Symposium on Constructive Mathematics
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Abstracts and Titles of Talks
August 26
The fan theorem and unique existence in constructive analysis

Josef Berger
Mathematisches Institut,
Ludwig-Maximilians Universität München

Abstract: Working in the framework of Bishop’s constructive mathematics, we give a survey of some existence theorems which have recently turned out to be equivalent to Brouwer’s fan theorem (for detachable bars):

• Each continuous function on a compact metric space with at most one minimum attains its minimum.

• Each continuous mapping on a compact metric space with at most one fixed point has a fixed point.

• Each infinite tree with at most one infinite branch has an infinite branch.

Thus uniqueness is the crucial point. These results were obtained together with Douglas Bridges, Hajime Ishihara, and Peter Schuster.

Depending on further developments, we might also bring the following question into discussion: Does the fan theorem imply that each pointwise continuous function from $2^N$ to $N$ is uniformly continuous? We stress that in this context we do not allow the full fan theorem (for arbitrary bars), nor do we allow the use of continuous choice.
Four slices of constructive analysis

Douglas S. Bridges
University of Canterbury
New Zealand

Abstract

This talk deals with four topics in constructive analysis, namely:

(i) The convexity of the numerical range of a bounded linear operator with an adjoint on a Hilbert space.
(ii) Ultraweakly continuous functionals in the abstract setting.
(iii) The existence of adjoints.
(iv) Commutative Banach algebras and local Nullstellensätze.

The results are some of the fruits of collaborations which I have been privileged to have with several mathematicians over many years.
A new proof of Gelfand representation theorem

Thierry Coquand
Department of Computer Science,
Chalmers University of Technology, Sweden

Abstract: We give a new proof to Gelfand representation theorem for commutative C*-algebra, in the framework of Bishop constructive mathematics.
Subsystems of constructive Zermelo-Fraenkel set theory

Laura Crosilla
Department of Philosophy,
University of Florence, Italy

Abstract: We shall give a rather informal presentation of variants of proof theoretically weak subsystems of constructive Zermelo Fraenkel set theory and motivate our interest on them. These systems combine in fact a very limited proof theoretic strength with a considerable expressiveness both on the mathematical and the metamathematical level.
Constructive reverse mathematics: compactness properties

Hajime Ishihara
JAIST, Japan

Abstract: We first propose a base formal system for constructive reverse mathematics aiming at classifying various theorems in intuitionistic, constructive recursive and classical mathematics by logical principles, function existence axioms and their combinations, which is weak enough to compare results in the system with the results in classical reverse mathematics and to prove theorems in Bishop’s constructive mathematics. Then we formalize some results on compactness properties, such as the Heine-Borel theorem, the Cantor’s intersection theorem, the Bolzano-Weierstraß theorem, and sequential compactness, in the base formal system as test cases of its adequacy and faithfullness for the purpose of constructive reverse mathematics, and also investigate of computability of function existence axioms and their combination with logical principles identifying them with closure conditions on a class of functions.
On constructing completions *

Peter Schuster
Mathematisches Institut,
Ludwig-Maximilians Universität München

Abstract: The Dedekind cuts in an ordered set form a set in the sense of constructive Zermelo-Fraenkel set theory. We deduce the existence of this order completion from a presumably new principle, which we call the principle of refinement. This is a consequence of the axiom of fullness, with which it is equivalent in the presence of the principle of exponentiation. A remarkable observation is that we do not need to suppose any property of the ordering, which may be an arbitrary binary relation, and only use a binary version of the principle of refinement when constructing that completion.

As a consequence, the Dedekind reals form a set; whence we also have refined an earlier result by Aczel and Rathjen, who invoked the full form of the axiom of fullness. To further generalise this, we look at Richman’s method to complete an arbitrary metric space without sequences, which he designed to avoid countable choice. The completion of a separable metric space turns out to be a set even if the original space is a proper class; in particular, every complete separable metric space automatically is a set.

*joint work with L. Crosilla and H. Ishihara
Program extraction in constructive analysis

Helmut Schwichtenberg
Mathematisches Institut,
Ludwig-Maximilians Universität München

July 9, 2004

Abstract: We consider a formal arithmetical system with type and predicate parameters. Moreover inductive definitions with their introduction and elimination axioms are allowed. For this system we define extracted terms and the notion of (modified) realizability, and prove a soundness theorem. Then constructive analysis is developed in this setting, with special emphasis on low type level witnesses (which is possible when we assume separability). As an example we discuss the intermediate value theorem, and the form and usefulness of extracted programs form its proof.
Abstract: Being inspired by the work of Hausdorff, Mahlo presents in 1911 two hierarchies of regular cardinals: the \( p_\alpha \)-numbers and the \( \rho_\alpha \)-numbers; in modern terms: \( \alpha \)-weakly inaccessible and ”\( \rho \)-weakly Mahlo” cardinals, respectively.

In 1908, Veblen answered partly Hardy’s question ”How to associate with each countable limit ordinal a unique fundamental sequence” by defining the Veblen functions.

In my talk I will connect Mahlo’s hierarchies to Veblen’s. For this end I will present the constructive analogues of inaccessibility and mahloness in CZF and determine the proof-theoretic strength of some CZF-extensions with existence axioms for set-inaccessibles and set-Mahlos. The resulting ordinal plays an important role in the Veblen hierarchy.

This work is a result of my PhD-thesis and the PhD done by Ben Gibbons. Both doctorates have been submitted at the University of Leeds (UK) under the supervision of Michael Rathjen.

*Joint work with Ben Gibbons.*
Validated numerics and the art of dividing by zero

Warwick Tucker
Uppsala University, Sweden

Abstract: We will discuss a modern approach to numerical computations, based on interval analysis. Although the theory has been well known since the mid 60’s, it is not until recently that fast and efficient implementations of interval algorithms have appeared. Today there exist many good interval packages for e.g. Maple, Matlab, C++, and Fortran, much due to the fact that there now is one globally accepted standard for floating point computations.

Interval analysis is the mathematical foundation of so called auto-validating algorithms. By computing with intervals instead of single points, important properties of the real line can be captured and used in the algorithms. This leads to very robust methods, well suited for ill-conditioned problems.

Auto-validating algorithms produce mathematically correct results, incorporating not only the computer’s internal representation of the floating point numbers and its rounding procedures, but also all discretisation errors of the underlying numerical method. Thus the result comes equipped with guaranteed error bounds. With today’s computing speeds, this appears to be the only reasonable way to keep track of error propagation.

There are many situations in which the interval algorithm returns a guaranteed result faster than the floating point version delivers an ”approximation” (which can be very wrong indeed). Throughout the talk, several such applications will be presented, e.g. root-finding, implicit curve generation, and chaos theory.
The fan theorem as an axiom

Wim Veldman,
Subfaculteit Wiskunde,
Katholieke Universiteit Nijmegen,
the Netherlands

We make a cautious start with Intuitionistic Reverse Mathematics. Introducing a basic system BIM for intuitionistic analysis we study the force of the fan theorem as an additional assumption. In BIM the second-order objects are not sets of natural numbers, but infinite sequences of natural numbers. It turns out that the fan theorem has a number of surprising equivalents. One of them is the statement that two enumerable subsets of $\mathbb{N}$ that are constructively inseparable must have a point in common. Another one is the constructively correct approximate version of Brouwer’s own fixed point theorem.
Aspects of apartness

Luminița Simona Vîță
University of Canterbury
New Zealand

Abstract

Starting with a primitive notion of pre-apartness on Boolean algebras, we develop a constructive approach (in Bishop’s style) to the classical structures of proximity spaces. We also show how our model may be linked to similar structures like the Boolean Connection Algebras used in the theory of Region Connection Calculus.