UPPSALA UNIVERSITET
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PROV I MATEMATIK
Tillämpad logik DV1
2001-10-17

Skrivtid: 9 – 14. Tillåtna hjälpmedel:

- Fullständighetssatsen för predikatlogiken via sekventkalkyl. Kompendium av I. Sigstam, Matematiska institutionen, Uppsala universitet.
- Term rewriting systems. Utdrag från artikel av W. Klop i Handbook of Logic in Computer science, vol. 2. Oxford University Press.

The hours for the test are 9.00 – 14.00. No utilities except the listed articles are allowed. The maximum number of points for each problem is indicated within parentheses. For the grade "Godkänd" (passed) 18 points are required, for the grade "Väl Godkänd" (passed with distinction) 28 points will be required. The solutions may be written in English or Swedish.

1. Provide BHK-interpretations for the following formulas

(a) $A \wedge ((A \wedge B) \to C) \to (B \to C)$

(b) $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$

(c) $\neg\exists x\, A(x) \to \forall x\, \neg A(x)$ (6p)

2. Which of the following formulas are provable in intuitionistic propositional logic? Give a proof or a counter-model.

(a) $\sim(P \vee Q) \supset (\sim P \wedge \sim Q)$

(b) $(P \supset Q) \supset (\sim P \vee Q)$ (5p)

3. State the Gödel-Gentzen negative interpretation of $\forall, \wedge, \to$-formulas of classical logic into intuitionistic logic, and the main theorem for this interpretation. Mention some applications or consequences of the intepretation. (4p)

4. State the introduction and elimination rules for the Σ-types. Explain the role of the Σ-type in the propositions-as-types interpretation. (4p)

5. Find using the proof method for the completeness theorem a model falsifying the following sequent

$$\exists x\, R(x,x) \longrightarrow \exists x\, \forall y\, R(x,y).$$

(R is a binary relation symbol.) (4p)

(Please turn over)

6. Show that the following term rewriting system is strongly normalising by using one of the orderings in Klop's article.

$$\begin{aligned} f(B(x,y),B(u,v)) &\rightarrow f(x,f(y,f(B(x,y),v))) \\ f(B(x,y),z) &\rightarrow f(x,B(z,z)) \\ f(L,x) &\rightarrow B(x,x) \end{aligned}$$

(The signature is $\Sigma = \{B(\cdot,\cdot), L, f(\cdot,\cdot)\}$.) Is the system complete? (5p)

7. Let L be the language $\{<,+,\cdot,0,1\}$. We consider the L-structure $\mathcal{N} = \langle \mathbb{N}; <,+,\cdot,0,1\rangle$ of natural numbers. Let $\varphi(x)$ be an L-formula with x as the only possible free variable. It defines the following subset of $\mathbb{N}$:

$$S_\varphi = \{n \in \mathbb{N} : \mathcal{N} \models \varphi(\overline{n})\}$$

where $\overline{n} = 0+1+1+\cdots+1$ (n ones). For instance, if $\varphi(x)$ is $x \cdot x = x \vee \overline{2000} < x$, then $S_\varphi = \{0,1,2001,2002,2003,\ldots\}$.

(a) Show that if $M \subseteq \mathbb{N}$ is finite, or its complement $\mathbb{N} \setminus M$ is finite, then $M = S_\varphi$ for some quantifier-free L-formula $\varphi(x)$.

(b)* Prove that for quantifier-free L-formulas $\varphi(x)$, either S_φ or $\mathbb{N} \setminus S_\varphi$ is finite.

(c) Does $\mathcal{N}$ have quantifier elimination? I.e. is it the case that for every L-formula $\varphi(x_1,\ldots,x_n)$, there is a quantifier-free L-formula $\psi(x_1,\ldots,x_n)$ such that

$$\mathcal{N} \models \forall x_1 \cdots \forall x_n\, [\varphi(x_1,\ldots,x_n) \leftrightarrow \psi(x_1,\ldots,x_n)].$$

(d) The same question as in (c) if the language L is restricted to $\{+,0,1\}$. (7p)

8. Let L be a first-order language. Suppose that T is a consistent L-theory. Prove that T is complete if, and only if, $\mathcal{A} \equiv \mathcal{B}$ for any two models $\mathcal{A}$ and $\mathcal{B}$ of T. (Recall that $\mathcal{A} \equiv \mathcal{B}$ is defined as: for all closed L-formulas φ:

$$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.)$$

Why are complete theories T useful in connection with automatic theorem proving? (5p)

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