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EXERCISES
BONUS PROBLEMS 6
APPLIED LOGIC
2009-11-25

Modal Logic and Computation Tree Logic

The BONUS PROBLEMS are marked (+) below. Solutions of these are to be handed in at the latest on 7 December. Maximum bonus for this set of problems is 2.5%. LCS below denotes the course book *Logic in Computer Science* by M. Huth and M. Ryan.

1. Suppose the intended meaning of $\Box\phi$ is 'Agent A knows ϕ'. Formalise the following statements in modal logic:

   (a) If ϕ is true, then it is consistent with what A knows, that A knows it.

   (b) If it is consistent with what A knows that ϕ, and it is consistent with what A knows that ψ, then it is consistent with what A knows that $\phi \wedge \psi$.

   (c) If A knows ϕ, then it is consistent with what A knows that ϕ.

   (d) If it is consistent with what A knows that it is consistent with what A knows that ϕ, then it is consistent with what A knows that ϕ.

   Which of these statements seems plausible principles concerning knowledge and consistency.

2. Show that the following formulas are valid (with respect to the class of all frames)

   (a) $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$,

   (b) $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$,

   (c) $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$.

3. Consider the basic temporal language and the following frames $(\mathbb{Z}, <)$, $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$, where $<$, in each case, is the usual less-than relation. Which of the following formulas are valid on these frames?

   (a) $GGp \rightarrow p$,

   (b) $(p \wedge Hp) \rightarrow FHp$.

4. Show that the following formulas are non-valid by constructing a counterexample in each case:

(a) $\Box\bot$,

(b) $\Diamond p \to \Box p$,

(c) $p \to \Box\Diamond p$,

(d) $\Diamond\Box p \to \Box\Diamond p$,

(e) $\Box p \to p$.

5. (+) Show the following

   (a) Frame-validity of B: $\phi \to \Box\Diamond\phi$ corresponds to symmetry of R.

   (b) Frame-validity of D: $\Box\phi \to \Diamond\phi$ corresponds to R being serial.

6. Let $\mathbf{F}$ be a class of frames. Show that $\Lambda_{\mathbf{F}} = \{\varphi \mid \mathcal{F} \models \varphi \text{ for all } \mathcal{F} \in \mathbf{F}\}$ is a normal modal logic.

7. Consider a modal language with two boxes [1] and [2]. Show that $p \to [2]\langle 1\rangle p$ is valid on precisely those frames for the language that satisfy the condition
$$\forall xy(xR_2y \to yR_1x).$$
What sort of frames does $p \to [1]\langle 1\rangle p$ define?

8. Consider a language with three boxes $[1], [2]$ and $[3]$. Show that the modal formula $\langle 3\rangle p \leftrightarrow \langle 1\rangle\langle 2\rangle p$ is valid on a frame for this language if and only if the frame satisfies the condition
$$\forall xy(xR_3y \leftrightarrow \exists z(xR_1z \wedge zR_2y)).$$

9. (+) Consider a language with two boxes [1] and [2]. Prove that the class of frames in which $R_1 = R_2^*$, where R_2^* is the reflexive transitive closure of R_2, is defined by the formulas

   (a) $\langle 1\rangle p \to (p \vee \langle 1\rangle(\neg p \wedge \langle 2\rangle p))$,

   (b) $\langle 1\rangle p \leftrightarrow (p \vee \langle 2\rangle\langle 1\rangle p)$.

10. Suppose $\mathcal{T} = (T, <)$ is a bidirectional frame (where we write $y < x$ instead of $x\check{<}y$) such that $<$ is transitive, irreflexive and satisfies $\forall xy(x < y \vee x = y \vee y < x)$. Show that
$$\mathcal{T} \models \{G(Gp \to p) \to Gp, H(Hp \to p) \to Hp\}$$
implies that $\mathcal{T}$ is finite.

11. Show that Grzegorczyk's formula
$$\Box(\Box(p \to \Box p) \to p) \to p$$
characterizes the class of frames $\mathcal{F} = (W, R)$ satisfying

(*i*) R is reflexive,

(*ii*) R is transitive,

(*iii*) there are no infinite paths $x_0 R x_1 R x_2 R \dots$ such that for all i, $x_i \neq x_{i+1}$.

12. (+) Exercise 3.4.6 (a) and (b) (only CTL formulas) in Chapter 3 of LCS
13. Exercise 3.4.7 (f)-(h) in Chapter 3 of LCS
14. (+) Exercise 3.4.8 in Chapter 3 of LCS
15. (+) Exercise 3.4.10 (c)-(g) in Chapter 3 of LCS

Some of the exercises are taken from the book *Modal Logic* by Patrick Blackburn, Maarten de Rijke and Yde Venema, which is a very good reference for Modal Logic.