Exercise 12 (a)

We set $A(x) \equiv (\forall y)(y < x \to B(y))$. We then want to show (C-IND):

$$(\forall x)(A(x) \to B(x)) \to (\forall x)B(x).$$

If we can show that $A(0)$ and $(\forall x)(A(x) \to A(\mathbf{S}(x)))$ we have $(\forall x)A(x)$ using (IND) and then it is easy to see that (C-IND) follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(0) \land (\forall x)(A(x) \to A(\mathbf{S}(x))) \quad \text{IND}}{(\forall x)A(x)} \to E}{A(x)} \forall E \qquad \dfrac{(\forall x)(A(x) \to B(x))^1}{A(x) \to B(x)} \forall E}{B(x)} \to E}{(\forall x)B(x)} \forall I}{(\forall x)(A(x) \to B(x)) \to (\forall x)B(x)} \to I, 1$$

Hence, we start by showing $A(0)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\forall y)\neg y < 0}{\neg y < 0} \forall E \qquad y < 0^2}{\bot} \to E}{B(y)} \bot E}{y < 0 \to B(y)} \to I, 2}{A(0)} \forall I$$

Then we show the induction step (note that we are actually building a whole proof tree for (C-IND), which explains the use of assumption 1 in the following tree):

$$\frac{\text{H2}}{(\forall y)(y < \mathbf{S}(x) \leftrightarrow y = x \lor y < x)}\ \forall E$$

$$\frac{(\forall y)(y < \mathbf{S}(x) \leftrightarrow y = x \lor y < x)}{y < \mathbf{S}(x) \leftrightarrow y = x \lor y < x}\ \forall E$$

$$\frac{y < \mathbf{S}(x) \leftrightarrow y = x \lor y < x \qquad y < \mathbf{S}(x)^{4}}{y = x \lor y < x}\ \leftrightarrow E$$

$$\frac{\text{H4 with P=B}}{(\forall y)(x = y \land B(x) \to B(y))}\ \forall E$$

$$\frac{(\forall y)(x = y \land B(x) \to B(y))}{x = y \land B(x) \to B(y)}\ \forall E$$

$$\frac{(\forall x)(A(x) \to B(x))^{1}}{A(x) \to B(x)}\ \forall E$$

$$\frac{A(x) \to B(x) \qquad A(x)^{3}}{B(x)}\ \to E$$

$$\frac{x = y^{5} \qquad B(x)}{x = y \land B(x)}\ \land I$$

$$\frac{x = y \land B(x) \to B(y) \qquad x = y \land B(x)}{B(y)}\ \to E$$

$$\frac{A(x)^{3}}{y < x \to B(y)}\ \forall E$$

$$\frac{y < x^{6} \qquad y < x \to B(y)}{B(y)}\ \to E$$

$$\frac{y = x \lor y < x \qquad B(y) \qquad B(y)}{B(y)}\ \lor E, 5, 6$$

$$\frac{B(y)}{y < \mathbf{S}(x) \to B(y)}\ \to I, 4$$

$$\frac{y < \mathbf{S}(x) \to B(y)}{A(\mathbf{S}(x))}\ \forall I$$

$$\frac{A(\mathbf{S}(x))}{A(x) \to A(\mathbf{S}(x))}\ \to I, 3$$

$$\frac{A(x) \to A(\mathbf{S}(x))}{(\forall x)(A(x) \to A(\mathbf{S}(x)))}\ \forall I$$