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EXERCISES 1
Tillämpad Logik DV1, ht-00
2000-09-04

Exercises [1] 1, Tillämpad Logik DV1

The following problems concern mainly Chapter 1-4 of "Konstruktiv logik" and Section 3 of "Oavgörbara problem i elementär aritmetik".

1. (A nonconstructive proof.) Let $S \subseteq \{0,1\}^*$ be an infinite set of strings such that if $u = a_1 a_2 \cdots a_n \in S$, then every initial segment $a_1 a_2 \cdots a_k$ of u $(0 \leq k < n)$ also belongs to S. Show that there is an inifinite sequence $b_1 b_2 b_3 \cdots$ such that all its finite initial segments belong to S.

2. Construct a typed lambda term $add2$ of type $\mathbb{N} \to \mathbb{N}$ that adds 2 to each natural number. Show that $\mathsf{apply}(add2, \mathsf{S}(0)) = \mathsf{S}(\mathsf{S}(\mathsf{S}(0)))$.

3. Construct a typed lambda term add of type $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ that implements addition. Show that $\mathsf{apply}(\mathsf{apply}(add, \mathsf{S}(0)), \mathsf{S}(0)) = \mathsf{S}(\mathsf{S}(0))$.

   Hint: Consider the type of add, i.e. add can be considered as function that given a number n, returns a function that adds n to any number. Higher functions are required.

4. For each formula below do the following:

   - Prove the formula using the rules of intuitionistic logic.
   - Construct a certificate for the formula using typed lambda calculus.

   (a) $A \to A$
   (b) $A \wedge B \to B \wedge A$
   (c) $A \vee B \to B \vee A$
   (d) $(A \to B) \wedge (B \to C) \to (A \to C)$
   (e) $(A \to \neg A) \to \neg A$
   (f) $A \to \neg\neg A$

5. In intuitionistic logic $\neg A$ is defined as $A \to \bot$. In the exercise above we saw that $A \to \neg\neg A$ has a BHK-interpretation. What would a BHK-interpretation of $\neg\neg A \to A$ be?

6. Does $\bot \to A$ have a certificate? Why or why not?.

7. (Definability) Show that the following subsets, relations or functions are definable in the elementary language over $\langle \mathbb{N}; +, \cdot, 0, 1 \rangle$ using as simple formulas as seems possible:

[1]Exercises 2-6 courtesy of Lars Lindqvist.

(a) x is odd

(b) $y = x(x+1)/2$

(c) $x \leq y$

(d) x divides y

(e) x is the sum of two prime numbers

(f) $z = \max(x, y)$

(g) $y = x!$

(h) $y = 2^{2^{2^{2}}}$

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