A note on domain representability and formal topology

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Abstract

Two common approaches to constructive and effective topology are connected by showing that formal topologies have canonical representation in terms of Scott domains. Moreover a map lifting theorem for the representation is proved.

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The theory of domain representations initiated by Stoltenberg-Hansen and Tucker [9] gives a method for introducing notions of computability on abstract mathematical structures. These structures may be of countable as well as uncountable cardinality. The compact elements of the representing Scott domain form the finitistic objects of the computation theory. In [10] and later in [1, 2] the method was applied to classes of topological spaces. Another theory which can serve the same purpose is the theory of formal spaces, which is concerned with representations of locales. It comes in a predicatively constructive variant [5, 4], and in an effective variant [7]. The purpose of the present paper is to demonstrate one basic connection between the two methods for representing spaces. We recommend [4] as a background.

A topological space X is *domain representable by* (D, D^R, φ) if D is a Scott domain and $D^R \subseteq D$ is a subset, *the representing elements,* equipped with the Scott topology induced by D, and where $\varphi : D^R \longrightarrow X$ is a continuous map which is onto. Explicitly this means that the topology on D^R has as a base the sets $B_a = \{x \in D^R : a \sqsubseteq x\}$ where a varies over the compact elements of D. One may consider further conditions [2] on the map φ. In the examples of the present paper the map will be a homeomorphism, but having full generality is useful in some cases [3].

Formal topology includes a constructive theory of domains [7, 6, 4]. In this note we show that every formal topology $\mathcal{S}$ automatically gives a domain representation of its space of points $X = \mathrm{Pt}(\mathcal{S})$. This is done by taking the *Scott compactification* [4] $\mathcal{S}_0$ of $\mathcal{S}$ and letting the representing domain be $D = \mathrm{Pt}(\mathcal{S}_0)$. We also show that each continuous map between formal spaces extends to a continuous map between

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their domain representations. An interesting feature of the extended function is that it can be applied not only to partial elements but also to non-constructive elements. The latter is in general not possible for the original function in a constructive setting.

1 Formal topology

We use standard definitions for formal topology on pre-ordered neighbourhoods; see [4, Def. 2.1]. For such a topology $\mathcal{X}$ we indicate components as $\mathcal{X} = (X, \leq_{\mathcal{X}}, \lhd_{\mathcal{X}})$, where X is the set of basic neighbourhoods ordered by $\leq_{\mathcal{X}}$, and where $\lhd_{\mathcal{X}}$ is the cover relation.

The points of a formal topology $\mathcal{X}$ form a class $\mathrm{Pt}(\mathcal{X})$. For $a \in X$ let $\mathrm{ext}(a) = \mathrm{ext}_{\mathcal{X}}(a)$ denote the subclass of points α in $\mathcal{X}$ where $a \in \alpha$. For a subset $U \subseteq X$, let $\mathrm{ext}_{\mathcal{X}}(U)$ denote the union of all the subclasses $\mathrm{ext}_{\mathcal{X}}(a)$ where $a \in U$. The point topology on $\mathrm{Pt}(\mathcal{X})$ is the topology $\tau_{\mathcal{X}}$ which has $\mathrm{ext}_{\mathcal{X}}(a)$ for a base.

2 Domain representability

By ordering the points of a formal topology by inclusion we can obtain some familiar DCPOs and domains. A subset S of a pre-order $(X, \sqsubseteq)$ is *directed* in case it is inhabited and for any $a, b \in S$, there is $c \in S$ with $a \sqsubseteq c$ and $b \sqsubseteq c$.

Theorem 2.1 *If $\mathcal{X}$ is a formal topology, then $(\mathrm{Pt}(\mathcal{X}), \subseteq)$ is a directed complete partial order.*

Proof. Let α_i be a directed family of points indexed by the set I. Then $\cup_{i \in I} \alpha_i$ is a point, which is the supremum of the family. □

For many spaces $\mathcal{X}$ the inclusion relation between points is trivial since all points are maximal. This is, for example, the case for regular formal topologies. However, for a big class of T_0-spaces it has an interesting structure. A formal topology $\mathcal{X} = (X, \leq_{\mathcal{X}}, \lhd_{\mathcal{X}})$ is *unary* if

$$a \lhd_{\mathcal{X}} U \Longrightarrow (\exists b \in U)\, a \leq_{\mathcal{X}} b.$$

In such a topology $a\!\uparrow = \{b \in X : a \leq_{\mathcal{X}} b\}$ is a point, for any $a \in X$. This is usually a non-maximal point.

We need a notion of semi-lattice adapted to pre-orders. A pre-order $(X, \leq)$ is a *lower semi-lattice pre-order* (LSP) if it has a largest element $\top$ and an operation $\wedge : X \times X \longrightarrow X$ so that

(a) $x \wedge y \leq x$ and $x \wedge y \leq y$,

(b) $z \leq x$ and $z \leq y$ implies $z \leq x \wedge y$.

A weaker notion is the following. A pre-order $(X, \leq)$ is a *consistently complete lower semi-lattice pre-order* (CLSP) if it has a largest element $\top$ and for any consistent pair $x, y \in X$, i.e. which has a lower bound $t \leq x$ and $t \leq y$, there is an element $x \wedge y$ satisfying (a) and (b) above.

Theorem 2.2 ([7, 6]) *If $\mathcal{X}$ is a unary formal topology, where $(X, \leq_{\mathcal{X}})$ is a CLSP, then $D = (\mathrm{Pt}(\mathcal{X}), \subseteq)$ is a Scott domain.*

We define the *Scott compactification $\mathcal{X}_0$ of $\mathcal{X}$:* For a formal topology $\mathcal{X}$ define $\leq_{\mathcal{X}_0}$ by

$$a \leq_{\mathcal{X}_0} b \Longleftrightarrow a \lhd_{\mathcal{X}} \{b\}$$

and define $\lhd_{\mathcal{X}_0}$ by

$$a \lhd_{\mathcal{X}_0} U \Longleftrightarrow (\exists b \in U) a \leq_{\mathcal{X}_0} b.$$

Let $\mathcal{X}_0 = (X, \leq_{\mathcal{X}_0}, \lhd_{\mathcal{X}_0})$. We have the following basic result relating a topology and its Scott compactification.

Theorem 2.3 *If $\mathcal{X}$ is a formal topology, such that $(X, \leq_{\mathcal{X}_0})$ is a CLSP, then*

(a) $\mathcal{X}_0$ *is a unary formal topology,*

(b) $\mathrm{Pt}(\mathcal{X}) \subseteq \mathrm{Pt}(\mathcal{X}_0)$,

(c) $\mathrm{Pt}(\mathcal{X}) \cap \mathrm{ext}_{\mathcal{X}_0}(U) = \mathrm{ext}_{\mathcal{X}}(U)$.

Proof. (a) It is easily verified that $\lhd_{\mathcal{X}_0}$ is a covering relation. The unary property of covers follows by definition.

(b) Suppose that α is a point in $\mathcal{X}$. Then α is inhabited. If $a, b \in \alpha$, then there is $c \in \alpha$ with $c \leq_{\mathcal{X}} a$ and $c \leq_{\mathcal{X}} a$. We have $c \leq_{\mathcal{X}_0} a$, $c \leq_{\mathcal{X}_0} b$, since $\leq_{\mathcal{X}_0}$ is an extension of $\leq_{\mathcal{X}}$. Suppose now that $a \lhd_{\mathcal{X}_0} U$ and $a \in \alpha$. Since $\mathcal{X}_0$ is unary, there is some $b \in U$ with $a \leq_{\mathcal{X}_0} b$. Hence $a \lhd_{\mathcal{X}} \{b\}$, and because α is a point in $\mathcal{X}$, we have $b \in \alpha$. We have shown that α is a point in $\mathcal{X}_0$.

(c, $\subseteq$): For $\alpha \in \mathrm{Pt}(\mathcal{X})$ we have by (b) that: $\alpha \in \mathrm{ext}_{\mathcal{X}}(a)$ iff $\alpha \in \mathrm{ext}_{\mathcal{X}_0}(a)$. □

We can now state the first main observation as an easy corollary.

Corollary 2.4 *For any formal topology $\mathcal{X}$ its space $\mathbb{X} = \mathrm{Pt}(\mathcal{X})$ is represented by the domain $D = \mathrm{Pt}(\mathcal{X}_0)$ with representing elements $D^R = \mathrm{Pt}(\mathcal{X})$ and representation map φ being the identity. This is called the* canonical domain representation *of* $\mathbb{X}$.

Proof. From Theorem 2.2 and 2.3 it follows that the domain $D = (\mathrm{Pt}(\mathcal{X}_0), \subseteq)$ represents the space $\mathbb{X} = (\mathrm{Pt}(\mathcal{X}), \tau_{\mathcal{X}})$, where $D^R = \mathrm{Pt}(\mathcal{X})$ and $\varphi : D^R \longrightarrow \mathbb{X}$ is the identity map. The topology on D^R is the relative topology induced by the topology of $\mathrm{Pt}(\mathcal{X}_0)$. Thus it has base sets of the form $\mathrm{Pt}(\mathcal{X}) \cap \mathrm{ext}_{\mathcal{X}_0}(a)$. By Theorem 2.3(c) these are just the sets $\mathrm{ext}_{\mathcal{X}}(a)$. Hence D^R and $\mathbb{X}$ have the same topology, so φ is a homeomorphism and in particular an onto quotient map. □

We remark that in [8] representations of locally compact spaces by domains and by formal topologies were considered simultaneously, and that the Corollary is implicit in their paper.

3 Lifting of continuous mappings

Suppose that (D, D^R, φ) and (E, E^R, ψ) are domain representations of X and Y, respectively. An important problem for domain representations is whether it is possible to extend, or lift, a continuous function $X \longrightarrow Y$ to a continuous function $D \longrightarrow E$. By this we mean more precisely the following. Is it possible to associate to each continuous function $f : X \longrightarrow Y$, a continuous function $\bar{f} : D \longrightarrow E$ such that $\bar{f}[D^R] \subseteq E^R$ and $f \circ \varphi = \psi \circ \bar{f}$? Such a lifting is possible under certain conditions on the representations of X and Y; see [2].

Here we show that the lifting result (Theorem 3.3) is almost automatic for continuous maps between formal spaces and their canonical representations. We recall the notion of a continuous map. Consider formal topologies $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{T} = (T, \leq', \lhd')$. A relation $F \subseteq S \times T$ is a *continuous mapping* $\mathcal{S} \longrightarrow \mathcal{T}$ if

(A1) $a\,F\,b,\ b \lhd' V \Longrightarrow a \lhd F^{-1}\,V$,

(A2) $a \lhd U,\ x\,F\,b$ for all $x \in U \Longrightarrow a\,F\,b$,

(A3) $S \lhd F^{-1}\,T$,

(A4) $a\,F\,b,\ a\,F\,c \Longrightarrow a \lhd F^{-1}(b_{\leq'} \cap c_{\leq'})$.

Here $F^{-1}Z = \{x \in S : (\exists y \in Z)\, x\,F\,y\}$ and $z_{\leq'}$ is $\{x \in T : x \leq' z\}$.

Each continuous mapping F induces a point function $f = \mathrm{Pt}(F) : \mathrm{Pt}(\mathcal{S}) \longrightarrow \mathrm{Pt}(\mathcal{T})$ given by

$$f(\alpha) = \{b \in T : (\exists a \in \alpha)\, a\,F\,b\}.$$

It satisfies: $a\,F\,b \Rightarrow f[\mathrm{ext}(a)] \subseteq \mathrm{ext}(b)$.

Note that if both $\mathcal{S}$ and $\mathcal{T}$ are unary then the axioms for a continuous mapping simplify to

(I1) $c \leq a,\ a\,F\,b,\ b \leq' d \Longrightarrow c\,F\,d$,

(I2) $(\forall a \in S)(\exists b \in T)\, a\,F\,b$

(I3) $a\,F\,b,\ a\,F\,c \Longrightarrow (\exists d \leq' b, c)\, a\,F\,d$.

Proposition 3.1 *Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be a continuous map between arbitrary formal topologies. Then the point function $f = \mathrm{Pt}(F) : \mathrm{Pt}(\mathcal{S}) \longrightarrow \mathrm{Pt}(\mathcal{T})$ preserves directed suprema, i.e.*

$$f(\cup_{i \in I} \alpha_i) = \cup_{i \in I} f(\alpha_i)$$

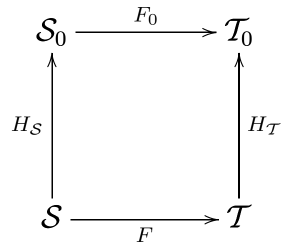
for directed sets α_i $(i \in I)$ of points.

We prove two lifting theorems. First we note that $H : \mathcal{S} \longrightarrow \mathcal{S}_0$ defined by

$$a\,H\,b \Longleftrightarrow a \lhd_{\mathcal{S}} \{b\}.$$

is a continuous map whose point function is the identity embedding. That is $\mathrm{Pt}(H)(\alpha) = \alpha$ for all $\alpha \in \mathrm{Pt}(\mathcal{S})$. The theorems will pertain to a large class of formal topologies. Define a formal topology $\mathcal{S} = (S, \leq, \lhd)$ to be an *LSP formal topology,* if $(S, \leq)$ is a LSP pre-order.

Theorem 3.2 (Formal Lifting) *Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be a continuous map between LSP formal topologies. Then $F_0 = F$ is also a continuous map from $\mathcal{S}_0$ to $\mathcal{T}_0$ with makes this lifting square commute:*



Proof. We check that F satisfies (I1), (I2) and (I3).

(I1): Suppose $a \leq_{\mathcal{S}_0} b$ and $b \, F \, c$. Thus $a \lhd_\mathcal{S} \{b\}$. Hence by (A2) $a \, F \; c$. Now suppose $b \, F \, c$ and $c \leq_{\mathcal{T}_0} d$. Thus $c \lhd_\mathcal{T} \{d\}$, and by (A1), $b \lhd_\mathcal{S} F^{-1}(d)$. Using (A2) it follows that $b \, F \, d$.

(I2): We have by (A3), $S \lhd_\mathcal{S} F^{-1} T$. As T has a top element, we have $T \lhd \{\top\}$ and so $S \lhd F^{-1}\top$. Thus by (A2), $s \, F \top$ for all $s \in S$. This verifies (I2).

(I3): Suppose $a \, F \, b$ and $a \, F \, c$. By (A4) we have $a \lhd F^{-1}(b_{\leq_\mathcal{T}} \cap c_{\leq_\mathcal{T}})$. Now $F^{-1}(b_{\leq_\mathcal{T}} \cap c_{\leq_\mathcal{T}}) \lhd F^{-1}(b \wedge c)$, so $a \, F \, (b \wedge c)$. But $b \wedge c \leq b, c$, so we have $b \wedge c \leq_{\mathcal{T}_0} b, c$.

That the square commutes is a straightforward verification. □

The Lifting Theorem, in the sense of domain representability, is the second main observation.

Corollary 3.3 (Lifting Theorem) *Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be a continuous map between LSP formal topologies. Then $f_0 = \mathrm{Pt}(F_0) : \mathrm{Pt}(\mathcal{S}_0) \longrightarrow \mathrm{Pt}(\mathcal{T}_0)$ is an extension of $f = \mathrm{Pt}(F) : \mathrm{Pt}(\mathcal{S}) \longrightarrow \mathrm{Pt}(\mathcal{T})$.*

Proof. By Theorem 3.2 we have $F_0 \circ H_\mathcal{S} = H_\mathcal{T} \circ F$. Now because Pt is a functor we have for all $\alpha \in \mathrm{Pt}(\mathcal{S})$,

$$\mathrm{Pt}(F_0)(\alpha) = \mathrm{Pt}(F_0)(\mathrm{Pt}(H_\mathcal{S})(\alpha)) = \mathrm{Pt}(H_\mathcal{T})(\mathrm{Pt}(F)(\alpha)) = \mathrm{Pt}(F)(\alpha)$$

showing that f_0 is an extension of f. □

There is no serious restriction in considering just LSP formal topologies, since every formal topology is isomorphic to an LSP formal topology; see [7, Lemma 1.1.17].

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