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PROV I MATEMATIK
Tillämpad logik 10 hp
(Applied Logic)
2007-12-17

The allowed time for the test is 5 hours and you may use:

- all course literature and hand-outs from the course,
- your own notes.

The maximum number of points for each problem is indicated within parentheses. For the grade "Godkänd" (passed) 18 points are required, for the grade "Väl Godkänd" (passed with distinction) 28 points will be required. Standard requirements for ECTS grades apply. The solutions may be written in English or Swedish.

1. Decide which of the following formulas that are provable in intuitionistic propositional logic. Let $P, Q, R, \ldots$ denote propositional variables. In case the formula is provable give a proof in some common system. If it is unprovable, either give a counter model, or show in some other way that it is unprovable. Explain which systems and methods you are using.

(a) $\neg(P \vee Q) \supset \neg P \wedge \neg Q$.

(b) $\neg\neg P \vee \neg P$.

(c) $\neg(P \wedge Q) \supset \neg P \vee \neg Q$. (6p)

2. *Undecidability.* Which of the following problems are decidable?

(a) Given a first order formula φ, is φ provable in first-order logic?

(b) Given a finite set S of first order formulas, is S consistent?

(c) Given a first order formula φ, does φ have a model with at most 2^n elements, where n is the length of the formula? (5p)

3. Consider the following formula in Heyting arithmetic

$$(\exists x)\,(A(x) \vee B(x)) \supset (\exists x)\,A(x) \vee (\exists x)\,B(x).$$

(a) Find a lambda term witness to the validity of the formula under the BHK-interpretation.

(b) Translate the formula into a type of Martin-Löf type theory. (4p)

4. Translate the following formula using the Gödel-Gentzen translation

$$(*) \qquad (\forall x)[(\forall y)(P(x) \vee Q(x,y)) \supset P(x) \vee (\forall y)Q(x,y)].$$

Here P and Q are predicate symbols. The formula (*) is valid in classical logic. Why is the translation of (*) provable in intuitionistic logic? (4p)

5. Let ψ denote the formula (*) in Problem 4. Transform ψ to a set of clauses and prove it using the resolution method. (5p)

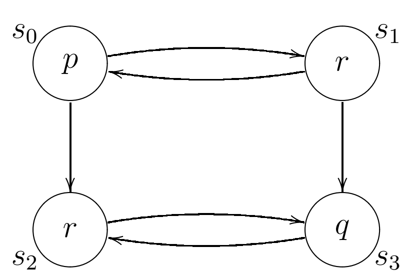
6. Describe briefly how a term rewriting system can be used to decide whether an equation is provable in an equational theory. Consider the following term rewriting system R over $\Sigma = \{0, +, \cdot, \mathsf{s}\}$ consisting of the following rules

$$\begin{aligned} x + 0 &\rightarrow x \\ x + \mathsf{s}(y) &\rightarrow \mathsf{s}(x+y) \\ x \cdot 0 &\rightarrow 0 \\ x \cdot \mathsf{s}(y) &\rightarrow x \cdot y + x \end{aligned}$$

What are its normal forms? Prove that R is a complete term rewriting system, by showing it is strongly normalising and has the following confluence property: for any terms r, s_1, s_2 with $r \rightarrow s_1$ and $r \rightarrow s_2$ there is some term t with $s_1 \rightarrow^* t$ and $s_2 \rightarrow^* t$. (6p)

7. Prove that the class of frames in which the accessibility relation is serial, is defined by the formula $\Box p \rightarrow \Diamond p$. That is, if $\mathcal{F} = (W, R)$ is a frame, prove that $\mathcal{F} \models \Box p \rightarrow \Diamond p$ if and only if R is serial. (A relation R is serial if it satisfies: $\forall x \exists y (xRy)$). (5p)

8. Let $\mathcal{M}$ be the following CTL model



Unwind the labeled transition system into an infinite tree starting at state s_0. For which states s_i, $i = 0, 1, 2, 3$ do we have

(a) $\mathcal{M}, s_i \models \mathrm{AG}(\mathrm{EF}\ r)$?

(b) $\mathcal{M}, s_i \models \mathrm{AG}(p \rightarrow \mathrm{EF}\ q)$?

(c) $\mathcal{M}, s_i \models q \vee (r \wedge \mathrm{EX}(\mathrm{E}[r\ \mathrm{U}\ q]))$? (5p)

—— Lycka till! / Good luck! ——