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PROV I MATEMATIK
Tillämpad logik DV1
2002-10-17

Skrivtid: 14–19. Tillåtna hjälpmedel:

- Fullständighetssatsen för predikatlogiken via sekventkalkyl. Kompendium av I. Sigstam, Matematiska institutionen, Uppsala universitet.
- Term rewriting systems: supplementary notes by E.Palmgren.
- Logical background to the resolution method: supplementary notes by E.Palmgren.
- 3 handwritten pages of your own notes.

5 hours are allowed for the test. No utilities except the listed items above are allowed. The maximum number of points for each problem is indicated within parentheses. For the grade "Godkänd" (passed) 18 points are required, for the grade "Väl Godkänd" (passed with distinction) 28 points will be required. The solutions may be written in English or Swedish.

1. Provide BHK-interpretations for the following formulas

   (a) $A \wedge B \to B \vee A$

   (b) $(A \to (B \to C)) \to (A \wedge B \to C)$

   (c) $\neg \exists x\, A(x) \to \forall x\, \neg A(x)$ (5p)

2. Is the following formula provable in intuitionistic propositional logic? Give a proof or a counter-model.

$$\sim P \vee \sim\sim P \qquad (4p)$$

3. Describe, briefly, the idea of model-checking, and some uses. (3p)

4. Consider the modal model given by the set of week days

$$W = \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$$

where the accessibility relation $B(x, y)$: is x is the day *immediately before* y, where we assume $B(\text{Sunday}, \text{Monday})$. Note that the accessibility relation is cyclic but not transitive (draw a diagram). Is there any difference between the modal operators $\Box$ and $\Diamond$ in this case? The truth-values for P, Q and R are as follows on each day.

$t =$	**M**	**Tu**	**W**	**Th**	**F**	**Sa**	**Su**
$V_t(P) =$	1	1	1	1	1	0	1
$V_t(Q) =$	0	0	1	1	1	1	1
$V_t(R) =$	1	1	1	0	1	1	0

For each day of the week t_0 try to find a formula A mentioning only P, Q, R which is true exactly on this day, i.e. such that $V_t(A) = 1$ if, and only if, $t = t_0$. Or argue that no such formula exists. (5p)

(Please turn over)

5. Translate each of the formulas in Problem 1 into types of Martin-Löf type theory. Explain the role of the Π-type in the propositions-as-types interpretation. (4p)

6. Let $F = P_{\text{fin}}(\mathbb{N})$ be the set of finite subsets of $\mathbb{N}$.

(a) Show that $(F, \subset)$ (strict inclusion) is a well-founded relation.

(b) Show that $(F, \subseteq)$ is *not* a well-quasi order. (4p)

7. Use the resolution method to prove the formula

$$\exists x\, \forall y\, R(x, y) \to \exists x\, R(x, x),$$

where R is a binary relation symbol. Account for all the necessary pre-processing steps (negation, Skolemization) as well as the deduction of the empty clause. (5p)

8. Which of the following pairs of terms that can be unified? Compute their most general unifier, or explain why they do not unify.

(a) $f(x, g(f(x, y)))$ and $f(g(u), g(f(u, v)))$

(b) $h(x, k(x, y), k(x, y))$ and $h(k(u, v), u, k(w, v))$

(4p)

9. *Quantifier-elimination.* Consider the rational numbers $(\mathbb{Q}, <)$ with the usual order relation. We consider modal models where $\mathbb{Q}$ is the set of worlds, thought of as points in time, and $<$ is the accessibility relation, interpreted as "later". Note that time is not discrete here. Consider the open intervals in $\mathbb{Q}$, i.e. those that have the forms

$$(-\infty, b), (a, b), (a, \infty),$$

and singleton sets

$$\{a\}.$$

The subsets of $\mathbb{Q}$ that can be formed by taking unions of finitely many such intervals and singletons, are here called *basic sets.* For instance, $\emptyset$ is the empty union, $[a, b] = \{a\} \cup (a, b) \cup \{b\}$, $\mathbb{Q} = (-\infty, 1) \cup (-1, \infty)$. The union of two basic sets is clearly basic.

(a) Show that the complement of a basic set is a basic set.

(b) Show that if A is modal formula such that

$$\{t \in \mathbb{Q} : V_t(A) = 1\}$$

is a basic set, then

$$\{t \in \mathbb{Q} : V_t(\Box A) = 1\}$$

is also a basic set.

(c) Argue that we can algorithmically decide the truth of modal formula in this model, provided that $\{t \in \mathbb{Q} : V_t(P) = 1\}$ is basic for each propositional variable P occuring in the formula. (6p)

—— Lycka till! / Good luck! ——