UPPSALA UNIVERSITET
Matematiska Institutionen
Anton Hedin

EXERCISES
BONUS PROBLEMS 5
APPLIED LOGIC
2009-10-26

Semantics of Intuitionistic Propositional Logic

The BONUS PROBLEMS are marked (+) below. Solutions of these are to be handed in at the latest on 2 November. Maximum bonus for this set of problems is 2.5%.

1. Recall that we defined a *lattice* as a partially ordered set $(L, \leq)$ in which every finite subset $F \subseteq L$ has both a least upper bound $\bigvee F$ and a greatest lower bound $\bigwedge F$, with respect to the partial order $\leq$. Another definition is given by the following:
   A *lattice* $(L, \wedge, \vee, 0, 1)$ consists of a set L with two distinguished elements 0 and 1, and two binary operations $\wedge$, $\vee$ satisfying the following laws

Idempotency	Commutativity	Associativity
$a \vee a = a$	$a \vee b = b \vee a$	$a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge a = a$	$a \wedge b = b \wedge a$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Absorption	Top	Bottom
$a \vee (a \wedge b) = a$	$a \vee 1 = 1$	$a \vee 0 = a$
$a \wedge (a \vee b) = a$	$a \wedge 1 = a$	$a \wedge 0 = 0$

   Show that the two definitions are equivalent. The latter (algebraic) definition can in fact be made more compact; show that Idempotency, $a \wedge 1 = a$ and $a \vee 0 = a$ follow from Absorption.

2. A lattice $(L, \wedge, \vee, 0, 1)$ is called *distributive* if the following two equalities hold

   D1: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

   D2: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

   Show that D2 follows from D1.

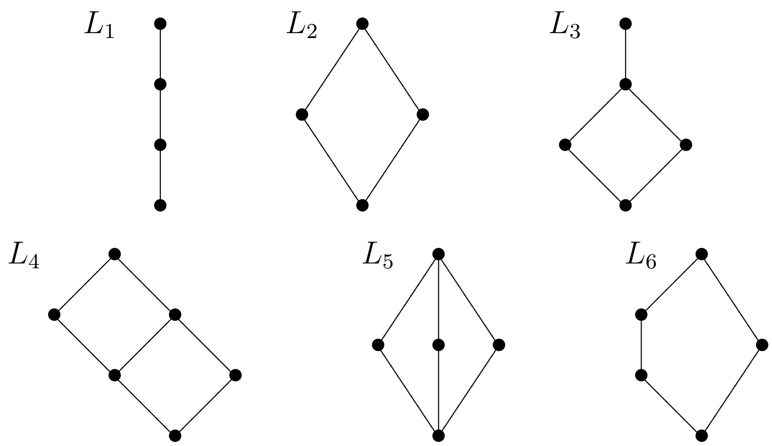
3. (+) Show that in a Heyting algebra: if $a \wedge b = \bot$ and $a \vee b = \top$, then $b = \neg a$. Thus every true complement is a pseudo-complement.

4. Show that every Boolean algebra is a Heyting algebra.

5. Show that every Heyting algebra in which $a \vee \neg a = \top$ for all a, is a Boolean algebra.

6. Let $(X, \mathcal{O})$ be a topological space. Show that the set $\mathcal{O}$ of open sets is a Heyting algebra with respect to intersection and union.

7. Consider the following lattices



Which of these are (i) distributive, (ii) Boolean algebras, (iii) Heyting algebras?

8. (+) For each of the lattices L_1-L_4, in exercise 7, find a finite set $S \subseteq L_i$ and a subset $M \subseteq \mathcal{P}(S)$ such that the Hasse diagram of $(M, \subseteq)$ is the same as that for the lattice L_i.

9. In IPC the negation $\neg A$ of a formula is defined as $A \to \bot$. Deduce the meaning of $i \Vdash \neg A$ for $i \in W$, some Kripke model $\mathcal{K} = (W, \leq, L)$.

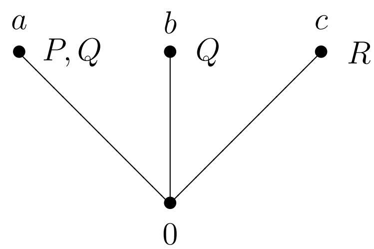
10. (+) Show (without giving derivations) that the following formulas are derivable in CPC but *not* derivable in IPC

    (i) $(P \to Q) \vee (Q \to P)$,

    (ii) $\neg(P \to Q) \to P \wedge \neg Q$, (*iii*) $(\neg Q \to \neg P) \to (P \to Q)$,

(*iv*) $((P \to Q) \to P) \to P$ (Peirce's law).

11. Consider the following IPC Kripke model $\mathcal{K}$:



Write down $\mathcal{K}$ formally as a tuple $(W, \leq, L)$. Construct from $\mathcal{K}$ an equivalent Heyting algebra model. That is, construct from the partial order $(W, \leq)$ a Heyting algebra H and from the labelling function L a H-valuation V, such that

$$V(A) = \top \iff 0 \Vdash A,$$

for all formulas A.

12. (+) Let $(P, \leq)$ be a partially ordered set. Show that $\mathsf{UC}(P) = \{U \subseteq P : U \text{ is upwards closed}\}$ is a Boolean algebra iff the partial order satisfies $p = q$ whenever $p \leq q$.