UPPSALA UNIVERSITET
Matematiska institutionen
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PROV I MATEMATIK
Tillämpad logik DV1
2000-10-20

Skrivtid: 8 – 13. Inga hjälpmedel är tillåtna.

The hours for the test are 8.00 – 13.00. No utilities except writing materials are allowed. The maximum number of points for each problem is indicated within parentheses. For the grade "Godkänd" (passed) 18 points are required, for the grade "Väl Godkänd" (passed with distinction) 28 points will be required. The solutions may be written in English or Swedish.

1. Provide BHK-interpretations for the following formulas of intuitionistic logic.

   (a) $A \to (B \to A \wedge B)$,

   (b) $((\exists x\, A(x)) \to B) \to \forall x\, (A(x) \to B)$, where x is not free in B. (4p)

2. State the Gödel-Gentzen negative interpretation of $\forall, \wedge, \to$-formulas of classical logic into intuitionistic logic. Apply the interpretation to the formula

$$\forall x [\forall y\, (P(x) \vee R(x,y)) \to (P(x) \vee \forall y\, R(x,y))]$$

after rewriting $\vee$ in a suitable way. Here P and R are relation symbols. Show that the interpreted formula is provable in intuitionistic logic. (4p)

3.(a) Show that for a propositional variable P, the formula $\sim P \vee \sim\sim P$ is unprovable in intuitionistic logic.

   (b) Prove that for any Kripke model $\mathcal{M} = (W, \leq, \{V_t\}_{t \in W})$ where $(W, \leq)$ is a *linear* order,

$$\mathcal{M} \models (P \supset Q) \vee (Q \supset P).$$

   Here P and Q are propositional variables. (6p)

4. Show that the following term rewriting system for the Ackermann function

$$\begin{aligned} a(0,x) &\to S(x) \\ a(S(x),0) &\to a(x,S(0)) \\ a(S(x),S(y)) &\to a(x,a(S(x),y)) \end{aligned}$$

is strongly normalising, by using the lexicographical path ordering. (The signature is $\Sigma = \{0, S, a\}$.) Is the system complete? (5p)

5. State the introduction and elimination rules for the Π-types. Explain the role of the Π-type in the propositions-as-types. (4p)

(P.T.O.!)

6. Let $L = \{0, S, \leq\}$ and let $\mathcal{N} = (\mathbb{N}, 0, S, \leq)$ be the L-structure of natural numbers with the usual interpretation of these symbols.

(a) Eliminate, within $\mathcal{N}$, the quantifiers from the following L-formula $\varphi(x, z)$:

$$\forall t \exists y((x = S(S(y)) \wedge y \leq z) \vee S(t) = S(S(y))),$$

i.e. find a quantifier free L-formula $\psi(x, z)$ such that $\mathcal{N} \models \forall x \forall z\,(\varphi(x, z) \leftrightarrow \psi(x, z))$.

(b) Let $\mathcal{M} = (\mathbb{N}, \leq, \{V_t\}_{t \in \mathbb{N}})$ be a modal model where for each propositional variable P either $\{t \in \mathbb{N} : V_t(P) = 0\}$ or $\{t \in \mathbb{N} : V_t(P) = 1\}$ is finite. Show that whether $V_t(A) = 1$ is algorithmically decidable, for each modal formula A.

(6p)

7. Let L be a first order language with finitely many symbols. Suppose that T is a consistent L-theory without infinite models.

(a) Prove that T is complete if, and only if, $\mathcal{A} \cong \mathcal{B}$ for all models $\mathcal{A}$ and $\mathcal{B}$ of T. (Note: $\cong$ instead of the usual $\equiv$.)

(b) Indicate two possible methods for deciding when a closed formula φ is a theorem of T, provided that T complete, recursively axiomatized and without infinite models. (6p)

8. *The proof of the completeness theorem.*

(a) Find using the proof method for the completeness theorem a model falsifying the following sequent

$$\forall x\, \exists y\, R(x, y) \longrightarrow \forall x\, R(x, x).$$

(R is a binary relation symbol.)

(b) Suppose that the first order formula φ is quantifier free, contains no function symbols, and that its free variables are among $x_1, \ldots, x_n$. Prove that it is (algorithmically) decidable whether $\forall x_1 \cdots \forall x_n \varphi$ is valid or not. (5p)
