Constructive embeddings of intermediate logics

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The Gödel-Tarski-McKinsey embedding of intuitionistic logic into S4

1. Gödel (1933): $\vdash_{Int} A \rightarrow \vdash_{S4} A^*$ soundness
2. McKinsey & Tarski (1948): $\vdash_{S4} A^* \rightarrow \vdash_{Int} A$ faithfulness
3. Dummett & Lemmon (1959): $\vdash_{Int+Ax} A$ iff $\vdash_{S4+Ax^*} A^*$

A modal logic **M** is a **modal companion** of a superintuitionistic logic **L** if $\vdash_{\mathbf{L}} A$ iff $\vdash_{\mathbf{M}} A^*$. So **S4** is a modal companion of **Int**, **S4+Ax*** is a modal companion of **Int+Ax**.

2. and 3. are proved semantically. We look closer at McKinsey & Tarski (1948):

5. Relations between the Heyting calculus and the Lewis system. In this section we shall prove three theorems[16] which provide methods of "translating" the Heyting calculus into the Lewis system. In view of the decision method for the Lewis system given in McKinsey [8], each of these theorems provides a new decision method for the Heyting calculus.

THEOREM 5.1. Let T be a function defined over all formulas of the Heyting calculus, assuming as values formulas of the Lewis calculus, and satisfying the following conditions (where v_i is any sentential variable, and α and β are arbitrary formulas):

(i) $T(v_i) = \sim\Diamond\sim v_i$;
(ii) $T(\alpha \vee \beta) = T(\alpha) \vee T(\beta)$;
(iii) $T(\alpha \wedge \beta) = T(\alpha) \wedge T(\beta)$;
(iv) $T(\alpha \rightarrow \beta) = T(\alpha) \supset T(\beta)$;
(v) $T(\sim\alpha) = \sim\Diamond T(\alpha)$.

Then, for any formula α of the Heyting calculus, α is provable in the Heyting calculus if and only if $T(\alpha)$ is provable in the Lewis system.

Proof. Let α be any formula of the Heyting calculus; suppose that α is of index n. By Theorem 1.4 we see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(T(\alpha))}(x_1, \ldots, x_n) = 1 \tag{1}$$

is true for all elements $x_1, \ldots, x_n$ of every closure algebra. By condition (i) of the hypothesis of our theorem, it is then seen that $T(\alpha)$ is provable in the Lewis system if and only if (1) is true for all open elements of every closure algebra. By means of conditions (ii)-(v) of the hypothesis of our theorem, the principle of duality for closure algebras, and 1.14** and 1.15**, we then see that $T(\alpha)$ is provable in the Lewis system if and only if the equation

$$f^{(\alpha)}(x_1, \ldots, x_n) = 0$$

is true for all elements of every Brouwerian algebra. Our theorem now follows by Theorem 4.3.

Proof by McKinsey & Tarski (1948) uses:

- 1. Completeness of intuitionistic logic wrt Heyting algebras (Brouwerian algebras) and of S4 wrt topological Boolean algebras (closure algebras)
- 2. Representation of Heyting algebras as the opens of topological Boolean algebras.
- 3. The proof is **indirect** because of 1. and **non-constructive** because of 2. (Uses Stone representation of distributive lattices, in particular Zorn's lemma)

The result was generalized to intermediate logics by Dummett and Lemmon (1959).
No syntactic proof of faithfulness in the literature except the complex proof of the embedding of **Int** into **S4** in Troelstra & Schwichtenberg (1996).

Goal: give a direct, constructive, syntactic, and uniform proof.

Background on labelled sequent systems

Method for formulating systems of contraction-free sequent calculus for basic modal logic and its extensions and for systems of non-classical logics in Negri (2005).

General ideas for basic modal logic K and other systems of normal modal logics.

Syntax embodies the possible world semantics:

- Rules for $\Box$, $\Diamond$ through meaning explanation and inversion principles.
- Frame properties in the form of "non-logical rules" added to the basic sequent calculus. Extensions are modular.

Relation of this approach to the extensive literature on labelled deductive systems of various forms is detailed in Negri (2011).

Here explicit labelling and contraction-free systems.

Sequent calculus for the basic modal logic K

- Start with the calculus **G3c** for classical propositional logic
- Enrich the language: Sequents $\Gamma \Rightarrow \Delta$ where Γ and Δ consist of expressions xRy and $x : A$ (corresponding to $x \Vdash A$ of Kripke models), with $x, y, z, ...$ ranging in a set W and with A any formula in the language of propositional logic extended with the modal operators of necessity and possibility, $\Box$ and $\Diamond$.
- Rules for basic modal logic obtained from the inductive definition of validity in a Kripke frame.

From

$$x : \Box A \textit{ iff for all } y,\ xRy \textit{ implies } y : A$$

obtain the rules

- If $y : A$ can be derived for an arbitrary y accessible from x, then $x : \Box A$ can be derived

$$\frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A} R\Box$$

  arbitrariness of y becomes the variable condition y not (free) in Γ, Δ

- If $x : \Box A$ and y is accessible from x, then $y : A$

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta} L\Box$$

The rules for $\Diamond$ are obtained similarly from the semantical explanation

$$x : \Diamond A \text{ iff for some } y,\ xRy \text{ and } y : A$$

- If $x : \Diamond A$, there exists y is accessible from x such that $y : A$

  $$\frac{xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\, L\Diamond$$

  variable condition: y not (free) in Γ, Δ
- If y is accessible from x and $y : A$, then $x : \Diamond A$ gives the rule

  $$\frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}\, R\Diamond$$

Initial sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\&} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, {\scriptstyle R\&}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, {\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\supset} \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, {\scriptstyle R\supset}$$

$$\frac{}{x : \perp, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\perp}$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\Box} \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, {\scriptstyle R\Box}$$

$$\frac{xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\, {\scriptstyle L\Diamond} \qquad \frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}\, {\scriptstyle R\Diamond}$$

Modal systems and frame properties

- Modal logic K characterized by arbitrary frames.
- Restrictions of the class of frames amount to adding certain frame properties to the calculus.
  - In Kripke frames for $S4$ the accessibility relation is reflexive
    $\forall x. xRx$
    and transitive
    $\forall x \forall y \forall z ((xRy \ \& \ yRz) \supset xRz)$
  - For $S4.2$ also directed $\forall xyz (xRy \,\&\, xRz \supset \exists w (yRw \,\&\, zRw))$

Axioms as rules (cont.)

- We use the method developed in NvP (1998) and in N (2003)
- Universal axioms are transformed into conjunctive normal form, that is, conjunctions of formulas of the form $P_1 \& \dots \& P_m \supset Q_1 \vee \cdots \vee Q_n$
- Each conjunct is then converted into the *regular rule*,

$$\frac{Q_1, P_1, \dots, P_m, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, P_1, \dots, P_m, \Gamma \Rightarrow \Delta}{P_1, \dots, P_m, \Gamma \Rightarrow \Delta}\ Reg$$

- Other rules may be added by the closure condition. Those that are instances of contraction are admissible.

Examples of universal axioms

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\ xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\ xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness

	Frame property	Rule
T	$\forall x\ xRx$ reflexivity	$\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$
4	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}$
E	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
B	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}$
3	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness	$\dfrac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta \quad zRy, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
D	$\forall x \exists y\ xRy$ seriality	
2	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	

Axioms as rules (cont.)

- Method extended (N2003) to *geometric theories*, that is, theories axiomatized by formulas

$$\forall \overline{z}(A \supset B)$$

where A and B do not contain $\supset$ or $\forall$.

- These can be reduced to conjunctions of

$$\forall \overline{z}(P_1 \& \dots \& P_m \supset \exists x_1 M_1 \vee \dots \vee \exists x_n M_n)$$

where M_j is conjunction of atoms Q_j

- and turned into the *geometric rule*, with the y_i's not in the conclusion

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\, GR$$

Examples of geometric implications

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\ xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy\ \&\ yRz \supset xRz)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy\ \&\ xRz \supset yRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy\ \&\ xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\ xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy\ \&\ xRz \supset \exists w(yRw\ \&\ zRw))$ directedness

	Frame property	Rule
T	$\forall x\ xRx$ reflexivity	$\dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$
4	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ trans.	$\dfrac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}$
E	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclid.	$\dfrac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
B	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}$
3	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$	$\dfrac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta \quad zRy, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$
D	$\forall x \exists y\ xRy$ seriality	$\dfrac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ y$
2	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$	$\dfrac{yRw, zRw, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\ w$

Structural properties of the basic system

- **Theorem** *The structural rules are admissible in extensions of* **G3c** *with regular or geometric rules satisfying the closure condition. Weakening and contraction are* height-preserving *admissible. All the rules are invertible*.
- The same structural properties hold in **G3K**, in addition substitution of labels is height-preserving admissible.

Results

Let **G3K*** be any extension of **G3K** with universal or geometric rules rules for the accessibility relation.

- All the structural rules–weakening, contraction, and cut–are admissible in the system **G3K***.
- The characteristic axioms are derivable.
- The necessitation rule is admissible.
- Indirect completeness, through equivalence with the corresponding axiomatic system.
- Direct completeness: proof search in the system either gives a derivation or a Kripke countemodel.
- Answers to questions of undefinability through conservativity theorems.
- Answers to decidability questions through algorithms of terminating proof search.

Intuitionistic logic

Kripke semantics for intuitionistic logic was inspired by the modal embedding. We use the semantics to get a syntactic proof of the embedding.

The accessibility relation is a *pre-order*

$\forall x.x \leqslant x$ (Ref) $\qquad \forall xyz(x \leqslant y \& y \leqslant z \supset x \leqslant z)$ (Trans)

Forcing relation as in basic modal logic except for implication:

$x \Vdash A \supset B$ iff $\forall y(x \leqslant y \ \& \ y \Vdash A \supset y \Vdash B)$

Gives the rules:

$$\frac{x \leqslant y, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \Rightarrow \Delta} \ L\supset$$

$$\frac{x \leqslant y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \supset B} \ R\supset$$

Rule $R\supset$ has the condition that y is not in the conclusion.

Intuitionistic logic

Initial sequents of **G3K** modified to have *monotonicity* of the forcing relation. Enough to have monotonicity with respect to atomic formulas

$$x \leqslant y, x : P, \Gamma \Rightarrow \Delta, y : P$$

Monotonicity w.r.t. arbitrary formulas admissible.

The mathematical rules for the accessibility relation $\leqslant$ are the rules *Ref* and *Trans*.

The system **G3I**

Initial sequents: $x \leqslant y, x : P, \Gamma \Rightarrow \Delta, y : P$

Logical rules: As in **G3K** for &, $\vee$, $\bot$,

$$\frac{x \leqslant y, x : A \supset B, \Gamma \Rightarrow y : A, \Delta, \quad x \leqslant y, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \Rightarrow \Delta} \; L\supset$$

$$\frac{x \leqslant y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \supset B} \; R\supset$$

Order rules:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; \mathit{Ref} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta} \; \mathit{Trans}$$

Rule $R\supset$ has the condition that y must not be in Γ, Δ.

Intermediate logics (joint work with Roy Dyckhoff)

G3I can be extended with rules expressing additional properties of the pre-order $\leqslant$ exactly as done for modal logic.

For example, Gödel-Dummett logic has a *strongly connected* accessibility relation $\forall xyz((x \leqslant y \; \& \; x \leqslant z) \supset (y \leqslant z \vee z \leqslant y))$. This becomes the rule

$$\frac{x \leqslant y, x \leqslant z, y \leqslant z, \Gamma \Rightarrow \Delta \quad x \leqslant y, x \leqslant z, z \leqslant y, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta} \; GD$$

Add the rule to **G3I** and obtain a (labelled) sequent system for Gödel-Dummett logic.

Denote by **G3I*** any extension of **G3I** with rules following the geometric rule scheme. Below more examples of intermediate logics.

Structural properties of **G3I***

All sequents of the following form are derivable in **G3I***:

1. $x \leqslant y, x : A, \Gamma \Rightarrow \Delta, y : A$
2. $x : A, \Gamma \Rightarrow \Delta, x : A$

The substitution rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ (y/x)$$

is hp-admissible in **G3I***.

The rules of *Weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : A}\ RW \qquad \frac{\Gamma \Rightarrow \Delta}{x \leqslant y, \Gamma \Rightarrow \Delta}\ LW_{\leqslant}$$

are hp-admissible in **G3I***.

Structural properties of **G3I*** (cont.)

All the rules of **G3I*** are hp-invertible.

The rules of *Contraction*

$$\frac{x : A, x : A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ \textit{L-Ctr} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : A}{\Gamma \Rightarrow \Delta, x : A}\ \textit{R-Ctr}$$

$$\frac{x \leqslant y, x \leqslant y, \Gamma \Rightarrow \Delta}{x \leqslant y, \Gamma \Rightarrow \Delta}\ \textit{L-Ctr} \leqslant$$

are hp-admissible in **G3I***.

The *Cut* rule

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Cut}$$

is admissible in **G3I***.

The rule

$$\frac{\Rightarrow x\!:\!A \supset B \quad \Rightarrow x\!:\!A}{\Rightarrow x\!:\!B}\ MP$$

is admissible in **G3I***.

The axioms corresponding to the frame properties are derivable in **G3I***.

Each system in **G3I*** is equivalent to the intermediate logic obtained by adding to **Int** the axiom(s) that correspond((s)) to the frame property(ies)

$$\vdash_{Int+Ax} A \text{ iff } \mathbf{G3I}^*_{\mathbf{Ax}} \vdash \Rightarrow x : A$$

Interpolable logics: axioms, frame conditions, and rules

- **Jan** *Jankov-De Morgan Logic*: The relation $\leqslant$ is *directed* or *convergent*, i.e.

  $$\forall xyz((x \leqslant y \;\&\; x \leqslant z) \;\supset\; \exists w(y \leqslant w \;\&\; z \leqslant w)).$$

  Known as **KC** or as "logic of weak excluded middle".
  Axiomatised by $\neg A \vee \neg\neg A$ or $\neg(A\&B) \supset (\neg A \vee \neg B)$.

  $$\frac{y \leqslant w, z \leqslant w, x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, x \leqslant z, \Gamma \Rightarrow \Delta}\; Jan \quad (w \textit{ fresh})$$

- $\mathbf{Bd}_2$: Bounded *depth at most* 2

  $$\forall xyz((x \leqslant y \leqslant z) \;\supset\; (y \leqslant x \vee z \leqslant y)).$$

  Axiomatised by $A \vee (A \supset (B \vee \neg B))$.

  $$\frac{y \leqslant x, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta \quad x \leqslant y, y \leqslant z, z \leqslant y, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\; Bd_2$$

- **GSc**:Depth at most 2 and at most 2 final elements

  $$\forall xyz\exists v((x \leqslant v \,\&\, y \leqslant v) \vee (y \leqslant v \,\&\, z \leqslant v) \vee (x \leqslant v \,\&\, z \leqslant v)).$$

  This logic is axiomatised by, for example, $(A \supset B) \vee (B \supset A) \vee ((A \supset \neg B) \& (\neg B \supset A))$ and $A \vee (A \supset (B \vee \neg B))$. Rules are the rule for $\mathbf{Bd}_2$ and (with v fresh)

  $$\frac{x \leqslant v, y \leqslant v, \Gamma \Rightarrow \Delta \quad y \leqslant v, z \leqslant v, \Gamma \Rightarrow \Delta \quad x \leqslant v, z \leqslant v, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ F_2$$

- **Sm**: Smetanich logic, also known as $\mathbf{LC}_2$ or **HT**, the "logic of here and there", or as Gödel's 3-valued logic. The accessibility relation is strongly connected and has depth at most 2, i.e.

  $$\forall xyz((x \leqslant y \;\&\; y \leqslant z) \supset (y \leqslant x \vee z \leqslant y))$$

  Axiomatised by the **GD** axiom plus the $\mathbf{Bd}_2$ axiom, or, equivalently, by $(\neg B \supset A) \supset (((A \supset B) \supset A) \supset A)$. Rules as rules for **GD** and $\mathbf{Bd}_2$ above.

- **Cl** *Classical logic*: The accessibility relation is *symmetric*,

$$\forall xy(x \leqslant y \ \supset \ y \leqslant x).$$

Axiomatised by $A \vee \neg A$ or by $\neg\neg A \supset A$. The rule is

$$\frac{x \leqslant y, y \leqslant x, \Gamma \Rightarrow \Delta}{x \leqslant y, \Gamma \Rightarrow \Delta} \ Cl$$

Expressive power

Not only the interpolable ones can be treated by this method: Several variants of these logics are non-interpolable but still have geometric frame conditions:
Bd_n for $n > 2$ ("Bounded depth n") and btw_n for $n > 2$ (approximately, "bounded top-width" n). For $n = 3$, frame condition is For $n = 3$, for example, the latter's characteristic frame condition is the geometric implication

$$\forall x_0 x_1 x_2 x_3 (\bigwedge_{i=1}^{3} x_0 R x_i \supset \exists y (\bigvee_{i \neq j} x_i R y \ \& \ x_j R y)).$$

a geometric implication

Displayable logics

Properly displayable logics are captured by the extension with rules for geometric implications.
By Kracht's results, displayable extensions of basic modal logic are characterized by *primitive* frame conditions, of the form

$$(\forall^R)(\exists^R)A$$

$$\forall^R \ldots\ldots \forall u(wRu \supset Au)$$

$$\exists^R \ldots\ldots \exists u(wRu \,\&\, Au)$$

A is built from conjunction and disjunctions of
$w = u, wRu, wR^{-1}u$ where w and u not both in the scope of an $\exists$

Displayable logics (cont.)

Through standard conversions of first order logic, primitive frame conditions convert to the form of a geometric implication:

$$\forall w_1 (At_1(w_1) \supset (\forall w_2 At_2(w_1, w_2) \supset \ldots \exists u_1 (Bt_1(u_1) \& (\exists u_2 Bt_2(u_2) \& \ldots)))$$

$$\rightsquigarrow$$

$$\forall w_1 \forall w_2 \ldots \forall w_n (At_1(w_1) \& At_2(w_1, w_2) \supset \exists u_1 \exists u_2 Bt_1(u_1) \& Bt_2(u_1, u_2) \ldots .))$$

Not every geometric implication satisfies the additional conditions on variables, but those that are needed in our context do: the existential label licences additional steps if related to a universal label. If both labels in an atom were bound by the existential quantifier they would be both fresh in the geometric rule scheme and thus useless.

Analyticity

- Rules not analytic in a strong sense (each expression in a premiss is a subexpression of the conclusion).
- Less suffices to ensure the consequences of strong analiticity.
- We can transform derivations so that they satisfy the **Subterm property**: *All terms (variables, worlds) in a derivation are either eigenvariables or terms (variables, worlds) in the conclusion*.
- **Weak subformula property**: *All formulas in a derivation are either subformulas of (formulas in) the endsequent or atomic formulas of the form xRy.*
- Analyticity properties + structural constraints (e.g., no duplications) permit proofs of decidability by terminating proof search algorithms.
- Completeness proof (Negri 2009) gives a method for either finding a proof or generating a countermodel.

Given an extension **G3I*** of **G3I** with rules for $\leqslant$, we denote by **G3S4*** the corresponding extension of **G3S4**.

Soundness If **G3I*** $\vdash\ \Gamma \Rightarrow \Delta$ then **G3S4*** $\vdash\ \Gamma^{\Box} \Rightarrow \Delta^{\Box}$.

Proof: By induction on the structure of the derivation.

$x \leqslant y, \Gamma, x : P \Rightarrow y : P, \Delta \quad \rightsquigarrow$

$$\dfrac{\dfrac{\dfrac{\overline{\ldots, \Gamma^{\Box}, x : \Box P, z : P \Rightarrow z : P, \Delta^{\Box}}\ Ax}{x \leqslant y, y \leqslant z, x \leqslant z, \Gamma^{\Box}, x : \Box P \Rightarrow z : P, \Delta^{\Box}}\ L\Box}{x \leqslant y, y \leqslant z, \Gamma^{\Box}, x : \Box P \Rightarrow z : P, \Delta^{\Box}}\ \mathit{Trans}}{x \leqslant y, \Gamma^{\Box}, x : \Box P \Rightarrow y : \Box P, \Delta^{\Box}}\ R\Box$$

$$\frac{x \leqslant y, \Gamma, y : A \Rightarrow y : B, \Delta}{\Gamma \Rightarrow x : A \supset B, \Delta} \; R\supset \quad \rightsquigarrow \quad \dfrac{\dfrac{x \leqslant y, \Gamma^{\Box}, y : A^{\Box} \Rightarrow y : B^{\Box}, \Delta^{\Box}}{x \leqslant y, \Gamma^{\Box} \Rightarrow y : A^{\Box} \supset B^{\Box}, \Delta^{\Box}} \; R\supset}{\Gamma^{\Box} \Rightarrow x : \Box(A^{\Box} \supset B^{\Box}), \Delta^{\Box}} \; R\Box$$

$L\supset$ similar; conjunction, disjunction and absurdity routine.

Frame rules identical in the two systems, so nothing to prove for them.

Faithfulness If **G3S4*** $\vdash\ \Gamma^{\Box} \Rightarrow \Delta^{\Box}$ then **G3I*** $\vdash\ \Gamma \Rightarrow \Delta$.

Follows as a special case from:

If Γ, Δ are multisets of labelled formulas (with relational atoms also possibly in Γ) and Γ', Δ' are multisets of labelled atomic formulas, and **G3S4*** $\vdash\ \Gamma^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'$, then **G3I*** $\vdash\ \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Proof: By induction on the derivation.
By induction on the derivation of $\Gamma^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'$. If it is an initial sequent, then some atom $x : P$ is in Γ' and in Δ'; the conclusion then follows in **G3I*** by *Refl* from the initial sequent $x \leqslant x, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$. If it is a conclusion of $L\bot$, so also is $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.
If it is derived by a rule for & or for $\vee$, the inductive hypothesis applies to the premisses and then the corresponding rule in **G3I*** gives the conclusion.

If it is derived by a modal rule, the principal formula, being a translated formula, can only be of the form $\Box P$ or of the form $\Box(A^{\Box} \supset B^{\Box})$. There are four cases:

- 1. With $\Box P$ principal on the left, the step

$$\frac{x \leqslant y, y : P, x : \Box P, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'}{x \leqslant y, x : \Box P, \Gamma''^{\Box}, \Gamma' \Rightarrow \Delta^{\Box}, \Delta'}\ L\Box$$

  is translated to the admissible **G3I*** step

$$\frac{x \leqslant y, y : P, x : P, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}{x \leqslant y, x : P, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}$$

- 2. With $\Box P$ principal on the right, the step (with y fresh)

$$\frac{x \leqslant y, \Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', y : P}{\Gamma^{\Box}, \Gamma' \Rightarrow \Delta''^{\Box}, \Delta', x : \Box P}\ R\Box$$

  is translated (using admissibility of substitution) to the **G3I*** steps

$$\cfrac{\cfrac{x \leqslant y, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', y : P}{x \leqslant x, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : P}\ (x/y)}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : P}\ Refl$$

- 3. With $\Box(A^\Box \supset B^\Box)$ principal on the left, the step

$$\frac{x \leqslant y, x:\Box(A^\Box \supset B^\Box), y:A^\Box \supset B^\Box, \Gamma''^\Box, \Gamma' \Rightarrow \Delta^\Box, \Delta'}{x \leqslant y, x:\Box(A^\Box \supset B^\Box), \Gamma''^\Box, \Gamma' \Rightarrow \Delta^\Box, \Delta'} \; L\Box$$

Observe that $A^\Box \supset B^\Box$ is not a translated formula, nor an atomic one. By hp-invertibility of $L\supset$ in **G3S4*** we have

$$x \leqslant y, x:\Box(A^\Box \supset B^\Box), \Gamma''^\Box, \Gamma' \Rightarrow \Delta^\Box, \Delta', y:A^\Box$$

and

$$x \leqslant y, x:\Box(A^\Box \supset B^\Box), y:B^\Box, \Gamma''^\Box, \Gamma' \Rightarrow \Delta^\Box, \Delta'$$

Now the inductive hypothesis applies. We therefore have the derivation in **G3I***

$$\frac{\overset{\mathcal{I.H.}}{x \leqslant y, x:A \supset B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', y:A} \quad \overset{\mathcal{I.H.}}{x \leqslant y, x:A \supset B, y:B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}}{x \leqslant y, x:A \supset B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}$$

- 4. If $\Box(A^\Box \supset B^\Box)$ is principal on the right, the step is

$$\frac{x \leqslant y, \Gamma^\Box, \Gamma' \Rightarrow \Delta''^\Box, \Delta', y : A^\Box \supset B^\Box}{\Gamma^\Box, \Gamma' \Rightarrow \Delta''^\Box, \Delta', x : \Box(A^\Box \supset B^\Box)} \; R\Box$$

  from which, by hp-invertibility of $R\supset$ in **G3S4***, we have a derivation in **G3S4*** of

$$x \leqslant y, y : A^\Box, \Gamma^\Box, \Gamma' \Rightarrow \Delta''^\Box, \Delta', y : B^\Box$$

  to which the inductive hypothesis applies. An $R\supset$ step in **G3I*** gives us the desired conclusion.

QED

The proof is direct, constructive, syntactic, and uniform for all the intermediate logics obtained as geometric extensions of intuitionistic logic

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