual cover relation $U \sim V$ to hold if, and only if, $U \lhd V$ and $V \lhd U$. A subset $Z \subseteq X$ is *down-closed* if $Z_{\leq} = Z$. Let $Z_{\lhd} = \{x \in X : x \lhd Z\}$. A subset $Z \subseteq X$ is *saturated* if $Z_{\lhd} = Z$. The saturated subsets corresponds to elements in the associated locale. These may always be represented by mere subsets, up to mutual covering, since $U \sim U_{\lhd}$. Any subset represents an open set in this way.

1.2 Equivalent forms of set-presentation

A pair (a,U), where $a \in X$ and $U \subseteq X$, is called a *covering axiom*. We say that a formal topology $\mathcal{X}$ is *generated by a family of covering axioms* (a_i, U_i) $(i \in I)$, if $\lhd_{\mathcal{X}}$ is the smallest cover relation so that

$$a_i \lhd_{\mathcal{X}} U_i$$

for all $i \in I$.

One can show that a formal topology is set-presented if, and only if, it is generated by a set-indexed family of covering axioms.

1.3 Points

A *point* of $\mathcal{X}$ is a non-void subset $\alpha \subseteq S$ which is

(Fil) $\leq$-filtering, i.e. for $a, b \in \alpha$, there is $c \in \alpha$ with $c \in a \wedge b$,

(Spl) such that α contains a neighbourhood from U, whenever $a \lhd U$ and $a \in \alpha$. (This is often expressed as: "a point splits any cover").

Note that the quantification over all subsets U can be replaced by quantification over the family $\{C_{\mathcal{X}}(a,i)\}_{i \in I(a)}$. If $\mathcal{X}$ is inductively generated by the axioms $(a_j, U_j)_{j \in J}$, then (Spl) is equivalent to

$$a_j \in \alpha \Longrightarrow (\exists b \in U_j)\, b \in \alpha. \tag{1}$$

This is often the most useful version. The points of a formal topology $\mathcal{X}$ forms a class $\mathrm{Pt}(\mathcal{X})$, which sometimes is a set. For $a \in X$ let a^* denote the subclass of points α in $\mathcal{X}$ satisfying $a \in \alpha$. For a subset $U \subseteq X$, let U^* denote the union of all the subclasses a^* for $a \in U$.

Lemma 1.2 *Any formal cover of $\mathcal{X}$ is a point-wise cover:*

$$a \lhd_{\mathcal{X}} U \Longrightarrow a^* \subseteq U^*. \quad \square$$

We say the covers of formal topology $\mathcal{X}$ are *order conservative,* if $a \leq_{\mathcal{X}} b$ whenever $a \lhd_{\mathcal{X}} \{b\}$. The covers are *point-wise order conservative* if $a^* \subseteq b^*$ implies $a \leq_{\mathcal{X}} b$. In view of Lemma 1.2 the latter is a stronger property.

1.4 Continuous morphisms

Let $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{T} = (T, \leq', \lhd')$ be formal topologies. A relation $F \subseteq S \times T$ is a *continuous mapping* $\mathcal{S} \to \mathcal{T}$ if

(A1) $a F b, b \lhd' V \Longrightarrow a \lhd F^{-1} V$,

(A2) $a \lhd U, x F b$ for all $x \in U \Longrightarrow a F b$,

(A3) $S \lhd F^{-1} T$,

(A4) $a F b, a F c \Longrightarrow a \lhd F^{-1}(b \wedge c)$.

Here $F^{-1} Z = \{x \in S : (\exists y \in Z) x R y\}$. Note that $b \wedge c$ in (A4) is $\{b\}_{\leq'} \cap \{c\}_{\leq'}$.

Some equivalent versions of the above axioms are

(A1') $b \lhd' V \Longrightarrow F^{-1} b \lhd F^{-1} V$,

(A2') $a \lhd F^{-1} b \Longrightarrow a F b$,

(A4') $F^{-1} U \cap F^{-1} V \lhd F^{-1}(U \wedge V)$.

We have for any continuous F that

$$U \lhd' V \Longrightarrow F^{-1} U \lhd F^{-1} V. \tag{2}$$

Hence $F^{-1} U \sim F^{-1} V$ whenever $U \sim' V$. Also by (A1) $F^{-1}(U_{\lhd'}) \sim F^{-1} U$. By (A2) it follows that each $F^{-1} Z$ is down-closed.

Lemma 1.3 *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are formal topologies, where $\mathcal{T}$ is generated by $\{(a_i, U_i)\}_{i \in I}$. If $F \subseteq S \times T$ is a relation which satisfies (A2-A4) and*

(a) $F^{-1}a_i \lhd_{\mathcal{S}} F^{-1}U_i$, *for all* $i \in I$,

(b) $F^{-1}x \lhd_{\mathcal{S}} F^{-1}y$, *whenever* $x \leq y$ *and* $x, y \in S$,

then $F : \mathcal{S} \to \mathcal{T}$ *is a continuous mapping.*

Proof. To prove (A1') it suffices by minimality of $\lhd_{\mathcal{S}}$ to prove that the relation K defined below is a cover relation, which satisfies the generating axioms. Let

$$a\,K\,U \Longleftrightarrow_{\mathrm{def}} F^{-1}a \lhd_{\mathcal{S}} F^{-1}U.$$

Clearly the generating axioms are satisfied, according to (a). Assumption (b) yields axiom (Ext). The axiom (Ref) and (Tra) are straightforwardly checked. To verify (Loc) for K suppose $F^{-1}a \lhd_{\mathcal{S}} F^{-1}U$ and $F^{-1}a \lhd_{\mathcal{S}} F^{-1}V$. Let $x \in F^{-1}a$. Thus by (Loc) for $\lhd_{\mathcal{S}}$,

$$x \lhd_{\mathcal{S}} F^{-1}U \wedge F^{-1}V.$$

Let y be a member of the right hand side. Thus $y \leq y_1$ and $y \leq y_2$ where $y_1\,F\,u$ and $y_2\,F\,v$ with $u \in U$ and $v \in V$. By (A2) it follows that $y\,F\,u$ and $y\,F\,v$. By (A4) then

$$y \lhd_{\mathcal{S}} F^{-1}(u \wedge v).$$

But $F^{-1}(u \wedge v) \subseteq F^{-1}(U \wedge V)$, so transitivity gives the desired

$$y \lhd F^{-1}(U \wedge V).$$

□

Each continuous mapping $F : \mathcal{S} \to \mathcal{T}$ induces a point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) F(a,b)\} : \mathrm{Pt}(\mathcal{S}) \to \mathrm{Pt}(\mathcal{T}).$$

It satisfies: $a\,F\,b \Rightarrow f[a^*] \subseteq b^*$.

The composition of two continuous morphisms $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ is given as follows

$$a\,(G \circ F)\,c \Longleftrightarrow a \lhd_{\mathcal{X}} F^{-1}[G^{-1}(c)].$$

The one-point formal topology is the terminal object in the category of formal topologies. It is constructed as $\mathbf{1} = (\{*\}, \leq_{\mathbf{1}}, \lhd_{\mathbf{1}})$, where $* \leq_{\mathbf{1}} *$ and $a \lhd_{\mathbf{1}} U$ iff U is inhabited. The terminal map $!_{\mathcal{Y}}$ from $\mathcal{Y}$ to $\mathbf{1}$ is defined by letting the relation $y\ !_{\mathcal{Y}}\ a$ be true for all y and a.

Now any point $\alpha \in \mathrm{Pt}(\mathcal{X})$ in a formal topology, defines a morphism $F_\alpha : \mathbf{1} \to \mathcal{X}$ given by

$$a\,F_\alpha\,x \Longleftrightarrow x \in \alpha.$$

A map $\hat{\alpha} : \mathcal{Z} \to \mathcal{X}$ which is constant α is defined by the composition $F_\alpha \circ !_{\mathcal{Z}}$. More explicitly, the map is given by the relation

$$z \hat{\alpha} x \Longleftrightarrow z \lhd_{\mathcal{Z}} \{u \in Z : x \in \alpha\}.$$

In particular, if z covered by the empty set, then $z \hat{\alpha} x$ holds for any x.

Let $\mathcal{X}$ be a formal topology. Two neighbourhoods $p, q \in X$ are *formally disjoint,* $p \perp q$, if $p \wedge q \lhd \emptyset$. Write $p^\perp = \{q \in X : p \perp q\}$ for the set of neighbourhoods formally disjoint from p. A neighbourhood p is *well-covered* by q if $X \lhd p^\perp \cup \{q\}$. We write $p \lll q$ in this case. The topology $\mathcal{X}$ is *regular* if for any $q \in X$ we have

$$q \lhd \{p \in X : p \lll q\}.$$

Theorem 1.4 *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ a formal topologies and that $\mathcal{Y}$ is regular. Let $F, G : \mathcal{X} \to \mathcal{Y}$ be continuous maps. If $F \subseteq G$ (as graphs), then $F = G$.*

Proof. See [7]. □

2 Localic completion of metric spaces

We review here a representation of complete metric spaces due to S. Vickers [13, 14]. However, we use formal topologies instead of locales.

For any metric space (X, d) define its *localic completion* $\mathcal{M} = \mathcal{M}_X$ as follows. This is a formal topology $\mathcal{M} = (M, \leq_{\mathcal{M}}, \lhd_{\mathcal{M}})$ where M is the set of *formal ball symbols* $\{\mathsf{b}(x, \delta) : x \in X, \delta \in \mathbb{Q}_+\}$. Here $\mathbb{Q}_+$ is the set of positive rational numbers. These symbols are ordered by inclusion and strict inclusion respectively

$$\mathsf{b}(x, \delta) \leq_{\mathcal{M}} \mathsf{b}(y, \varepsilon) \Longleftrightarrow d(x, y) + \delta \leq \varepsilon$$
$$\mathsf{b}(x, \delta) <_{\mathcal{M}} \mathsf{b}(y, \varepsilon) \Longleftrightarrow d(x, y) + \delta < \varepsilon.$$

The *radius* of a formal ball $\mathsf{b}(x, \delta)$ is by definition $\mathsf{r}(\mathsf{b}(x, \delta)) = \delta$, whereas its *centre* is $\mathsf{c}(\mathsf{b}(x, \delta)) = x$. The cover relation $\lhd_{\mathcal{M}}$ is generated by the axioms

(M1) $p \lhd \{s \in M : s < p\}$,

(M2) $M \lhd \{\mathsf{b}(x, \delta) : x \in X\}$ for any $\delta \in \mathbb{Q}_+$,

By using localisation we obtain that for $p \leq q$ and $p \leq r$

$$(\text{M1}') \qquad p \lhd \{s \in M : s < q \text{ and } s < r\}.$$

As the special case $p = q = r$ is exactly (M1), we have indeed an equivalent formulation to that of Vickers [13] for metric spaces.

A useful observation is that in order to prove $a \lhd U$, it is by (M1) sufficient to prove $b \lhd U$ for each $b < a$.

Using the density property of the rational numbers it is plain that

Proposition 2.1 *Let $\mathcal{M} = \mathcal{M}_X$. For any $p,q,r \in M$ with $p < q$ and $p < r$, there is $s \in M$ such that $p < s < q$ and $p < s < r$.*

Example 2.2 Consider the rational numbers $\mathbb{Q}$ as a metric space with metric $d(x,y) = |x-y|$. Then $\mathcal{M}_{\mathbb{Q}}$ is identical to the formal reals $\mathcal{R}$ [8] if we identify a formal ball $\mathsf{b}(x,\delta)$ with the formal interval $(x-\delta, x+\delta)$. The orders $\leq_{\mathcal{M}}$ and $\leq_{\mathcal{R}}$ then coincide. (M1) and (G1) are then the same axioms.

To derive (G2) $(a,b) \lhd \{(a,c),(b,d)\}$ for $a < b < c < d$ from (M2): Suppose $a < d < c < b$, and let $\delta = (c-d)/2$. Thus by (M2) and localisation,

$$(a,b) \lhd \{\mathsf{b}(x,\delta) : x \in \mathbb{Q}\} \wedge \{(a,b)\}.$$

The right hand side consists of intervals $(e,f) \leq (a,b)$ which are no longer than $2\delta = c-d$. Hence $(e,f) \leq (a,c)$ or $(e,f) \leq (d,b)$. Thus $(a,b) \lhd \{(a,c),(d,b)\}$, by transitivity.

To derive (M2) from (G2) is easy by subdividing an interval repeatedly, using (G2) of the form $(a,b) \lhd \{(a,(a+2b)/3),((2a+b)/3,b)\}$

Example 2.3 Consider the closed unit interval $X = [0,1]$ of real numbers with the usual metric. Note that in $\mathcal{M}_X$ we have $\mathsf{b}(0,3) \lhd \mathsf{b}(0,2)$, though $\mathsf{b}(0,3) \leq \mathsf{b}(0,2)$ is false. The former holds since by (M2) $\mathsf{b}(0,3) \lhd \{\mathsf{b}(x,1) : x \in X\}$ and $\mathsf{b}(x,1) \leq \mathsf{b}(0,2)$ for all x.

Let (X,d) and (Y,e) be metric spaces. A function $f : X \to Y$ is *metric preserving* if $e(f(x),f(u)) = d(x,u)$ for all $x,u \in X$. It is a *metric isomorphism* if it is has an inverse which is also metric preserving. We say that (X',d') is a *metric completion* of (X,d) if there is metric preserving $i : X \to X'$, such that if (Y,e) is any complete metric space, and if $f : X \to Y$ is metric preserving, then there is a unique metric preserving $f' : X' \to Y$ so that $f' \circ i = f$. We note immediately that if (X'',d'') is another metric completion of (X,d), then there is a metric isomorphism $h : X' \to X''$.

We recall one standard completion method of a metric space (X,d). Define $(\widetilde{X},\widetilde{d})$ as follows. The elements of $\widetilde{X}$ are sequences $(x_n)_n$ in X such that for all $m,n \geq 0$,

$$d(x_m,x_n) \leq 2^{-m} + 2^{-n},$$

so called *regular sequences*. Two regular sequences $(x_n)_n$ and $(y_n)_n$ are identified if, and only if, $\lim_n d(x_n,y_n) = 0$. Define the metric as

$$\widetilde{d}((x_n),(y_n)) = \lim_n d(x_n,y_n).$$

Define $i_X : X \to \widetilde{X}$ by $i(x) = (n \mapsto x)$, the constant sequence.

Let $X = (X,d)$ be an arbitrary metric space. Let $\mathcal{M} = \mathcal{M}_X$ be its localic completion. For a point $\alpha \in \mathrm{Pt}(\mathcal{M})$

$$\mathsf{b}(x,\delta), \mathsf{b}(y,\varepsilon) \in \alpha \Longrightarrow d(x,y) < \delta + \varepsilon \tag{3}$$

This follows since by the filter property (Fil) of α, the left hand side implies there is some formal ball $\mathsf{b}(z,\xi) \leq \mathsf{b}(x,\delta), \mathsf{b}(y,\varepsilon)$ in α. Hence

$$d(x,y) \leq d(x,z) + d(z,y) < d(z,x) + \xi + d(z,y) + \xi \leq \delta + \varepsilon.$$

An infinite sequence $(x_n)_n$ of elements in X is a *fundamental sequence* for a point α of $\mathcal{M}$ if $\mathsf{b}(x_n, 2^{-n}) \in \alpha$, for every $n \geq 0$. Clearly, by (3) $d(x_m, x_n) < 2^{-m} + 2^{-n}$ so $(x_n)_n \in \widetilde{X}$.

Suppose that $(x_n)_n$ and $(u_n)_n$ are two fundamental sequences for α. Then for $n \geq 0$, $\mathsf{b}(x_n, 2^{-n}), \mathsf{b}(u_n, 2^{-n}) \in \alpha$. Hence using (3)

$$d(x_n, u_n) < 2^{-n} + 2^{-n} = 2^{-n+1}.$$

Evidently, $\lim_n d(x_n, u_n) = 0$.

Given $\alpha \in \mathrm{Pt}(\mathcal{M})$ we can construct a fundamental sequence for α as follows. Take first $p \in \alpha$. Now by (M2) we have $p \lhd \{\mathsf{b}(x,\delta) : x \in X\}$ for every $\delta \in \mathbb{Q}_+$. Thus for $\delta = 2^{-n}$, there is $x_n \in X$ with $\mathsf{b}(x_n, 2^{-n}) \in \alpha$, by the splitting axiom for the point. We get by countable many choices a fundamental sequence $(x_n)_n$ for α.

Thus define $\varphi : \mathrm{Pt}(\mathcal{M}) \to \widetilde{X}$ by letting $\varphi(\alpha) = (x_n)_n$ where $(x_n)_n$ is some fundamental sequence for α. This is, by the above, a well-defined function. It is also injective: Suppose $\varphi(\alpha) = \varphi(\beta)$. Let $(x_n)_n$ and $(y_n)_n$ be fundamental sequences for α and β, respectively. Thus $\lim_n d(x_n, y_n) = 0$. Let $\mathsf{b}(x,\delta) \in \alpha$. (M1) gives some $\mathsf{b}(u,\varepsilon) \in \alpha$ with $d(u,x) + \varepsilon < \delta$. Take δ' with $d(u,x) + \varepsilon < \delta' < \delta$. We have by the triangle inequality, (3) and $\mathsf{b}(x_n, 2^{-n}) \in \alpha$ that

$$d(x, y_n) \leq d(x,u) + d(u, x_n) + d(x_n, y_n) < d(x,u) + (\varepsilon + 2^{-n}) + d(x_n, y_n).$$

Thus by choosing n so large that $d(x_n, y_n) \leq (\delta - \delta')/3$ and $2^{-n} \leq (\delta - \delta')/3$, we get

$$d(x, y_n) + 2^{-n} < \delta' + (\delta - \delta')/3 + (\delta - \delta')/3 + (\delta - \delta')/3 = \delta.$$

Thus $\mathsf{b}(y_n, 2^{-n}) < \mathsf{b}(x,\delta)$. Since $\mathsf{b}(y_n, 2^{-n}) \in \beta$, we have $\mathsf{b}(x,\delta) \in \beta$. This shows $\alpha \subseteq \beta$. The reverse inclusion is proved in the same way. Hence $\alpha = \beta$.

Next we prove that φ is surjective. Let $(x_n)_n \in \widetilde{X}$. Define a new sequence by $y_n = x_{n+2}$, which is again regular and equivalent to (x_n). We have

$$d(y_{n+1}, y_n) + 2^{-(n+1)} \leq 2^{-(n+3)} + 2^{-(n+2)} + 2^{-(n+1)} < 2^{-n}.$$

Thus

$$\mathsf{b}(y_{n+1}, 2^{-(n+1)}) < \mathsf{b}(y_n, 2^{-n}) \tag{4}$$

for all $n \geq 0$. Now define

$$\alpha = \{p \in M : (\exists n)\, p \geq \mathsf{b}(y_n, 2^{-n})\},$$

which is easily verified to be a point using (4) and (1). Obviously, $\mathsf{b}(y_n, 2^{-n}) \in \alpha$, so $(y_n)_n$ is a fundamental sequence for α. Thereby $\varphi(\alpha) = (y_n)_n = (x_n)_n$ as desired.

Thus φ is bijective. Define a metric $m : \mathrm{Pt}(\mathcal{M}) \times \mathrm{Pt}(\mathcal{M}) \to \mathbb{R}$ as follows. For $\alpha, \beta \in \mathrm{Pt}(\mathcal{M})$ let

$$m(\alpha, \beta) = \lim_n d(x_n, y_n),$$

where $(x_n)_n$ and $(y_n)_n$ are some fundamental sequences for α and β respectively. Note that by the construction of the bijection φ, we actually have

$$m(\alpha, \beta) = \widetilde{d}(\varphi(\alpha), \varphi(\beta)).$$

We have proved

Theorem 2.4 $(\mathrm{Pt}(\mathcal{M}_X), m)$ *is metrically isomorphic to the completion* $(\widetilde{X}, \widetilde{d})$ *of* (X, d).

Define $j = j_X : X \to \mathrm{Pt}(\mathcal{M})$ by

$$j(x) = \{\mathsf{b}(u, \delta) \in M : d(x, u) < \delta\}.$$

Then $\varphi(i(x)) = j(x)$, so we have

Corollary 2.5 $(\mathrm{Pt}(\mathcal{M}_X), m)$ *is a metric completion of* (X, d) *via* j_X.

Corollary 2.6 *If* (X, d) *is a complete metric space, then* $j_X : (X, d) \to (\mathrm{Pt}(\mathcal{M}_X), m)$ *is a metric isomorphism.*

Proof. Note that since $id_X : (X, d) \to (X, d)$ is trivially a metric completion, j_X must be a metric isomorphism. □

Since $j(y)$ is a point and $b(x, \delta)$ is a neighbourhood of this point, we have

$$\neg b(x, \delta) \lhd \emptyset \tag{5}$$

The points in $\mathsf{b}(x, \delta)^*$ are those $j(y)$ such that $d(x, y) < \delta$. We have that

$$j^{-1}(\mathsf{b}(x, \delta)^*) = B(x, \delta) = \{y \in X : d(x, y) < \delta\}.$$

Write

$$\mathsf{b}(x, \delta)_* = B(x, \delta).$$

For a set of formal balls U, let $U_* = \cup\{c_* : c \in U\}$. While formal covers give rise to point-wise covers as in Lemma 1.2, the converse is rarely true. To prove that a formal cover relation holds, we can try to use the axioms and closure conditions, but we can also use one of the stronger relations considered in the next section.

Theorem 2.7 *Let $X = (X,d)$ be a metric space, and let X' be a dense subset of X. Then $X' = (X',d)$ is a metric space with a homeomorphic localic completion*

$$\mathcal{M}_{X'} \cong \mathcal{M}_X.$$

Proof. Write $\mathcal{M}_X = (M_X, \lhd, \leq, <)$ and $\mathcal{M}_{X'} = (M_{X'}, \lhd', \leq', <')$. We have $M_{X'} \subseteq M_X$ and $\leq'$ and $<'$ are the restrictions of $\leq$ and $<$ to this subset. For any $a \in M_X$ define $O(a) = \{p \in M_{X'} : p < a\}$. By (M1) and density it follows that

$$a \lhd O(a). \tag{6}$$

We sketch the proof of the homeomorphism. By induction one can show

$$p \lhd' U \Longrightarrow p \lhd U. \tag{7}$$

Again by induction it follows

$$p \lhd U \Longrightarrow O(p) \lhd' \bigcup_{q \in U} O(q). \tag{8}$$

Then since $p \lhd' O(p)$, we have for any $p, q \in M_{X'}$,

$$p \lhd q \Longrightarrow p \lhd' q. \tag{9}$$

Define $F : \mathcal{M}_{X'} \to \mathcal{M}_X$ and $G : \mathcal{M}_X \to \mathcal{M}_{X'}$ by

$$\begin{aligned} p\,F\,q &\iff_{\text{def}} p \lhd q, \\ p\,G\,q &\iff_{\text{def}} O(p) \lhd' q. \end{aligned}$$

It is now straightforward to check that these are mutual inverses and indeed continuous morphisms. □

In particular, $\mathcal{R} = \mathcal{M}_{\mathbb{Q}}$ is homeomorphic to $\mathcal{M}_{\mathbb{R}}$.

3 Cover relations

In this section we fix a metric space $X = (X,d)$ and its localic completion $\mathcal{M}_X = (M, \leq, \lhd)$. We study some useful notions of covers for this formal topology. The first is a *refinement cover.* Define for $U, V \subseteq M$

$$U \leq V \iff (\forall p \in U)(\exists q \in V)\, p \leq q.$$

This is a reflexive and transitive relation. Write $p \leq V$ and $U \leq q$ for $\{p\} \leq V$ and $U \leq \{q\}$ respectively. Similar extensions can be made for the relation $<$ by replacing $\leq$ by $<$. We then have

Proposition 3.1 $U < V \Longrightarrow U \leq V$

Furthermore, since $U \leq V$ is equivalent to $U \subseteq (V_{\leq})$ we have immediately

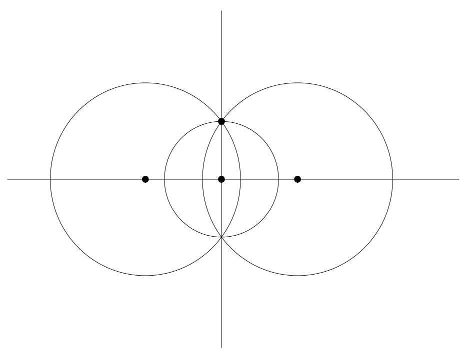
Proposition 3.2 $U \leq V \Longrightarrow U \lhd V$.

The next cover relations is the *ball-wise cover* (in contradistinction to point-wise cover). Let $R(\varepsilon) = \{p \in M : \mathsf{r}(p) \leq \varepsilon\}$, i.e. the set balls of radius at most ε. Define

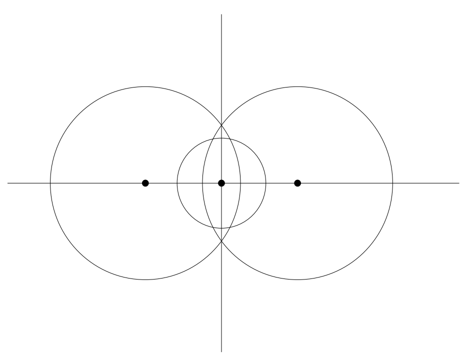
$$p \sqsubseteq_{\varepsilon} U \Longleftrightarrow (\forall q \in R(\varepsilon))(q \leq p \Rightarrow q \leq U) \tag{10}$$

Obviously, this relation becomes easier to satisfy when ε decreases.

Example 3.3 In the following figure the two formal balls B and C of radius 5 centred at $(-4,0)$ respectively $(4,0)$, covers the formal ball A of radius 3 at the origin, in the sense that $A \lhd \{B,C\}$. However, they do not cover A in the sense of $\sqsubseteq_{\varepsilon}$ for any ε. For any positive rational number ε, consider the formal ball of radius ε centred at $(0,3-\varepsilon)$, which is inside A, but neither inside B nor inside C.



If the radius of the formal ball A is decreased slightly to A', then $A' \sqsubseteq_{\varepsilon} \{B,C\}$ for some sufficiently small ε.



We have

Proposition 3.4 *If $p \sqsubseteq_\varepsilon U$, then $p \lhd U$.*

Proof. Suppose $p \sqsubseteq_\varepsilon U$. By (M2) and transitivity we have $p \lhd R(\varepsilon)$. By applying localisation with $p \lhd \{p\}$, we obtain

$$p \lhd R(\varepsilon) \wedge \{p\} = R(\varepsilon) \cap \{p\}_\leq.$$

For any $q \in R(\varepsilon) \cap \{p\}_\leq$, we have by $p \sqsubseteq_\varepsilon U$, that $q \leq U$. Hence $q \lhd U$ by Proposition 3.2. □

Extend $\sqsubseteq_\varepsilon$ to a relation between subsets as follows: $U \sqsubseteq_\varepsilon V$ if and only if $p \sqsubseteq_\varepsilon V$, for all $p \in U$.

Lemma 3.5 *(a) $\sqsubseteq_\varepsilon$ is a reflexive and transitive relation on subsets of M.*

(b) If $U \sqsubseteq_\delta V$ and $V \sqsubseteq_\varepsilon W$, then $U \sqsubseteq_{\min(\delta,\varepsilon)} W$.

(c) $U \leq V$ if, and only if, for all $\varepsilon \in \mathbb{Q}_+$: $U \sqsubseteq_\varepsilon V$.

Proof. (a) is immediate.

(b) is immediate from (a) using the observation after definition (10).

(c, $\Leftarrow$): For $p \in U$, consider $\varepsilon = \mathsf{r}(p)$.

(c, $\Rightarrow$): Suppose $U \leq V$. Let $p = \mathsf{b}(x,\delta) \in U$ and $\varepsilon \in \mathbb{Q}_+$. We have some $q = \mathsf{b}(y,\rho) \in V$ with $p \leq q$. Assume $s \leq p$ and $\mathsf{r}(s) \leq \varepsilon$. Thus by transitivity of $\leq$ we get $s \leq q$. □

Define the relation $a \sqsubseteq U$ to hold if, and only if, there is some $\varepsilon \in \mathbb{Q}_+$ such that $a \sqsubseteq_\varepsilon U$. Clearly, this relation implies $a \lhd U$.

The following shows that the well-cover relation extends the strict order relation on balls.

Lemma 3.6 *Let $p, q \in M$. If $p < q$, then $p \lll q$.*

Proof. Let $p = \mathsf{b}(x,\delta)$ and $q = \mathsf{b}(y,\varepsilon)$, and assume $p < q$. We have

$$d(x,y) + \delta < \varepsilon.$$

Then take $\theta \in \mathbb{Q}_+$ such that

$$d(x,y) + \delta + \theta < \varepsilon - \theta. \tag{11}$$

We have by (M2),

$$M \lhd \{\mathsf{b}(u,\theta) : u \in X\}.$$

Let $u \in X$ and consider the number $d(y,u)$. By (11) and co-transitivity, we get

$$d(x,y) + \delta + \theta < d(y,u) \quad \text{or} \quad d(y,u) < \varepsilon - \theta.$$

In the first case

$$d(x,y) + \delta + \theta < d(y,u) \leq d(y,x) + d(x,u),$$

so $\delta + \theta < d(x,u)$. This means that

$$\mathsf{b}(u,\theta) \in \mathsf{b}(x,\delta)^{\perp}.$$

In the second case

$$d(y,u) < \varepsilon - \theta,$$

which is equivalent to $\mathsf{b}(u,\theta) < \mathsf{b}(y,\varepsilon)$. This means that

$$\{\mathsf{b}(u,\theta) : u \in X\} \subseteq \mathsf{b}(x,\delta)^{\perp} \cup (\{\mathsf{b}(y,\varepsilon)\}_{\leq}) \lhd \mathsf{b}(x,\delta)^{\perp} \cup \{\mathsf{b}(y,\varepsilon)\}.$$

Hence by transitivity $M \lhd p^{\perp} \cup \{q\}$, that is $p \lll q$. □

Theorem 3.7 *$\mathcal{M}_X$ is regular for any metric space X.*

Proof. By (M2) we have $p \lhd \{q \in M : q < p\}$. By Lemma 3.6 we get $p \lhd \{q \in M : q \lll p\}$. □

4 Compactness

We first review some notions of compactness for formal topologies and for metric spaces. Then we show how these interact in the localic completion $\mathcal{M}_X$ of a metric space X. The main result, Theorem 4.16, is the characterisation of the cover relation of $\mathcal{M}_X$ for a locally compact X.

A subset $V \subseteq X$ is *subfinite* (or *finitely enumerable*) if $V = \{a_1, \ldots, a_n\}$ for some $a_1, \ldots, a_n \in X$, where $n \geq 0$. Let $\mathcal{P}_{\mathrm{subf}}(X)$ denote the set of subfinite subsets of X.

A formal topology $\mathcal{X}$ is *compact,* if whenever $X \lhd U$, then there exists $V \in \mathcal{P}_{\mathrm{subf}}(U)$ so that $X \lhd V$.

For a formal topology $\mathcal{X}$, with set-presentation C, I define the *way below* relation

$$a \ll b \Longleftrightarrow_{\mathrm{def}} (\forall i \in I(b))(\exists W \in \mathcal{P}_{\mathrm{subf}}(C(b,i)))\, a \lhd W.$$

The condition on the right hand side is in fact equivalent to the more useful: for any $U \subseteq X$

$$b \lhd U \Longrightarrow (\exists W \in \mathcal{P}_{\mathrm{subf}}(U))\, a \lhd W.$$

A formal topology $\mathcal{X}$ is *locally compact,* if $p \lhd \{q \in X : q \ll p\}$ for any $p \in X$.

Theorem 4.1 *[6, p.311] Using AC and PEM: locally compact locales have enough points.*

Corollary 4.2 *Using AC and PEM: For locally compact topologies:*

$$a \lhd U \Longleftrightarrow a^* \subseteq \cup U^*.$$

A metric space (Y, d) is *compact* if it is complete and totally bounded. Any compact metric space is bounded, so for every $\varepsilon > 0$ there is some formal ball $\mathsf{b}(x_0, \delta_0)$ with

$$R(\varepsilon) \leq \mathsf{b}(x_0, \delta_0).$$

Theorem 4.3 *Let X be a complete metric space. If $\mathcal{M}_X$ is compact, then X is compact.*

Proof. We need to show that X is totally bounded. Let δ be a positive rational number. By (M2) and compactness it follows that there are $x_1, \ldots, x_n \in X$ such that

$$M \lhd_{\mathcal{M}} \{\mathsf{b}(x_1, \delta), \ldots, \mathsf{b}(x_n, \delta)\}.$$

From this and Lemma 1.2 we get

$$M_* \subseteq \{\mathsf{b}(x_1, \delta)_*, \ldots, \mathsf{b}(x_n, \delta)_*\},$$

Thus for any point $x \in X$, there is some i with $x \in \mathsf{b}(x_i,\delta)_* = B(x_i,\delta)$, i.e. $d(x,x_i) < \delta$. $\square$

Bishop introduces a metric notion of local compactness which is stronger than the standard topological notion. A non-void metric space X is *locally compact,* if each bounded subset can be included in some compact subset of X. It may be shown that any such space is complete and separable [2]. Note that the open interval $(0,1)$ with the usual metric is not locally compact in Bishop's sense since it is not complete. We shall henceforth use Bishop's terminology keeping in mind that completeness is included in the definition. Clearly, any non-void compact metric space is locally compact.

A function $f : X \to Y$ between a locally compact metric space X and a metric space Y is *continuous,* if f is uniformly continuous on each compact subset of X. An equivalent requirement is that f is uniformly continuous on each open ball $B(x,\delta)$. Let $\omega^f_{\mathsf{b}(x,\delta)}$ denote the modulus of uniform continuity for f on the ball $B(x,\delta)$:

$$(\forall \varepsilon \in \mathbb{Q}_+)(\forall u,v \in B(x,\delta))[d(u,v) \leq \omega^f_{\mathsf{b}(x,\delta)}(\varepsilon) \Longrightarrow d(f(u),f(v)) \leq \varepsilon]$$

As the space X is separable and non-void, there is a countable dense subset $D \subseteq X$. Let $\mathsf{E}_X = \{\mathsf{b}(x,\delta) : x \in D, \delta \in \mathbb{Q}_+\}$. The following lemma is easily proved using density of D and $\mathbb{Q}_+$.

Lemma 4.4 *Let X be a locally compact metric space.*

(a) For any $a < b$ and $a < b'$ in M_X there is $c \in \mathsf{E}_X$ with $a < c$, $c < b$ and $c < b'$.

(b) For any $c \in M_X$ there are $a,b \in \mathsf{E}_X$ with $a < c < b$.

The locally compact spaces and continuous functions form a category which we denote by **LComp**.

Lemma 4.5 *Let $f : X \to Y$ be a continuous function from a locally compact metric space X to a metric space Y. If $A \subseteq X$ is bounded, then the image $f[A]$ is bounded.*

Proof. Let A be a bounded subset of X. Let K be a compact subset including A. Thus it suffices to show that $f[K]$ is bounded. Now f is uniformly continuous on K. For any $y \in Y$, the map $z \mapsto d(y,z)$ is uniformly continuous $Y \to \mathbb{R}$. Thus $d(y,\cdot) \circ f : K \to \mathbb{R}$ is uniformly continuous, and hence has a supremum for each y. Take $y = f(x_0)$ for some $x_0 \in K$. It follows that $f[K]$ is bounded. $\square$

Here is a basic lemma for locally compact metric spaces expressed in terms of formal balls.

Lemma 4.6 *Let X be a locally compact metric space. For any formal balls $p < q$ and any rational number $\delta > 0$, there is a subfinite set C of formal balls with*

$$p \sqsubseteq C < q$$

and whose elements have radius less than δ. In fact, C may be taken to be a subset of E_X, or the radii of all the balls C may be chosen to be identical.

Proof. Let $p = \mathsf{b}(y,\varepsilon)$, $q = \mathsf{b}(z,\rho)$ and $\delta \in \mathbb{Q}_+$. Suppose $p < q$. Thus $d(y,z) + \varepsilon < \rho$. Take $\gamma \in \mathbb{Q}_+$ so small that

$$\begin{aligned} 2\gamma &< \delta \\ 2\gamma &< \varepsilon \\ d(y,z) + \varepsilon + 4\gamma &< \rho. \end{aligned}$$

Thus

$$p = \mathsf{b}(y,\varepsilon) < \mathsf{b}(y,\varepsilon+4\gamma) < \mathsf{b}(z,\rho) = q.$$

By the local compactness of X there is a compact subset K of X with $B(z,\rho) \subseteq K$. Take a γ-net $\{x_1,\ldots,x_n\} \subseteq K$. Let $I = \{1,\ldots,n\}$. Then for every $i \in I$, we have by cotransitivity of real numbers,

$$\varepsilon + \gamma < d(y,x_i) \text{ or } d(y,x_i) < \varepsilon + 2\gamma.$$

Thus there is a function $f : I \longrightarrow \{0,1\}$ so that for any $i \in I$: either $\varepsilon+\gamma < d(y,x_i)$ and $f(i) = 0$, or $d(y,x_i) < \varepsilon + 2\gamma$ and $f(i) = 1$. Let $J = \{i \in I : f(i) = 1\}$ and $C = \{\mathsf{b}(x_i,2\gamma) : i \in J\}$. Since J is finite, C is subfinite. Also, $2\gamma < \delta$, so the radius condition for C is satisfied. Note that the radii of the balls are all identical. To establish the lemma it is now sufficient to prove

$$\mathsf{b}(y,\varepsilon) \sqsubseteq C < \mathsf{b}(y,\varepsilon+4\gamma). \tag{12}$$

We show first that $\mathsf{b}(y,\varepsilon) \sqsubseteq_\gamma C$. Suppose $\mathsf{b}(u,\theta) \le \mathsf{b}(y,\varepsilon)$ and $\theta \le \gamma$. Using the γ-net, we find $i \in I$ so that $d(u,x_i) < \gamma$. Since $d(u,y) + \theta \le \varepsilon$, it follows by the triangle inequality that $d(y,x_i) + \theta < \gamma + \varepsilon$. Thus $f(i) = 0$ is impossible, and hence $i \in J$. Further $d(u,x_i) + \gamma < 2\gamma$, so $\mathsf{b}(u,\gamma) < \mathsf{b}(x_i,2\gamma)$. Indeed, we have shown $\mathsf{b}(y,\varepsilon) \sqsubseteq_\gamma C$. For $i \in J$ we get, by adding 2γ, that $d(x_i,y) + 2\gamma < \varepsilon + 4\gamma$, i.e. $\mathsf{b}(x_i,2\gamma) < \mathsf{b}(y,\varepsilon+4\gamma)$. This proves (12).

To prove the second statement of the theorem, we apply the first part of the theorem to obtain $p \sqsubseteq D < q$, where

$$D = \{\mathsf{b}(w_1,\delta_1),\ldots,\mathsf{b}(w_n,\delta_n)\}$$

is a set of balls, with $\delta_1, \ldots, \delta_n < \delta/2$. By Lemma 4.4.(a) find, for each $i = 1, \ldots, n$, some $c_i \in \mathsf{E}_X$ with $\mathsf{b}(w_i, \delta_i) < c_i < \mathsf{b}(w_i, \delta/2)$ and $c_i < q$. Then $C = \{c_1, \ldots, c_n\}$ satisfies the requirements. □

Note that since $a \sqsubseteq U$ implies $a_* \subseteq U_*$, Lemma 4.6 and the below proposition provides an alternative characterisation of locally compact metric spaces.

Proposition 4.7 *Let X be a nonvoid complete metric space such that*

$$a < b,\ \delta \in \mathbb{Q}_+ \Longrightarrow (\exists U \in \mathcal{P}_{\mathrm{subf}}(X))\, a_* \subseteq U_* \,\&\, U < b \,\&\, \mathsf{r}(U) < \delta. \tag{13}$$

Here $\mathsf{r}(U) = \max\{r(p) : p \in U\}$. Then X is locally compact.

Proof. Let $a = \mathsf{b}(x_0, \varepsilon)$ be given, and put $b_n = \mathsf{b}(x_0, \varepsilon + 2^{-n})$. Using (13) we find a sequence $U_n \in \mathcal{P}_{\mathrm{subf}}(X)$, $n = 0, 1, 2, \ldots$, such that $(b_{n+1})_* \subseteq (U_n)_*$, $U_n < b_n$ and $r(U_n) < 2^{-n}$. It follows that $(U_m)_* \subseteq (U_n)_*$ whenever $m \geq n$. Now define C_n to be the set of centres of the balls in U_n, and let $C = \cup_{n \geq 0} C_n$. We claim that the closure $\overline{C}$ of C in X is a compact subset of X containing a_*. First we show $a_* \subseteq \overline{C}$. Let $x \in a_*$. For each n, we have by the above construction

$$a_* \subseteq (b_{n+1})_* \subseteq (U_n)_*.$$

Thus we find $x_n \in C_n$ with $d(x_n, x) < 2^{-n}$. This shows $x \in \overline{C}$. Next, we show that $\overline{C}$ is compact. It is closed by definition. Thus it suffices to show that $D_n = C_0 \cup C_1 \cup \cdots \cup C_{n+1}$ is a 2^{-n}-net. Let $x \in \overline{C}$ be arbitrary. There is therefore some $y \in C_m$ with $d(x, y) < 2^{-(n+1)}$, for some m. If $m \leq n+1$, then $y \in D_n$. Suppose instead that $m > n+1$. By the above we have $(U_m)_* \subseteq (U_{n+1})_*$. Thus $y \in (U_{n+1})_*$, so there is some $z \in C_{n+1}$ with $d(y, z) < 2^{-(n+1)}$. Hence also $d(x, z) \leq d(x, y) + d(y, z) < 2^{-n}$ as required. □

The following shows that Lemma 1.2 can "almost" be reversed for singleton U.

Lemma 4.8 *Let X be a locally compact metric space. Then for formal balls a, b, c of $\mathcal{M}_X$: if $a_* \subseteq b_*$ and $b < c$, then $a \lhd c$.*

Proof. We have by (M1) that $a \lhd \{d : d < a\}$. Hence it is sufficient to prove that

$$d < a, \quad a_* \subseteq b_*, \quad b < c \Longrightarrow d \lhd c.$$

Assume the conditions on left of the implication. Write $b = \mathsf{b}(y, \beta)$ and $c = \mathsf{b}(z, \gamma)$. Then, since $b < c$, we can find a positive rational number θ with

$$d(y, z) + \beta + \theta < \gamma. \tag{14}$$

By Lemma 4.6, and since $d < c$, there is a set U of formal balls, all of radius $< \theta$, such that

$$d \sqsubseteq U < a. \tag{15}$$

Since $d \sqsubseteq U$ implies $d \lhd U$, it suffices to show $U \lhd c$ to establish the lemma. Consider any ball $e = \mathsf{b}(v, \varepsilon) \in U$. By (15) we have $e < a$, so $v \in a_*$. Hence $v \in b_*$, which means that $d(v, y) < \beta$. By (14) we thus have

$$\begin{array}{rcl} d(v,z) + \varepsilon & \leq & d(v,y) + d(y,z) + \varepsilon \\ & < & \beta + (\gamma - \beta - \theta) + \theta = \gamma \end{array}$$

But this implies that $e < c$. □

For $p < q$ in M let

$$A(p,q) = \{V \in \mathcal{P}_{\mathrm{subf}}(M) : p \sqsubseteq V < q\}.$$

Observe that $\{p\} \in A(p,q)$, and also that $p \lhd V$ for $V \in A(p,q)$. Then define a new cover relation

$$a \lessdot U \Longleftrightarrow (\forall b < c < a)\,(\exists U_0 \in A(b,c))\, U_0 < U.$$

Lemma 4.9 *Let X be a complete metric space. If $a \lessdot U$, then $a \lhd U$.*

Proof. Suppose $a \lessdot U$. Let $b < a$. Then by Proposition 2.1 there is c with $b < c < a$. Hence $U_0 < U$ for some $U_0 \in A(b,c)$. Propositions 3.2 and 3.4 then implies $b \lhd U$. Since $b < a$ was arbitrary it follows by (M1) than $a \lhd U$. □

Lemma 4.10 *Let X be a complete metric space. Then the relation $\lessdot$ satisfies axiom (M1).*

Proof. To show $p \lessdot \{q \in M : q < p\}$, assume $b < c < p$. Then $\{b\} \in A(b,c)$, and clearly

$$\{b\} < \{q \in M : q < p\},$$

which establishes the validity of the axiom. □

Lemma 4.11 *Let X be a locally compact metric space. Then the relation $\lessdot$ satisfies axiom (M2).*

Proof. We have to show $r \lessdot \{\mathsf{b}(x,\delta) : x \in X\}$ for arbitrary $r \in M$ and $\delta \in \mathbb{Q}_+$. Suppose $p < q < r$. By Lemma 4.6, take a subfinite set of formal balls C whose diameter is less than δ and satisfying $p \sqsubseteq C < q$. Thus $C \in A(p,q)$ and furthermore $C < \{\mathsf{b}(x,\delta) : x \in X\}$. This proves the claim. □

Lemma 4.12 *Let X be a complete metric space. Then the relation $\lessdot$ satisfies axioms (Ref) and (Ext) .*

Proof. Since $a \lessdot U$ and $U \subseteq V$ obviously implies $a \lessdot V$, it suffices by the reflexivity of $\leq$ to check (Ext). Suppose $p \leq q$. In order to show $p \lessdot \{q\}$ assume $b < c < p$. We have then $\{b\} \in A(b,c)$ and $b < q$. Hence $\{b\} < \{q\}$ as required. □

Lemma 4.13 *Let X be a complete metric space. The relation $\lessdot$ satisfies the transitivity axiom (Tra).*

Proof. Suppose $a \lessdot U$ and $U \lessdot V$. Let $b < c < a$. By the first assumption there is $U_0 \in A(b,c)$ such that $U_0 < U$. Write $U_0 = \{e_1,\ldots,e_n\}$. Then pick, for each $i = 1,\ldots,n$, a $p_i \in U$ with $e_i < p_i$. Since $e_i < c$ we may, by Proposition 2.1, further pick $q_i < c$ so that

$$e_i < q_i < p_i \qquad (i = 1,\ldots,n).$$

Thus by the second assumption $p_i \lessdot V$, so there is $V_0^i < V$ with $V_0^i \in A(e_i,q_i)$ for each $i = 1,\ldots,n$. Let $V_0 = V_0^1 \cup \cdots \cup V_0^n$, which is subfinite. Clearly, $V_0 < c$. We show $b \sqsubseteq V_0$. There are ε_i $(i = 1,\ldots,n)$ with $e_i \sqsubseteq_{\varepsilon_i} V_0^i$. Also there is some γ with $b \sqsubseteq_\gamma U_0$. Put $\varepsilon = \min(\gamma,\varepsilon_1,\ldots,\varepsilon_n)$. Then for $s \in R(\varepsilon)$ and $s \leq b$, there is some i with $s \leq e_i$. This in turn yields $s \leq h$ for some $h \in V_0^i \subseteq V_0$. Hence $b \sqsubseteq_\gamma V_0$. Thus we have $V_0 \in A(b,c)$ as required. □

Lemma 4.14 *Let X be a locally compact metric space. Suppose $q \sqsubseteq U$ and $q \sqsubseteq V$ where U and V are subfinite. Then for any $p < q$ there is a subfinite W with $p \sqsubseteq W$ and $W \leq U$ and $W \leq V$.*

Proof. Let $\gamma \in \mathbb{Q}_+$ be sufficiently small that both $q \sqsubseteq_\gamma U$ and $q \sqsubseteq_\gamma V$. Apply Lemma 4.6 to get some subfinite W with $p \sqsubseteq W < q$ and where the elements of W have radius less than γ. For any $b \in W$ we have $\mathsf{r}(b) < \gamma$ and $b < q$, and hence $b \leq U$ and $b \leq V$. Thus $W \leq U$ and $W \leq V$. □

Lemma 4.15 *Let X be a locally compact metric space. Then the relation $\lessdot$ satisfies the localisation axiom (Loc).*

Proof. Suppose $a \lessdot U$ and $a \lessdot V$. Let $b < c < a$. Pick d with $b < d < c$. By assumption we have some $U_0 < U$ and $V_0 < V$ such that $U_0, V_0 \in A(d,c)$. Then by Lemma 4.14 we obtain $W_0 \in A(b,c)$ with $W_0 \leq U_0$ and $W_0 \leq V_0$. From this follows

$$W_0 < U \wedge V,$$

since for $x \in W_0$ there are $y \in U$ and $z \in V$ with $x < y$ and $x < z$ and hence by Proposition 2.1 there is some u with $x < u < y$ and $x < u < z$. □

The main result is the following characterisation of the cover relation on $\mathcal{M}_X$ for a locally compact metric X.

Theorem 4.16 *Let X be a locally compact metric space. Then:*

$$a \lessdot U \quad \Longleftrightarrow \quad a \lhd U.$$

Proof. (⇒) is Lemma 4.9.

(⇐) Lemmas 4.10, 4.11, 4.12, 4.13 and 4.15 together states that $\lessdot$ is a cover relation satisfying (M1) and (M2). The direction (⇐) now follows immediately since $\lhd$ is the least cover relation satisfying (M1) and (M2), and □

Theorem 4.17 *Let X be a compact metric space. Then*

(i) $\mathcal{M}_X$ *is compact*

(ii) (Lebesgue Lemma) If U is a covering of $\mathcal{M}_X$*, then there is a rational number* $\rho > 0$ *and subfinite* $U_1 \subseteq U$ *such that* $R(\rho) \leq U_1$.

Proof. (i): Suppose $M \lhd U$. By (M2) we have $M \lhd \{\mathsf{b}(x,1) : x \in X\}$. Since X is compact, it is also bounded and we find some sufficiently large $\mathsf{b}(x_0,\delta_0)$ that $\mathsf{b}(x,1) < \mathsf{b}(x_0,\delta_0)$, for any $x \in X$. Hence

$$\{\mathsf{b}(x,1) : x \in X\} \lhd \mathsf{b}(x_0,\delta_0),$$

We have thereby $M \lhd \mathsf{b}(x_0,\delta_0)$. Further the assumption $M \lhd U$ says in particular $\mathsf{b}(x_0,\delta_0+2) \lhd U$. By Theorem 4.16 thus

$$\mathsf{b}(x_0,\delta_0+2) \lessdot U.$$

Since $\mathsf{b}(x_0,\delta_0) < \mathsf{b}(x_0,\delta_0+1) < \mathsf{b}(x_0,\delta_0+2)$ we have

$$\mathsf{b}(x_0,\delta_0) \sqsubseteq U_0 \tag{16}$$

for some subfinite $U_0 < U$. Thus there is a subfinite $U_1 \subseteq U$ with $U_0 < U_1$. We get by Lemmas 3.4 and 3.2 that

$$\mathsf{b}(x_0,\delta_0) \lhd U_1,$$

and this gives $M \lhd U_1$ as required. This proves (i).

Next we prove (ii) by looking closer at the information obtained above. By (16) there is some $\delta \in \mathbb{Q}_+$ so that

$$\mathsf{b}(x_0,\delta_0) \sqsubseteq_\delta U_0$$

Let $\rho = \min(\delta, 1)$. Thus $R(\rho) \leq R(1) < b(x_0,\delta_0)$ and $R(\rho) \leq U_0$. Hence $R(\rho) \leq U_1$. □

Theorem 4.18 *$\mathcal{M}_X$ is locally compact, if X is a locally compact metric space.*

Proof. Suppose X is a locally compact metric space. By (M1) we have $p \lhd \{q \in M : q < p\}$. To prove local compactness it thus suffices to show

$$q < p \Longrightarrow q \ll p$$

Suppose $q < p$ and that $p \lhd U$. Thus by Theorem 4.16, $p \lessdot U$. Take now a with $q < a < p$, and obtain $U_0 \in A(q,a)$ with $U_0 < U$. Since U_0 is subfinite, there is a subfinite $W \subseteq U$, with $q \lhd W$. $\square$

From Theorem 4.17 and 4.18 follows

Corollary 4.19 (Vickers[14]) *Let X be a complete metric space. Then $\mathcal{M}_X$ is compact if, and only if, X is totally bounded.*

5 Functorial embedding

Theorem 5.1 *Let X be a locally compact metric space, and let Y be a complete metric space. If $F : \mathcal{M}_X \to \mathcal{M}_Y$ is a continuous morphism, then $f = j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X : X \to Y$ is continuous.*

Proof. Let $F : \mathcal{M}_X \to \mathcal{M}_Y$ is a continuous morphism. Let $z \in X$ and $\gamma \in \mathbb{Q}_+$. We show that f is uniformly continuous on $B(z,\gamma)$.

Let $\varepsilon \in \mathbb{Q}_+$. We have by (M2) and (2)

$$M_X \lhd F^{-1}[M_Y] \lhd F^{-1}\{\mathsf{b}(y,\varepsilon/2) : y \in Y\}. \tag{17}$$

Consider $\mathsf{b}(z,\gamma) < \mathsf{b}(z,\gamma') < \mathsf{b}(z,\gamma'') < \mathsf{b}(z,\gamma''')$ in M_X. By Theorem 4.16 and the above cover (17) we get

$$\mathsf{b}(z,\gamma''') \lessdot F^{-1}\{\mathsf{b}(y,\varepsilon/2) : y \in Y\}.$$

Thus we find $C \in A(\mathsf{b}(z,\gamma'),\mathsf{b}(z,\gamma''))$ with

$$C < F^{-1}\{\mathsf{b}(y,\varepsilon/2) : y \in Y\}. \tag{18}$$

Therefore there is some $\delta \in \mathbb{Q}_+$ with

$$\mathsf{b}(z,\gamma') \sqsubseteq_\delta C.$$

We may assume that $\delta < \gamma' - \gamma$. Suppose now that $x, x' \in B(z,\gamma)$ and $d(x,x') < \delta$. We shall prove that

$$d(f(x), f(x')) < \varepsilon.$$

Now $\delta + \gamma < \gamma'$ and $d(x,z) < \gamma$, yields $\mathsf{b}(x,\delta) < \mathsf{b}(z,\gamma')$, so indeed $\mathsf{b}(x,\delta) \in C$. Hence by (18)

$$\mathsf{b}(x,\delta)\, F\, \mathsf{b}(y,\varepsilon/2),$$

for some $y \in Y$. Thus $\mathrm{Pt}(F)(\mathsf{b}(x,\delta)^*) \subseteq \mathsf{b}(y,\varepsilon/2)^*$, and hence $f[B(x,\delta)] \subseteq B(y,\varepsilon/2)$. Then $d(f(x), f(x')) < \varepsilon$ is clear. □

We now proceed to define the functor $\mathcal{M}$ on morphisms. For a function $f : X \to Y$ between complete metric spaces, define a relation D_f between the basic neighbourhoods of $\mathcal{M}_X$ and $\mathcal{M}_Y$ as follows

$$a D_f b \Longleftrightarrow (\exists c < b) f[a_*] \subseteq c_*.$$

Thus for $a = \mathsf{b}(x,\delta)$, the righthand side is equivalent to

$$(\exists \mathsf{b}(y,\varepsilon) < b)\, f[B(x,\delta)] \subseteq B(y,\varepsilon).$$

Then define another relation

$$a A_f b \Longleftrightarrow_{\text{def}} a \lhd D_f^{-1} b.$$

Classically, $a A_f b$ in fact equivalent to $f[a_*] \subset b_*$, see Proposition 5.3. Note that by transitivity of covers we have

$$a \lhd A_f^{-1} U \Longleftrightarrow a \lhd D_f^{-1} U.$$

Theorem 5.2 *Let X and Y be complete metric spaces and suppose that X is locally compact. Let $f : X \to Y$ be a continuous function. Then $A_f : \mathcal{M}_X \to \mathcal{M}_Y$ is a continuous morphism.*

Proof. We check conditions (A2) – (A4) for A_f first.

(A2): this follows immediately by transitivity of covers.

(A3): We are to prove $M_X \lhd A_f^{-1} M_Y$. For any $\mathsf{b}(x, \delta) \in M_X$ there is by Lemma 4.5 some $y \in Y$ and $\varepsilon \in \mathbb{Q}_+$ such that

$$f[B(x, \delta)] \subseteq B(y, \varepsilon).$$

But $\mathsf{b}(y, \varepsilon) < b(y, 2\varepsilon) \in M_Y$, so clearly $\mathsf{b}(x, \delta) \lhd A_f^{-1} M_Y$.

(A4): Suppose $a A_f b_1$ and $a A_f b_2$. We are to show

$$a \lhd A_f^{-1}[b_1 \wedge b_2].$$

From the assumption and localisation we obtain

$$a \lhd (D_f^{-1} b_1) \wedge (D_f^{-1} b_2) = D_f^{-1} b_1 \cap D_f^{-1} b_2.$$

Since $D_f^{-1} x \subseteq A_f^{-1} x$ it now suffices to show

$$D_f^{-1} b_1 \cap D_f^{-1} b_2 \lhd D_f^{-1}[b_1 \wedge b_2].$$

Let d be an element of the lefthand set. There are then $c_1 < b_1$ and $c_2 < b_2$ with $f[d_*] \subseteq (c_1)_*$ and $f[d_*] \subseteq (c_2)_*$. Now to prove $d \lhd D_f^{-1}[b_1 \wedge b_2]$ it is enough by Theorem 4.16 and since X is locally compact, to establish the following: For any $d'' < d' < d$ there is $C \in A(d'', d')$ with $C < D_f^{-1}[b_1 \wedge b_2]$.

Assume $d'' < d' < d$. Write $c_i = \mathsf{b}(z_i, \gamma_i)$ and $b_i = \mathsf{b}(y_i, \beta_i)$ for $i = 1, 2$. Take $\varepsilon \in \mathbb{Q}_+$ sufficiently small that

$$\mathsf{b}(z_i, \gamma_i + 2\varepsilon) < \mathsf{b}(y_i, \beta_i).$$

Pick $\delta \in \mathbb{Q}_+$ with $\delta < \omega(\varepsilon)$, where ω is a continuity modulus of f on the open ball given by d. By Lemma 4.6 take a subfinite set C of formal balls so that $d'' \sqsubseteq C < d'$ and each ball in C has radius $< \delta$. Suppose

$$C = \{\mathsf{b}(x_1, \alpha_1), \ldots, \mathsf{b}(x_n, \alpha_n)\}.$$

We get since $C < d' < d$ and $f[d_*] \subseteq (c_i)_*$, that $f[B(x_j, \alpha_j)] \subseteq B(z_i, \gamma_i)$, for $i = 1, 2$ and $j = 1, \ldots, n$. Moreover, we have $\alpha_j < \delta < \omega(\varepsilon)$ so by uniform continuity

$$f[B(x_j, \alpha_j)] \subseteq B(f(x_j), 3\varepsilon/2). \tag{19}$$

Since $f(x_j) \in B(z_i, \gamma_i)$, we have $\mathsf{b}(f(x_j), 2\varepsilon) < \mathsf{b}(z_i, \gamma_i + 2\varepsilon)$. Combining this with $\mathsf{b}(z_i, \gamma_i + 2\varepsilon) < b_i$ we get

$$\mathsf{b}(f(x_j), 2\varepsilon) < b_i \tag{20}$$

for $j = 1, \ldots, n$ and $i = 1, 2$. Now $\mathsf{b}(f(x_j), 3\varepsilon/2) < b(f(x_j), 2\varepsilon)$, so by (19) and (20) we have

$$C \subseteq D_f^{-1}[b_1 \wedge b_2].$$

Which is more than required.

To conclude the proof we need only to check conditions (a) and (b) for Lemma 1.3.

(a), axiom M1: We need to check that

$$A_f^{-1} a \lhd A_f^{-1}\{s : s < a\}.$$

It suffices to verify that $D_f^{-1} a \lhd D_f^{-1}\{s : s < a\}$, since $D_f^{-1} y \subseteq A_f^{-1} y$. Suppose $u D_f a$. Thus $f[u_*] \subseteq c_*$ for some $c < a$. We find by Proposition 2.1 some s with $c < s < a$. Hence $u D_f s$, i.e. $u \in D_f^{-1}\{s : s < a\}$.

(a), axiom M2: Let $\varepsilon \in \mathbb{Q}_+$. We have to show

$$A_f^{-1} M_Y \lhd A_f^{-1}\{b(y, \varepsilon) : y \in Y\}.$$

Since $A_f^{-1} M_Y \subseteq M_X$, it is enough to prove that for any $p \in M_X$ we have $p \lhd A_f^{-1}\{b(y, \varepsilon) : y \in Y\}$. By Theorem 4.16, and since X is locally compact, it is thus sufficient to prove

$$p \lessdot A_f^{-1}\{b(y, \varepsilon) : y \in Y\}.$$

Let $a < b < p$ and suppose $a = \mathsf{b}(x, \delta)$, $b = \mathsf{b}(u, \rho)$ and $p = \mathsf{b}(v, \gamma)$. Let ω be a modulus of uniform continuity on $B(v, \gamma)$. Take $\delta \in \mathbb{Q}_+$ with $\delta < \omega(\varepsilon/3)$. Then, by Lemma 4.6, pick $C \in A(a, b)$ where each ball in C has radius $< \delta$. Write

$$C = \{\mathsf{b}(x_1, \alpha_1), \ldots, \mathsf{b}(x_n, \alpha_n)\}.$$

Next take $\theta \in \mathbb{Q}_+$ so small that $\alpha_i + \theta < \delta$ and $\mathsf{b}(x_i, \alpha_i + \theta) < b(u, \rho)$ for each $i = 1, \ldots, n$. The uniform continuity then gives

$$f[B(x_i, \alpha_i + \theta)] \subseteq B(f(x_i), 2\varepsilon/3).$$

Also $\mathsf{b}(f(x_i), 2\varepsilon/3) < \mathsf{b}(f(x_i), \varepsilon)$, so

$$\mathsf{b}(x_i, \alpha + \theta)\, D_f\, \mathsf{b}(f(x_i), \varepsilon).$$

Since $\mathsf{b}(x_i, \alpha) < \mathsf{b}(x_i, \alpha_i + \theta)$, we have then, as required,

$$C < A_f^{-1}\{b(y, \varepsilon) : y \in Y\}.$$

Condition (b): Suppose $a \leq b$. We have to show $A_f^{-1} a \lhd A_f^{-1} b$. Again it suffices to show $D_f^{-1} a \lhd D_f^{-1} b$. Suppose $u \in D_f^{-1} a$. Thus $f[u_*] \subseteq c_*$ for some $c < a$. But $a \leq b$ implies $c < b$, so $u \in D_f^{-1} b$. □

Proposition 5.3 *With the same assumptions as in Theorem 5.2:*

(i) If $a A_f b$, then $f[a_*] \subseteq b_*$.

(ii) Assuming AC and PEM, the converse of (i) holds.

Proof. (i): From $a A_f b$ follows by Lemma 1.2 that

$$a_* \subseteq (D_f^{-1} b)_*. \tag{21}$$

Now

$$(D_f^{-1} b)_* = f^{-1}[\cup\{c_* : c < b\}] = f^{-1}[b_*]. \tag{22}$$

Here the last equality follows from (M1) and Lemma 1.2. Thus $a_* \subseteq f^{-1}[b_*]$, i.e. $f[A_*] \subseteq b_*$.

(ii) Assume now AC and PEM. Then by Corollary 4.2 we can go back from (21) to $a A_f b$. □

Remark 5.4 A correction: In [8] the definition of $(a, b) A_f (c, d)$ for a continuous $f : \mathbb{R} \to \mathbb{R}$ was taken to be $f[(a, b)] \subseteq (c, d)$, which was incorrect. In view of the above it should be

$$(a, b) A_f (c, d) \iff$$
$$(a, b) \lhd_{\mathcal{R}} \{(a', b') : (\exists c', d')\, f[(a', b')] \subseteq (c', d') \,\&\, c < c' < d' < d\}. \square$$

For a continuous function $f : X \to Y$, where X is a locally compact metric space, an alternative, but equivalent, definition of A_f can be given. Define $H_f \subseteq M_X \times M_Y$ by letting the relation $a H_f b$ hold exactly when

$$(\exists \varepsilon \in \mathbb{Q}_+)(\exists c \in \mathsf{E}_X)\; a < c \,\&\, \mathsf{r}(a) < \omega_c^f(\varepsilon) \,\&\, \mathsf{b}(f(\mathsf{c}(a)), \varepsilon) < b. \tag{23}$$

Notice that since E_X is countable this relation is semidecidable.

Proposition 5.5 *If $f : X \to Y$ is a continuous function between metric spaces, and where X is locally compact, then*

$$a A_f b \Longleftrightarrow a \lhd H_f^{-1} b.$$

Proof. ($\Leftarrow$) It is enough to show $H_f \subseteq D_f$. Suppose $\mathsf{b}(x,\delta) H_f b$ with $\varepsilon \in \mathbb{Q}_+$ and $c \in \mathsf{E}_X$ as in (23) above. Take $\beta \in \mathbb{Q}_+$ so that $\mathsf{b}(f(x), \varepsilon + \beta) < b$. In order to establish $\mathsf{b}(x,\delta) D_f b$, it thus suffices to show $f[B(x,\delta)] \subseteq B(f(x), \varepsilon + \beta)$. Let $u \in B(x,\delta)$, i.e. $d(x,u) < \delta$. Since $\mathsf{b}(x,\delta) < c$, the function ω_c^f is a continuity modulus on $B(x,\delta)$, and since $\delta < \omega_c^f(\varepsilon)$ we get $d(f(x), f(u)) \leq \varepsilon$. Hence $f(u) \in B(f(x), \varepsilon + \beta)$.

($\Rightarrow$) By axiom (M1) it suffices to show the implication

$$a' < a,\; a D_f b \Longrightarrow a' \lhd H_f^{-1} b.$$

Suppose that $a' < a = \mathsf{b}(x,\alpha)$, $f[B(x,\alpha)] \subseteq B(y,\beta)$ and $\mathsf{b}(y,\beta) < b = \mathsf{b}(z,\gamma)$. By the last inequality, we can find $\theta \in \mathbb{Q}_+$ with

$$d(y,z) + \beta + \theta < \gamma. \tag{24}$$

By Lemma 4.4 we find $a'' \in \mathsf{E}_X$ with $a' < a'' < a$. Then by Lemma 4.6 take a subfinite $U \subseteq \mathsf{E}_X$ with $a' \sqsubseteq U < a''$ and such that the radii of the balls in U are $< \omega_{a''}^f(\theta)$. Now $a' \sqsubseteq U$ implies $a' \lhd U$. Therefore it suffices to show that $d H_f b$ for any $d = \mathsf{b}(u,\varphi) \in U$. For such a d we have $\varphi < \omega_{a''}^f(\theta)$ and $d < a'' < \mathsf{b}(x,\alpha)$. Thus $u \in B(x,\alpha)$ and hence $f(u) \in B(y,\beta)$. Then using (24)

$$\begin{aligned} d(f(u),z) + \theta &\leq d(f(u),y) + d(y,z) + \theta \\ &= \beta + d(y,z) + \theta \\ &< \gamma \end{aligned}$$

This means that $\mathsf{b}(f(u),\theta) < b$. Consequently $d H_f b$. $\square$

Next is a useful continuity result.

Lemma 5.6 *Let $f : X \to Y$ be a continuous function from a locally compact metric space X to a complete metric space Y. If $a < b$, $b\, D_f\, c$ and $c \sqsubseteq C$, then there exists $S \in A(a,b)$ so that for each $d \in S$ there is some $e \in C$ with $d\, D_f\, e$.*

Proof. Suppose that $a < b$, $f[b_*] \subseteq c'_*$, $c' < c$ and $c \sqsubseteq_\gamma C$ where $\gamma \in \mathbb{Q}_+$. Write $c' = \mathsf{b}(x,\alpha)$ and $c = \mathsf{b}(y,\beta)$. Since $c' < c$ we can find $\beta' \in \mathbb{Q}_+$ with

$$d(x,y) + \alpha < \beta' < \beta. \tag{25}$$

Let $\rho = \min(\beta - \beta', \gamma)$. Since f is uniformly continuous on b_* we find $\delta \in \mathbb{Q}_+$ so that

$$(\forall v, w \in b_*)[d(v,w) < \delta \Longrightarrow d(f(v), f(w)) < \rho]. \tag{26}$$

Moreover, since $a < b$ we find by Lemma 4.6 some $S \in A(a,b)$ such that the formal balls in S all have radius less than δ. Consider an arbitrary $d = \mathsf{b}(z,\theta) \in S$. Then since $d < b$ we have $z \in b_*$. Hence $f(z) \in c'_*$ and so $d(x, f(z)) < \alpha$. Thus using (25) we get

$$\begin{aligned} d(f(z),y) + \rho &\leq d(f(z),x) + d(x,y) + \rho \\ &< \alpha + d(x,y) + \rho \\ &< \beta' + \rho \leq \beta' + (\beta - \beta') = \beta. \end{aligned}$$

Hence $\mathsf{b}(f(z),\rho) < c$, and since $\rho \leq \gamma$, we have by $c \sqsubseteq_\gamma C$, that $\mathsf{b}(f(z),\rho) < e$ for some $e \in C$. It remains to show that e satisfies $d\, D_f\, e$. But by (26) we get $f[d_*] \subseteq \mathsf{b}(f(z),\rho)_*$ as required. □

We establish a bijection between continuous functions $X \to Y$ and continuous morphisms $\mathcal{M}_X \to \mathcal{M}_Y$, when X is locally compact.

Theorem 5.7 *Let X and Y be complete metric spaces, and suppose X is locally compact.*

(i) If $f : X \to Y$ is uniformly continuous, then

$$j_Y^{-1} \circ \mathrm{Pt}(A_f) \circ j_X = f.$$

(ii) If $F : \mathcal{M}_X \to \mathcal{M}_Y$ is a continuous mapping, then

$$A_{j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X} = F.$$

Proof. Part (i):

$$\begin{aligned} \mathrm{Pt}(A_f)(j_X(x)) &= \{q : (\exists p \in j_X(x))\, p \lhd D_f^{-1} q\} \\ &= \{q : (\exists p \in j_X(x))\, p\, D_f\, q\} \\ &= \{q : (\exists \mathsf{b}(u,\delta))(\exists \mathsf{b}(v,\varepsilon) < q) d(x,u) < \delta \;\&\; f[B(u,\delta)] \subseteq B(v,\varepsilon)\} \\ &= \{b(w,\rho) : d(f(x),w) < \rho\} = j_Y(f(x)). \end{aligned}$$

The second equality follows since points split covers. The next to last equality is straightforwardly verified using the point-wise continuity of f.

Part (ii): Since $\mathcal{M}_Y$ is a regular topology it is, by Theorem 1.4, enough to prove the inclusion

$$F \subseteq A_{j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X}.$$

Suppose $\mathsf{b}(x,\delta)\, F\, \mathsf{b}(y,\varepsilon)$. Then by axiom (M1) we get

$$\mathsf{b}(x,\delta) \lhd F^{-1}\{p : p < \mathsf{b}(y,\varepsilon)\}. \tag{27}$$

Assume that $\mathsf{b}(u,\rho)\, F\, \mathsf{b}(v,\gamma)$ where $\mathsf{b}(v,\gamma) < \mathsf{b}(y,\varepsilon)$, so that $\mathsf{b}(u,\rho)$ is in the right hand set of (27). Now this implies

$$\mathrm{Pt}(F)[b(u,\delta)^*] \subseteq \mathsf{b}(v,\gamma)^*.$$

But trivially $\mathrm{Pt}(F) = j_Y \circ (j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X) \circ j_X^{-1}$, so

$$\mathsf{b}(u,\rho)\, D_{j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X}\, \mathsf{b}(y,\varepsilon).$$

Thus

$$F^{-1}\{p : p < \mathsf{b}(y,\varepsilon)\} \subseteq D^{-1}_{j_Y^{-1} \circ \mathrm{Pt}(F) \circ j_X}\, \mathsf{b}(y,\varepsilon).$$

By transitivity of $\lhd$, this gives the desired inclusion. □

For locally compact metric spaces X and Y, and continuous $f : X \to Y$ define $\mathcal{M}(f) = A_f : \mathcal{M}_X \to \mathcal{M}_Y$.

Theorem 5.8 *$\mathcal{M}$ defines a full and faithful functor from the category of locally compact metric spaces to the category of formal topologies.*

Proof. Faithfulness and fullness follows from part (i) and (ii) of Theorem 5.7, respectively.

We prove the functoriality of $\mathcal{M}$. Consider the identity function $1_X : X \to X$. Then $\mathcal{M}(1_X) = A_{1_X}$ is by definition the relation given by

$$a A_{1_X} b \Longleftrightarrow a \lhd D_{1_X}^{-1} b \Longleftrightarrow a \lhd \{d : (\exists c < b)(d_* \subseteq c_*)\}$$

The right hand relation implies by Lemma 4.8, $a \lhd \{d : d \lhd b\}$, which in turn is equivalent to $a \lhd b$, i.e. $a\ 1_{\mathcal{M}(X)}\ b$. Conversely, from $a \lhd b$, follows by (M1) that $a \lhd \{c : c < b\}$, which implies $a \lhd \{d : (\exists c < b)(d_* \subseteq c_*)\}$. Thus $A_{1_X} = 1_{\mathcal{M}(X)}$.

Let $f : X \to Y$ and $g : X \to Z$ be continuous functions between locally compact metric spaces. Then we claim that for any c

$$D_{g \circ f}^{-1} c \lhd D_f^{-1} A_g^{-1} c \tag{28}$$

and

$$D_f^{-1}A_g^{-1}c \lhd D_{g\circ f}^{-1}c. \tag{29}$$

Since $a \lhd D_{g\circ f}^{-1}c$ if and only if $aA_{g\circ f}\,c$, and $a \lhd D_f^{-1}A_g^{-1}c$ if and only if $a \lhd A_f^{-1}A_g^{-1}c$ we obtain by the claims (28,29) that

$$aA_{g\circ f}\,c \Longleftrightarrow a\,(A_g \circ A_f)c.$$

Hence $\mathcal{M}$ defines a functor.

Proof of claim (28): By (M1) it suffices to show

$$a' < a,\ aD_{g\circ f}\,c \Longrightarrow a' \lhd D_f^{-1}A_g^{-1}c \tag{30}$$

Suppose $a' < a$, $(g\circ f)[a_*] \subseteq c'_*$, $c' < c$. Since f is locally compact, the images of bounded sets are bounded, so $f[a_*] \subseteq b_*$ for some b. Write $c = \mathsf{b}(z,\gamma)$, $c' = \mathsf{b}(z',\gamma')$. Pick $\gamma'' \in \mathbb{Q}_+$ with $d(z,z') + \gamma' < \gamma'' < \gamma$, and let $\varepsilon = \gamma - \gamma''$. The function g is uniformly continuous on b_* so there is $\delta \in \mathbb{Q}_+$ so that

$$(\forall x, y \in b_*)[d(x,y) < \delta \Longrightarrow d(g(x), g(y)) < \varepsilon.] \tag{31}$$

The function f is uniformly continuous on a_* so there is some $\rho \in \mathbb{Q}_+$ with

$$(\forall u, v \in a_*)[d(u,v) < \rho \Longrightarrow d(f(u), f(v)) < \delta/2.] \tag{32}$$

By Lemma 4.6 pick $U \in A(a',a)$ where $\mathsf{r}(s) < \rho$ for each $s \in U$. We have $a' \lhd U$ so it suffices to show $U \subseteq D_f^{-1}A_g^{-1}c$ to establish (30). Take $s = \mathsf{b}(x,\alpha) \in U$. Then $s < a$, so $x \in a_*$ and $f(x) \in f[a_*]$. By (32) we get

$$f[s_*] \subseteq \mathsf{b}(f(x), \delta/2)_*. \tag{33}$$

By (31) we obtain

$$g[\mathsf{b}(f(x),\delta)_*] \subseteq \mathsf{b}(g(f(x)),\varepsilon)_*. \tag{34}$$

Clearly, $g(f(x)) \in g[f[a_*]] \subseteq c'_*$ so $d(g(f(x)), z') < \gamma'$. We then find that

$$d(g(f(x)),z) + \varepsilon \leq d(g(f(x)),z') + d(z',z) + \varepsilon < \gamma' + d(z,z') + \varepsilon < \gamma'' + \varepsilon = \gamma.$$

Thus $\mathsf{b}(g(f(x)),\varepsilon) < \mathsf{b}(z,\gamma) = c$, which together with (34) gives $\mathsf{b}(f(x),\delta)\,D_g\,c$, and so $\mathsf{b}(f(x),\delta)\,A_g\,c$. According to (33) we get $s\,D_f\,\mathsf{b}(f(x),\delta)$. Thus $s \in D_f^{-1}A_g^{-1}x$ as required.

Proof of claim (29): Let $a \in D_f^{-1}[A_g^{-1}c]$. By (M1) it suffices to show $a' \lhd A_{g\circ f}^{-1}c$ for any $a' < a$. Suppose thus that $a\, D_f\, b$, $b \in A_g^{-1}c$ and $a' < a$. Using density we find $b' < b'' < b''' < b$ with $f[a_*] \subseteq b'_*$. Hence $a\, D_f\, b''$. Now $b \in A_g^{-1}c$ means $b \lhd D_g^{-1}c$, so by Theorem 4.16 and the definition of $\lessdot$ we find $U \in A(b'', b''')$ with $U < D_g^{-1}c$. Thus $b'' \sqsubseteq U < b'''$, so by Lemma 5.6 there is $V \in A(a, a')$ such that $(\forall d \in V)(\exists e \in U)\, d\, D_f\, e$. Furthermore, for each $e \in U$ there is $e' \in D_g^{-1}c$ with $e < e'$. For any $d \in V$ there are therefore e and e' with $d\, D_f\, e$, $e < e'$ and $e'\, D_g\, c$. Thus $f[d_*] \subseteq e'_*$ and $g[e'_*] \subseteq c'_*$ for some $c' < c$. Then $g[f[d_*]] \subseteq g[e'_*] \subseteq c'_*$, so $d\, D_{g\circ f}\, c$. We conclude that $V \subseteq A_{g\circ f}^{-1}c$, and since $V \in A(a', a)$ we have $a' \lhd V$. Therefore $a' \lhd A_{g\circ f}^{-1}c$. □

Note that the theorem implies in particular that $f = g$ if, and only if, $\mathcal{M}(f) = \mathcal{M}(g)$, and $F = G$ if and only if $\mathrm{Pt}(F) = \mathrm{Pt}(G)$ for $F, G : \mathcal{M}(X) \to \mathcal{M}(Y)$. This means of course that equalities of such continuous morphisms can be checked by pointwise calculation.

6 Products

Suppose that (X_1, d_1) and (X_2, d_2) are two locally compact metric spaces. We form the product $(X_1 \times X_2, d)$ by letting d be the metric

$$d((x,y),(x',y')) = \max(d_1(x,x'), d_2(y,y')).$$

It is straightforward to check that $(X_1 \times X_2, d)$ is a locally compact metric space. The projection $\pi_k : X_1 \times X_2 \to X_k$ is uniformly continuous. For Z a locally compact metric space, it is easy to verify that if $f_k : Z \to X_k$, $k = 1, 2$, are continuous, then $h : Z \to X_1 \times X_2$ given by $h(z) = (f_1(z), f_2(z))$, is also continuous. It follows that $(X_1 \times X_2, d)$ with the projections is a categorical product of X_1 and X_2 in **LComp**.

For a locally compact metric space (X, d), the metric d is itself a uniformly continuous function between locally compact metric spaces $X \times X \to \mathbb{R}$.

We recall the construction of the binary product of formal topologies $\mathcal{X}_1 = (X_1, \leq_1, \lhd_1)$ and $\mathcal{X}_2 = (X_2, \leq_2, \lhd_2)$. Their product is $\mathcal{X} = (X_1 \times X_2, \leq, \lhd)$ where $(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq_1 y_1$ and $x_2 \leq y_2$, and where $\lhd$ is the smallest cover relation such that

(PC1) $x_1 \lhd_1 U$ implies $(x_1, x_2) \lhd U \times \{x_2\}$

(PC2) $x_2 \lhd_2 V$ implies $(x_1, x_2) \lhd \{x_1\} \times V$.

The projections $P_1 : \mathcal{X} \to \mathcal{X}_1$ and $P_2 : \mathcal{X} \to \mathcal{X}_2$ are given by

$$\begin{aligned} (x_1,x_2)\, P_1\, u &\Leftrightarrow (x_1,x_2) \lhd \{u\} \times X_2 \\ (x_1,x_2)\, P_2\, v &\Leftrightarrow (x_1,x_2) \lhd X_1 \times \{v\}. \end{aligned}$$

If $F : \mathcal{Z} \to \mathcal{X}_1$ and $G : \mathcal{Z} \to \mathcal{X}_2$ are continuous, then $\langle F, G\rangle : \mathcal{Z} \to \mathcal{X}$ given by

$$z \langle F, G\rangle (x_1, x_2) \Longleftrightarrow_{\text{def}} z F x_1 \text{ and } z G x_2$$

is the unique continuous map $\mathcal{Z} \to \mathcal{X}$ such that $P_1 \circ \langle F, G\rangle = F$ and $P_2 \circ \langle F, G\rangle = G$.

The following is Lemma 3.9 from [9]:

Lemma 6.1 *Let X_1 and X_2 be complete metric spaces. Suppose $(p, q) \lhd W$ in $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$. Then:*

(i) if $y \in q_*$, then $p \lhd \{a \in M_{X_1} : (\exists b \in M_{X_2})(y \in b_* \;\&\; (a, b) \in W)\}$,

(ii) if $x \in p_*$, then $q \lhd \{b \in M_{X_2} : (\exists a \in M_{X_1})(x \in a_* \;\&\; (a, b) \in W)\}$.

Corollary 6.2 *Let X_1 and X_2 be complete metric spaces. In $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$ we have*

$$(p, q) \lhd (a, b) \Longleftrightarrow p \lhd a \;\&\; q \lhd b.$$

Proof. The direction ($\Leftarrow$) follows by (PC1) and (PC2). To prove the direction ($\Rightarrow$) let $W = \{(a, b)\}$, $y = \mathsf{c}(q)$ in Lemma 6.1. Then trivially, $y \in q_*$, so $(p, q) \lhd (a, b)$ implies

$$p \lhd \{a' \in M_{X_1} : (\exists b' \in M_{X_2})(y \in b'_* \;\&\; (a', b') \in \{(a, b)\})\} \subseteq \{a\}.$$

Thus $p \lhd a$. Similarly $q \lhd b$. □

Lemma 6.3 *The order relations in $\mathcal{M}_{X_1 \times X_2}$, $\mathcal{M}_{X_1}$ and $\mathcal{M}_{X_2}$ are related as follows*

(i) $\mathsf{b}((x_1, x_2), \varepsilon) \le \mathsf{b}((y_1, y_2), \delta)$ iff $\mathsf{b}(x_1, \varepsilon) \le \mathsf{b}(y_1, \delta)$ and $\mathsf{b}(x_2, \varepsilon) \le \mathsf{b}(y_2, \delta)$.

(ii) $\mathsf{b}((x_1, x_2), \varepsilon) < \mathsf{b}((y_1, y_2), \delta)$ iff $\mathsf{b}(x_1, \varepsilon) < \mathsf{b}(y_1, \delta)$ and $\mathsf{b}(x_2, \varepsilon) < \mathsf{b}(y_2, \delta)$.

Proof. Immediate by the definition of the metric on $X_1 \times X_2$. □

Lemma 6.4 *The following holds for covering relations on $\mathcal{M}_{X_1 \times X_2}$, $\mathcal{M}_{X_1}$ and $\mathcal{M}_{X_2}$.*

(i) If $\mathsf{b}((x, y), \varepsilon) \sqsubseteq_\gamma \{\mathsf{b}((x_i, y_i), \varepsilon_i) : i \in I\}$, then $\mathsf{b}(x, \varepsilon) \sqsubseteq_\gamma \{\mathsf{b}(x_i, \varepsilon_i) : i \in I\}$ and $\mathsf{b}(y, \varepsilon) \sqsubseteq_\gamma \{\mathsf{b}(y_i, \varepsilon_i) : i \in I\}$.

(ii) Suppose that X_1 and X_2 are both locally compact. If $\mathsf{b}((x, y), \varepsilon) \lhd \{\mathsf{b}((x_i, y_i), \varepsilon_i) : i \in I\}$, then $\mathsf{b}(x, \varepsilon) \lhd \{\mathsf{b}(x_i, \varepsilon_i) : i \in I\}$ and $\mathsf{b}(y, \varepsilon) \lhd \{\mathsf{b}(y_i, \varepsilon_i) : i \in I\}$.

Proof. (i): Suppose that $\mathsf{b}((x,y),\varepsilon) \sqsubseteq_\gamma \{\mathsf{b}((x_i,y_i),\varepsilon_i) : i \in I\}$. We show $\mathsf{b}(x,\varepsilon) \sqsubseteq_\gamma \{\mathsf{b}(x_i,\varepsilon_i) : i \in I\}$. Let $\mathsf{b}(u,\delta) \leq \mathsf{b}(x,\varepsilon)$ and $\delta \leq \gamma$. Then by Lemma 6.3.(i) we get $\mathsf{b}((u,y),\delta) \leq \mathsf{b}((x,y),\varepsilon)$. According to the assumption $\mathsf{b}((u,y),\delta) \leq \mathsf{b}((x_i,y_i),\varepsilon_i)$ for some $i \in I$. Again by Lemma 6.3.(i), $\mathsf{b}(u,\delta) \leq \mathsf{b}(x_i,\varepsilon_i)$, which was to be proved. The proof of $\mathsf{b}(y,\varepsilon) \sqsubseteq_\gamma \{\mathsf{b}(y_i,\varepsilon_i) : i \in I\}$ is symmetric.

(ii): Suppose $\mathsf{b}((x,y),\varepsilon) \lhd \{\mathsf{b}((x_i,y_i),\varepsilon_i) : i \in I\}$. Then by Theorem 4.16

$$\mathsf{b}((x,y),\varepsilon) \lessdot \{\mathsf{b}((x_i,y_i),\varepsilon_i) : i \in I\}.$$

By the same theorem it suffices to show that $\mathsf{b}(x,\varepsilon) \lessdot \{\mathsf{b}(x_i,\varepsilon_i) : i \in I\}$ to establish the first conclusion. Thus assume that $\mathsf{b}(u,\delta) < \mathsf{b}(v,\theta) < \mathsf{b}(x,\varepsilon)$. Then $\mathsf{b}((u,y),\delta) < \mathsf{b}((v,y),\theta) < \mathsf{b}((x,y),\varepsilon)$. Thus there is some $U = \{\mathsf{b}((w_k,z_k),\rho_k) : k = 1,\ldots,n\} < \{\mathsf{b}((x_i,y_i),\varepsilon_i) : i \in I\}$ with $\mathsf{b}((u,y),\delta) \sqsubseteq_\gamma U < \mathsf{b}((v,y),\theta)$ for some γ. Let $V = \{\mathsf{b}(w_k,\rho_k) : k = 1,\ldots,n\}$. Thus by Lemma 6.3.(ii) we obtain $V < \{\mathsf{b}(x_i,\varepsilon_i) : i \in I\}$ and $V < \mathsf{b}(v,\theta)$. By part (i) we get also $\mathsf{b}(u,\delta) \sqsubseteq_\gamma V$ as required. Hence $\mathsf{b}(x,\varepsilon) \lessdot \{\mathsf{b}(x_i,\varepsilon_i) : i \in I\}$.

The covering $\mathsf{b}(y,\varepsilon) \lessdot \{\mathsf{b}(y_i,\varepsilon_i) : i \in I\}$ is established in a symmetric way. □

A neighbourhood of the form (a_1,a_2), i.e. a 'rectangle', will be called a *square* if $\mathsf{r}(a_1) = \mathsf{r}(a_2) = \rho$; by abuse of terminology, ρ will be called its *radius.* The following is a version of (M1) for products which shows that a rectangle can be approximated from within by arbitrary small squares.

Lemma 6.5 *Let X_1 and X_2 be locally compact metric spaces. Then in $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$: for $\theta \in \mathbb{Q}_+$,*

$$(a,b) \lhd \{(c,d) : \mathsf{r}(c) = \mathsf{r}(d) < \theta, c < a, d < b\}. \tag{35}$$

Proof. The weaker statement

$$(a,b) \lhd \{(c,d) : c < a, d < b\}.$$

follows by applying PC1 and PC2 to M1 twice. In view of this, it suffices to show that for $a' < a$ and $b' < b$

$$(a',b') \lhd \{(c,d) : \mathsf{r}(c) = \mathsf{r}(d) < \theta, c < a, d < b\}. \tag{36}$$

Pick b'' with $b' < b'' < b$, and write $b'' = \mathsf{b}(x'',\delta'')$ and $b = \mathsf{b}(x,\delta)$. Then take $\varepsilon \in \mathbb{Q}_+$ so that

$$d(x'',x) + \delta'' + \varepsilon < \delta. \tag{37}$$

Since $a' < a$ we may, by Lemma 4.6, find $U \in A(a',a)$ whose balls all have radius ε', and where $\varepsilon' < \min(\theta,\varepsilon)$. Next pick balls $V = \{\mathsf{b}(x_1,\varepsilon''),\ldots,\mathsf{b}(x_n,\varepsilon'')\} \in A(b',b'')$ where $\varepsilon'' < \varepsilon'$. Increase the size of these to radius ε' as well by letting

$$V' = \{\mathsf{b}(x_1,\varepsilon'),\ldots,\mathsf{b}(x_n,\varepsilon')\}.$$

The product $U \times V'$ hence consists only of squares of radius ε'. Thus $b' \lhd V'$. Since $a' \lhd U$, we get $(a', b') \lhd U \times V'$. We have $U < a$, so it suffices to show $V' < b$ to prove (35). For any k we need to show $\mathsf{b}(x_k, \varepsilon') < \mathsf{b}(x, \delta)$, i.e. $d(x_k, x) + \varepsilon' < \delta$. We have using $\mathsf{b}(x_k, \varepsilon'') < b'' = \mathsf{b}(x'', \delta'')$,

$$d(x_k, x) + \varepsilon' \leq d(x_k, x'') + d(x'', x) + \varepsilon' < \delta'' - \varepsilon'' + d(x'', x) + \varepsilon'.$$

But since $\varepsilon' - \varepsilon'' < \varepsilon$ we get by (37) that the right hand side is less than δ. □

We define a strict order relation of formal neighbourhoods of the product $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$ by letting $(a_1, a_2) < (b_1, b_2)$ if, and only if, $a_1 < b_1$ and $a_2 < b_2$. The ball cover relation for such products is defined by $(p_1, p_2) \sqsubseteq_\varepsilon U$ if, and only if,

$$(\forall q_1 \in M_{X_1})(\forall q_2 \in M_{X_1})(\mathsf{r}(q_1), \mathsf{r}(q_2) \leq \varepsilon \,\&\, (q_1, q_2) \leq (p_1, p_2) \Rightarrow (q_1, q_2) \leq U).$$

Then define $(p_1, p_2) \sqsubseteq U$ if, and only if, for some $\varepsilon \in \mathbb{Q}_+$: $(p_1, p_2) \sqsubseteq_\varepsilon U$.

Lemma 6.6 *For $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$, where X_1 and X_2 are locally compact metric spaces, we have*

$$(p_1, p_2) \sqsubseteq U \Longrightarrow (p_1, p_2) \lhd U.$$

Proof. Suppose that $(p_1, p_2) \sqsubseteq_\varepsilon U$. Consider $(q_1, q_2) < (p_1, p_2)$, where $\mathsf{r}(q_1) = \mathsf{r}(q_2) = \varepsilon$, then $(q_1, q_2) \leq U$. Thus $(q_1, q_2) \lhd U$. By Lemma 6.5 it follows that $(p_1, p_2) \lhd U$. □

Further define $(a_1, a_2) <: U$ to hold if, and only if,

$$(\forall p < q < (a_1, a_2))(\exists U_0 \in A_2(p, q))\, U_0 < U, \tag{38}$$

where for $p = (p_1, p_2)$, $q = (q_1, q_2)$,

$$A_2(p, q) = \{U_0 \in \mathcal{P}_{\mathrm{subf}}(M_{X_1} \times M_{X_2}) : (p_1, p_2) \sqsubseteq U_0 < (q_1, q_2)\}.$$

Lemma 6.7 *Let X_1 and X_2 be locally compact metric spaces. If $(p_1, p_2) < (q_1, q_2)$ and $\delta \in \mathbb{Q}_+$, then there is some subfinite $C \subseteq M_{X_1} \times M_{X_2}$ with*

$$(p_1, p_2) \sqsubseteq C < (q_1, q_2)$$

such that $\max(\mathsf{r}(s_1), \mathsf{r}(s_2)) < \delta$ *for all* $(s_1, s_2) \in C$.

Proof. Apply Lemma 4.6 to get C_k with $p_k \sqsubseteq C_k < q_k$, $k = 1, 2$, and whose balls all have radius $< \delta$. Then it is easy to see that $C = C_1 \times C_2$ satisfies the conditions. □

Theorem 6.8 *Let X_1 and X_2 be locally compact metric spaces. Then the covering relation on the product $\mathcal{M}_{X_1} \times \mathcal{M}_{X_2}$ can be characterised as follows*

$$(a_1, a_2) \lhd U \iff (a_1, a_2) <: U.$$

Proof. ($\Leftarrow$) This follows from Lemma 6.6.

($\Rightarrow$) We follow the standard procedure and show that $<:$ is a covering relation satisfying the axioms (PC1) and (PC2). By minimality of $\lhd$ the implication then ensues.

The verification of the cover relation conditions is very similar to Lemma 4.12 – 4.15, but we include it for completeness.

(Ref): In the notation of (38): if $(a_1, a_2) \in U$, we may take $U_0 = \{(a_1, a_2)\}$ to verify $(a_1, a_2) <: U$.

(Ext): Again, in the notation of (38): if $(a_1, a_2) \leq (b_1, b_2)$, we may take $U_0 = \{(a_1, a_2)\}$ to verify $(a_1, a_2) <: \{(b_1, b_2)\}$.

(Tra): Suppose $(a_1, a_2) <: U$ and $U <: V$. Let $(p_1, p_2) < (q_1, q_2) < (a_1, a_2)$. We thus have some $U_0 = \{(b_{1,k}, b_{2,k}) : k = 1, \ldots, n\}$ in $A_2((p_1, p_2), (q_1, q_2))$ with $U_0 < U$. Using the latter pick $(c_{1,k}, c_{2,k}) \in U$ with $(b_{1,k}, b_{2,k}) < (c_{1,k}, c_{2,k})$, $k = 1, \ldots, n$. Next pick, using Proposition 2.1, $r_{i,k}$ so that $(r_{1,k}, r_{2,k}) < (q_1, q_2)$ and

$$(b_{1,k}, b_{2,k}) < (r_{1,k}, r_{2,k}) < (c_{1,k}, c_{2,k}).$$

Now $U <: V$ gives, for each $k = 1, \ldots, n$, some $V_0^k < V$ with $V_0^k \in A_2((b_{1,k}, b_{2,k}), (r_{1,k}, r_{2,k}))$. Let $V_0 = V_0^1 \cup \cdots \cup V_0^n$. Thus $V_0 < (q_1, q_2)$. We show $(p_1, p_2) \sqsubseteq V_0$. By the above, there are $\gamma, \varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Q}_+$ so that $(p_1, p_2) \sqsubseteq_\gamma U_0$ and $(b_{1,k}, b_{2,k}) \sqsubseteq_{\varepsilon_k} V_0^k$, $k = 1, \ldots, n$. Letting $\varepsilon = \min(\gamma, \varepsilon_1, \ldots, \varepsilon_n)$, it now follows easily that $(p_1, p_2) \sqsubseteq_\varepsilon V_0$.

(Loc): Suppose $(a_1, a_2) <: U$ and $(a_1, a_2) <: V$. Consider arbitrary $(p_1, p_2) < (q_1, q_2) < (a_1, a_2)$. Pick further r_1, r_2 so that $(p_1, p_2) < (r_1, r_2) < (q_1, q_2)$. By definition of $<:$ we now obtain $U_0, V_0 \in A_2((r_1, r_2), (q_1, q_2))$ so that $U_0 < U$ and $V_0 < V$. Take $\gamma \in \mathbb{Q}_+$ sufficiently small that

$$(r_1, r_2) \sqsubseteq_\gamma U_0 \qquad (r_1, r_2) \sqsubseteq_\gamma V_0.$$

Then by Lemma 6.7 we find a subfinite W_0 so that $(p_1, p_2) \sqsubseteq W_0 < (r_1, r_2)$, and $\max(\mathsf{r}(s_1), \mathsf{r}(s_2)) \leq \gamma$ for any $(s_1, s_2) \in W_0$. From this follows $W_0 \leq U_0$ and $W_0 \leq V_0$. Hence $W_0 \in A_2((p_1, p_2), (q_1, q_2))$ and $W_0 \leq (U_\leq \cap V_\leq)$ as required.

(PC1): Suppose $a_1 \lhd_{\mathcal{M}_X} V$. Consider $(p_1, p_2) < (q_1, q_2) < (a_1, a_2)$. Then $p_1 < q_1 < a_1$. By Theorem 4.16 we have some $U_0 \in A(p_1, q_1)$, with $U_0 < V$. We may assume that γ in $p_1 \sqsubseteq_\gamma U_0$ is smaller than $\mathsf{r}(p_2)$. Thus $(p_1, p_2) \sqsubseteq_\gamma U_0 \times \{p_2\} < (q_1, q_2)$ and $U_0 \times \{p_2\} < V \times \{a_2\}$. Hence $(a_1, a_2) <: V \times \{a_2\}$.

(PC2): similar to (PC1). □

Lemma 6.9 *Let X_1 and X_2 be a locally compact metric spaces. Let U_0 be a subfinite subset of $M_{X_1} \times M_{X_2}$. If $U_0 < W$, and $U_0 < (q_1, q_2)$, then there is $\theta \in \mathbb{Q}_+$, so that for any $\rho \in \mathbb{Q}_+$ with $\rho \leq \theta$ there is a subfinite set of squares $V_0 < (q_1, q_2)$, all of equal radius ρ, such that*

$$U_0 \sqsubseteq V_0 < W.$$

Proof. Suppose $U_0 = \{(\mathsf{b}(x_{1,k}, \varepsilon_{1,k}), \mathsf{b}(x_{2,k}, \varepsilon_{2,k})) : k = 1, \ldots, n\}$ and $U_0 < (q_1, q_2) = (\mathsf{b}(v_1, \beta_1), \mathsf{b}(v_2, \beta_2))$. Further, suppose that $U_0 < W$. Thus there are n neighbourhoods in W such that

$$(\mathsf{b}(x_{1,k}, \varepsilon_{1,k}), \mathsf{b}(x_{2,k}, \varepsilon_{2,k})) < (\mathsf{b}(w_{1,k}, \delta_{1,k}), \mathsf{b}(w_{2,k}, \delta_{2,k})) \in W.$$

Using this property, take $\theta \in \mathbb{Q}_+$ so small that

$$d(x_{i,k}, w_{i,k}) + \varepsilon_{i,k} + 2\theta < \delta_{i,k} \quad (i = 1, 2; \ell = 1, \ldots, n) \tag{39}$$

and

$$d(x_{i,k}, v_i) + \varepsilon_{i,k} + 2\theta < \beta_i \quad (i = 1, 2; k = 1, \ldots, n_i). \tag{40}$$

Now consider any $\rho \leq \theta$. By Lemma 4.6 we find $Z_{i,k} \in \mathcal{P}_{\mathrm{subf}}(M_{X_i})$ and $\gamma_{i,k}$ such that

$$\mathsf{b}(x_{i,k}, \varepsilon_{i,k}) \sqsubseteq_{\gamma_{i,k}} Z_{i,k} < \mathsf{b}(x_{i,k}, \varepsilon_{i,k} + \rho) \quad (i = 1, 2; k = 1, \ldots, n) \tag{41}$$

and so that all balls in $Z_{i,k}$ have radius $< \rho$. Let $V_{i,k} = \{\mathsf{b}(z, \rho) : \mathsf{b}(z, \alpha) \in Z_{i,k}\}$, i.e. increase the radii of balls in $Z_{i,k}$ to ρ. Then clearly

$$(\mathsf{b}(x_{1,k}, \varepsilon_{1,k}), \mathsf{b}(x_{2,\ell}, \varepsilon_{2,\ell})) \sqsubseteq_{\min(\gamma_{1,k}, \gamma_{2,\ell})} V_{1,k} \times V_{2,\ell}.$$

Put $\gamma = \min(\gamma_{1,1}, \ldots, \gamma_{1,n}, \gamma_{2,1}, \ldots, \gamma_{2,n})$ and

$$V_0 = \bigcup_{k=1}^{n} \bigcup_{\ell=1}^{n} V_{1,k} \times V_{2,\ell}.$$

Hence $U_0 \sqsubseteq_\gamma V_0$. By construction all neighbourhoods in V_0 are squares of radius ρ. It suffices now to show $V_{i,k} < \mathsf{b}(w_{i,k}, \delta_{i,k})$, and $V_{i,k} < q_i$, for $i = 1, 2$ and $k = 1, \ldots, n_i$, in order to establish $V_0 < W$ and $V_0 < (q_1, q_2)$.

For $\mathsf{b}(z, \rho) \in V_{i,k}$ there is $\mathsf{b}(z, \alpha) \in Z_{i,k}$ with $\alpha < \rho$. Hence $d(z, x_{i,k}) + \alpha < \varepsilon_{i,k} + \rho$ by (41), so $d(z, x_{i,k}) < \varepsilon_{i,k} + \rho$. We have then, using (39) in the last step,

$$\begin{aligned} d(z, w_{i,k}) + \rho &\leq d(z, x_{i,k}) + d(x_{i,k}, w_{i,k}) + \rho \\ &< \varepsilon_{i,k} + \rho + d(x_{i,k}, w_{i,k}) + \rho \\ &\leq \varepsilon_{i,k} + d(x_{i,k}, w_{i,k}) + 2\theta \\ &< \delta_{i,k}. \end{aligned}$$

This gives $\mathsf{b}(z,\rho) < \mathsf{b}(w_{i,k},\delta_{i,k})$ as required. Similarly,

$$\begin{array}{rcl} d(z,v_i)+\rho & \leq & d(z,x_{i,k})+d(x_{i,k},v_i)+\rho \\ & < & \varepsilon_{i,k}+\rho+d(x_{i,k},v_i)+\rho \\ & \leq & d(x_{i,k},v_i)+\varepsilon_{i,k}+2\theta \\ & < & \beta_i \end{array}$$

where the last step is (40). This shows $\mathsf{b}(z,\rho) < q_i = \mathsf{b}(v_i,\beta_i)$ finishing the proof. $\square$

For a subset U of $M_{X_1\times X_2}$, we define its corresponding set of squares $U^\dagger = \{(\mathsf{b}(u,\varepsilon),\mathsf{b}(v,\varepsilon)) : \mathsf{b}((u,v),\varepsilon)\in U\}$.

Lemma 6.10 *Let X_1 and X_2 be locally compact metric spaces. Then the cover relations in $\mathcal{M}_{X_1\times X_2}$ and $\mathcal{M}_{X_1}\times\mathcal{M}_{X_2}$ compare as follows:*

$$\mathsf{b}((u_1,u_2),\varepsilon)\lhd U \iff (\mathsf{b}(u_1,\varepsilon),\mathsf{b}(u_2,\varepsilon))\lhd U^\dagger.$$

Proof. ($\Rightarrow$): We use Lemma 6.5. Suppose $\mathsf{b}(w,\delta) < \mathsf{b}(u_1,\varepsilon)$ and $\mathsf{b}(z,\delta) < \mathsf{b}(u_2,\varepsilon)$. Then take $\delta' > \delta$ so that $\mathsf{b}(w,\delta') < \mathsf{b}(u_1,\varepsilon)$ and $\mathsf{b}(z,\delta') < \delta(u_2,\varepsilon)$. Let $p=\mathsf{b}((w,z),\delta)$ and $q=\mathsf{b}((w,z),\delta')$. Then by Lemma 6.3

$$p<q<\mathsf{b}((u_1,u_2),\varepsilon).$$

From $\mathsf{b}((u_1,u_2),\varepsilon)\lhd U$ follows now by Theorem 4.16 that there is some $U_0\in A(p,q)$ with $U_0<U$. Hence also $U_0\leq U$. By Lemma 6.3 we get $U_0^\dagger\leq U^\dagger$, and thereby $U_0^\dagger\lhd U^\dagger$. We have $p\sqsubseteq_\theta U_0<q$ for some θ. We show $(\mathsf{b}(w,\delta),\mathsf{b}(z,\delta))\lhd U_0^\dagger$ using Lemma 6.5. Consider any $\mathsf{b}(x,\alpha)<\mathsf{b}(w,\delta)$, $\mathsf{b}(y,\alpha)<\mathsf{b}(z,\delta)$ with $\alpha<\theta$. Then $\mathsf{b}((x,y),\alpha)\leq p$, so $\mathsf{b}((x,y),\alpha)\leq U_0$. Thus by Lemma 6.3, $(\mathsf{b}(x,\alpha),\mathsf{b}(y,\alpha))\leq U_0^\dagger$, and hence $(\mathsf{b}(x,\alpha),\mathsf{b}(y,\alpha))\lhd U_0^\dagger$, which suffices by Lemma 6.5.

($\Leftarrow$): Assume $(\mathsf{b}(u_1,\varepsilon),\mathsf{b}(u_2,\varepsilon))\lhd U^\dagger$, where $U=\{\mathsf{b}((z_{1,i},z_{2,i}),\delta_i) : i\in I\}$. Suppose $\mathsf{b}((x_1,x_2),\varepsilon')<\mathsf{b}((u_1,u_2),\varepsilon)$. Thus $q_1=\mathsf{b}(x_1,\varepsilon')<p_1=\mathsf{b}(u_1,\varepsilon)$ and $q_2=\mathsf{b}(x_2,\varepsilon')<p_2=\mathsf{b}(u_2,\varepsilon)$. Take s_1 and s_2 so that $q_1<s_1<p_1$ and $q_2<s_2<p_2$. By Theorem 6.8, we have $V_0\in A_2((q_1,q_2),(s_1,s_2))$ with $V_0<U^\dagger$. Using Lemma 6.9 we find a subfinite set of squares W_0, all of radius ρ, such that $(q_1,q_2)\sqsubseteq V_0\sqsubseteq W_0<U^\dagger$. Thus $W_0=Z_0^\dagger$ for some $Z_0=\{\mathsf{b}((w_{1,k},w_{2,k}),\delta_k) : k=1,\ldots,n\}$. By Lemma 6.3, $Z_0<U$. Also it is easy to see using this lemma that $\mathsf{b}((x_1,x_2),\varepsilon')\sqsubseteq Z_0$ as required. We have shown $\mathsf{b}((u_1,u_2),\varepsilon)\lhd U$. $\square$

Lemma 6.11 *For the projection $\pi_k : X_1\times X_2\to X_k$ we have*

$$\mathsf{b}((x_1,x_2),\varepsilon)\ \mathcal{M}\ (\pi_k)\ \mathsf{b}(y,\delta)\iff \mathsf{b}(x_k,\varepsilon)\lhd b(y,\delta).$$

Proof. The proof is a straightforward application of Lemmas 6.3 and 6.4 using Theorem 4.16. □

Theorem 6.12 *The functor $\mathcal{M}$ preserves binary products, in fact we have an isomorphism $\mathcal{M}(X_1 \times X_2) \cong \mathcal{M}(X_1) \times \mathcal{M}(X_2)$ given by the map $G = \langle \mathcal{M}(\pi_1), \mathcal{M}(\pi_2) \rangle$, i.e.*

$$\mathsf{b}((x_1,x_2),\varepsilon)\ G\ (p,q) \Longleftrightarrow_{\text{def}} \mathsf{b}(x_1,\varepsilon) \lhd p \ \&\ \mathsf{b}(x_2,\varepsilon) \lhd q$$

and whose explicit inverse F is given by

$$(p,q)\ F\ \mathsf{b}((x_1,x_2),\varepsilon) \Longleftrightarrow_{\text{def}} (p,q) \lhd (\mathsf{b}(x_1,\varepsilon), \mathsf{b}(x_2,\varepsilon)).$$

Proof. That F and G are mutual inverses follows by Corollary 6.2 and Lemma 6.10.

It remains to show that F is continuous. We check (A1-A4).

(A1): Suppose $(a_1,a_2)\ F\ \mathsf{b}((x_1,x_2),\varepsilon)$ and $\mathsf{b}((x_1,x_2),\varepsilon) \lhd U = \{\mathsf{b}((w_i,z_i),\varepsilon_i) : i \in I\}$. By Lemma 6.10 we have $(\mathsf{b}(x_1,\varepsilon),\mathsf{b}(x_2,\varepsilon)) \lhd U^\dagger$. Since $F^{-1}U \supseteq U^\dagger$, transitivity gives $\mathsf{b}((x_1,x_2),\varepsilon) \lhd F^{-1}U$.

(A2) is immediate.

(A3) follows from $(\mathsf{b}(y_1,\delta_1),\mathsf{b}(y_2,\delta_2))\ F\ \mathsf{b}((y_1,y_2),\max(\delta,\delta_2))$ via Lemma 6.10.

(A4) Suppose that $(a_1,a_2)\ F\ \mathsf{b}((x_1,x_2),\varepsilon)$ and $(a_1,a_2)\ F\ \mathsf{b}((y_1,y_2),\delta)$. Then using localisation and the definition of F we obtain.

$$(a_1,a_2) \lhd (\mathsf{b}(x_1,\varepsilon),\mathsf{b}(x_2,\varepsilon)) \wedge (\mathsf{b}(y_1,\delta),\mathsf{b}(y_2,\delta)).$$

It suffices to show

$$(\mathsf{b}(x_1,\varepsilon),\mathsf{b}(x_2,\varepsilon)) \wedge (\mathsf{b}(y_1,\delta),\mathsf{b}(y_2,\delta)) \lhd F^{-1}[\mathsf{b}((x_1,x_2),\varepsilon) \wedge \mathsf{b}((y_1,y_2),\delta)].$$

Take $p_k \leq \mathsf{b}(x_k,\varepsilon), \mathsf{b}(y_k,\delta)$, $k = 1,2$. By Lemma 6.5 we have

$$(p_1,p_2) \lhd \{(c,d) : \mathsf{r}(c) = \mathsf{r}(d), c < p_1, d < p_2\}. \tag{42}$$

Taking $(c,d) = (\mathsf{b}(w_1,\rho),\mathsf{b}(w_2,\rho))$ as in the righthand set of (42) we get $\mathsf{b}((w_1,w_2),\rho) \leq \mathsf{b}((x_1,x_2),\varepsilon), \mathsf{b}((y_1,y_2),\delta)$. Thus $(c,d) \in F^{-1}[\mathsf{b}((x_1,x_2),\varepsilon) \wedge \mathsf{b}((y_1,y_2),\delta)]$ as required. □

Remark 6.13 As we have seen in Theorem 5.8 the functor $\mathcal{M}$ is a kind of constructively refined version of Ω. Corresponding to Theorem 6.12, it is also well-known that there is a homeomorphism $\Omega(X \times Y) \cong \Omega(X) \times \Omega(Y)$ when at least one of X or Y is a locally compact topological space. Otherwise, this is not true [6, p. 61].

Corollary 6.14 *The functor $\mathcal{M}$ preserves all finite products.*

Proof. By Theorem 6.12, it is enough to check that $\mathcal{M}_X$ is a terminal formal topology, when X is a one-point metric space $T = (\{\bullet\}, d)$. It is easy to see that T is complete and locally compact. By (Ext) and (M2) it follows that $\mathsf{b}(\bullet, \delta) \lhd \mathsf{b}(\bullet, \varepsilon)$ for any $\delta, \varepsilon \in \mathbb{Q}_+$. The only point of $\mathcal{M}_T$ is $\{\mathsf{b}(\bullet, \delta) : \delta \in \mathbb{Q}_+\}$. Therefore $\mathsf{b}(\bullet, \delta) \lhd U$ implies that U is inhabited. It is then straightforward to check that $\mathcal{M}_T$ is a terminal formal topology (cf. **1** in Section 1). □

7 Ordering of real-valued maps

Let $\mathcal{X} = (X, \leq, \lhd)$ be a formal topology. A subset $U \subseteq X$ defines an open set in the topology. It also defines a *closed subspace* by its formal complement as follows. Let $\mathcal{X} \dot{-} U = (X, \leq, \lhd')$ where

$$a \lhd' V \Longleftrightarrow a \lhd U \cup V.$$

(Note that $\lhd'$ is generated by the covering axioms for $\lhd$ and the pairs $(a, \emptyset)$ for $a \lhd U$.)

We shall consider inclusion mappings between closed subspaces of a formal topology $\mathcal{X}$. For subsets $V \subseteq U \subseteq X$, let $E_{U,V} : \mathcal{X} \dot{-} U \to \mathcal{X} \dot{-} V$ be defined by

$$x E_{U,V}\, y \Longleftrightarrow_{\text{def}} x \lhd_{(\mathcal{X} \dot{-} U)} \{y\}.$$

The right hand side is thus equivalent to $x \lhd_{\mathcal{X}} U \cup \{y\}$, and hence we have

$$a \lhd_{\mathcal{X} \dot{-} U} E_{U,V}^{-1} W \Longleftrightarrow a \lhd_{\mathcal{X}} U \cup W. \tag{43}$$

Each morphism $E_{U,V}$ is a monomorphism in the category of formal topologies. Furthermore it follows that

$$E_{V,W} \circ E_{U,V} = E_{U,W} \tag{44}$$

for $W \subseteq V \subseteq U \subseteq X$. We shall write E_U for $E_{U,\emptyset} : (\mathcal{X} \dot{-} U) \to \mathcal{X}$.

The following lemma gives useful characterisation of when a map is continuous into a closed subspace.

Lemma 7.1 *Let $F : \mathcal{X} \to \mathcal{Y}$ be a continuous morphism. Let $W \subseteq Y$ and $E_W : (\mathcal{Y} \dot{-} W) \to \mathcal{Y}$ the corresponding embedding. The following statements are equivalent:*

(a) F factors through E_W.

(b) $F^{-1}W \sim_{\mathcal{X}} \emptyset$.

(c) $F : \mathcal{X} \to (\mathcal{Y} \dot{-} W)$ *is continuous.*

Proof. (a) ⇒ (b): Suppose $G : \mathcal{X} \to (\mathcal{Y} \dot{-} W)$ is continuous with $E_W \circ G = F$. Thus

$$x \lhd_{\mathcal{X}} G^{-1}E_W^{-1}y \Longleftrightarrow xF\,y. \tag{45}$$

Suppose $xF\,y$ and $y \in W$. Then $E_W^{-1}y \lhd_{\mathcal{Y}} W \lhd_{\mathcal{Y} \dot{-} W} \emptyset$. By the continuity of G we get

$$G^{-1}E_W^{-1}y \lhd_{\mathcal{X}} G^{-1}\emptyset.$$

But $G^{-1}\emptyset = \emptyset$, so $x \lhd_{\mathcal{X}} \emptyset$. Hence $F^{-1}W \sim_{\mathcal{X}} \emptyset$ is demonstrated.

(b) ⇒ (c): Suppose $F^{-1}W \sim_{\mathcal{X}} \emptyset$. The only clause to check is (A1). (A2-A4) follows directly from the continuity of $F : \mathcal{X} \to \mathcal{Y}$. To check (A1), assume $a F b$ and $b \lhd_{\mathcal{Y} \dot{-} W} V$, i.e. $b \lhd_{\mathcal{Y}} W \cup V$. Then by the continuity of $F : \mathcal{X} \to \mathcal{Y}$, we get

$$a \lhd_{\mathcal{X}} F^{-1}(W \cup V) = F^{-1}W \cup F^{-1}V.$$

But then (b) gives, by transitivity, the desired $a \lhd_{\mathcal{X}} F^{-1}V$.

(c) ⇒ (a): Let $F : \mathcal{X} \to (\mathcal{Y} \dot{-} W)$ be continuous as well. For the factorisation it suffices to check

$$x \lhd_{\mathcal{X}} F^{-1}E_W^{-1}y \Longleftrightarrow xF\,y.$$

From right to left is clear, since $y \in E_W^{-1}y$. For the other direction, suppose $x \lhd_{\mathcal{X}} F^{-1}E_W^{-1}y$. Let $a \in F^{-1}E_W^{-1}y$, i.e. $aF\,b$ and $b \lhd_{\mathcal{Y}} W \cup \{y\}$. Then by (c) and axiom (A1), we have $a \lhd_{\mathcal{X}} F^{-1}\{y\}$, i.e. $aF\,y$. Since a was arbitrary, we have shown that $F^{-1}E_W^{-1}y \subseteq F^{-1}y$. Hence $x \lhd_{\mathcal{X}} F^{-1}y$, i.e. $xF\,y$. □

Let $\mathcal{R} = \mathcal{M}_{\mathbb{R}}$. Then define

$$L = \{(\mathsf{b}(x,\delta), \mathsf{b}(y,\varepsilon)) : y + \varepsilon < x - \delta\}.$$

For real-valued continuous morphisms $F, G : \mathcal{X} \to \mathcal{R}$ we say that F *is majorized by* G (in symbols $F \leq G$) if $\langle F, G\rangle : \mathcal{X} \to \mathcal{R}^2$ factors through the embedding $E_L : (\mathcal{R}^2 \dot{-} L) \to \mathcal{R}^2$. By Lemma 7.1 we easily obtain

Lemma 7.2 *For continuous $F, G : \mathcal{X} \to \mathcal{R}$, the inequality $F \leq G$ is equivalent to the implication*

$$p F\,\mathsf{b}(x,\delta)\ \&\ p G\,\mathsf{b}(y,\varepsilon)\ \&\ y + \varepsilon < x - \delta \Rightarrow p \lhd_{\mathcal{X}} \emptyset.$$

Theorem 7.3 *Let X be a locally compact metric space. For continuous functions $f, g : X \to \mathbb{R}$ we have*

$$f \leq g \Longleftrightarrow \mathcal{M}(f) \leq \mathcal{M}(g).$$

Proof. ($\Rightarrow$): Suppose $f(u) \leq g(u)$ for all $u \in X$. We use the characterisation of Lemma 7.2. Let $p = \mathsf{b}(u,\gamma) \in M_X$ and suppose $p\,\mathcal{M}\,(f)\,\mathsf{b}(x,\delta)$, $p\,\mathcal{M}\,(g)\,\mathsf{b}(y,\varepsilon)$ where $y+\varepsilon < x-\delta$. Thus $f[B(u,\gamma)] \subseteq (x-\delta, x+\delta)$ and $g[B(u,\gamma)] \subseteq (y-\varepsilon, y+\varepsilon)$. But then $g(u) < y+\varepsilon < x-\delta < f(u)$, which contradicts the first assumption. Hence $p \lhd \emptyset$.

($\Leftarrow$): Suppose $\mathcal{M}\,(f) \leq \mathcal{M}\,(g)$. Assume $g(u) < f(u)$ for some $u \in X$. Then for some $\delta \in \mathbb{Q}_+$, $g(u)+\delta < f(u)-\delta$. By continuity there is $\gamma \in \mathbb{Q}_+$ so that $g[B(u,\gamma)] \subseteq (g(u)-\delta/2, g(u)+\delta/2)$ and $f[B(u,\gamma)] \subseteq (f(u)-\delta/2, f(u)+\delta/2)$. Thus for $p = \mathsf{b}(u,\gamma)$: $p\,\mathcal{M}\,(f)\,\mathsf{b}(f(u),\delta)$ and $p\,\mathcal{M}\,(g)\,\mathsf{b}(g(u),\delta)$. By $\mathcal{M}\,(f) \leq \mathcal{M}\,(g)$ follows then $p \lhd \emptyset$, which is impossible in $\mathcal{M}_X$. Hence we must have $f(u) \leq g(u)$ for all $u \in X$. □

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