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EXERCISES 1
Tillämpad Logik DV1, ht-02
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Exercises 1, Tillämpad Logik DV1

The following problems concern mainly Sections 3-5 of "Oavgörbara problem i elementär aritmetik" and Chapter 1-4 of "Konstruktiv logik".

1. (Definability) Show that the following subsets, relations or functions are definable in the elementary language over $\langle \mathbb{N}; +, \cdot, 0, 1 \rangle$ using as simple formulas as seems possible:
   (a) x is odd
   (b) $y = x(x+1)/2$
   (c) $x \leq y$
   (d) x divides y
   (e) x is the sum of two prime numbers
   (f) $z = \max(x, y)$
   (g) $y = x!$
   (h) $y = 2^{2^{2^{2^2}}}$

2. (Definability and decidability) Recall that in automata theory one studies languages as subsets of strings over a fixed alfabet. Let $\Sigma = \{a, b\}$ be an alfabet, and let Σ^* be the set of finite strings. Thus

$$\Sigma^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

   Here ϵ is the empty string. Let & denote concatenation of strings, so $baba\&bba = bababba$. We may now regard $\langle \Sigma^*; \; \& \rangle$ as a first-order structure with concatenation as the only operation. Find elementary propositions (formulas) over $\langle \Sigma^*; \; \& \rangle$ that definies the following properties (note that $=$ may be used)
   (a) x is a substring of y
   (b) x is an empty string (you may not mention ϵ)
   (c) x is a string of length 1 (you may not mention 0 or 1) (Hint: use (a) and (b). How many substrings can such a string have?)
   (d) x is a string of length 4.

   Consider now an extended structure $\langle \Sigma^*; \; \&, *, a, b, \epsilon \rangle$ where a, b, ϵ are constants (so they may be mentioned in elementary propositions) and moreover there is a "string duplicator" $*$ that satisfies the following

$$\begin{array}{rcll} u * \epsilon & = & \epsilon & \text{(erase)} \\ u * (a\&v) & = & u * v & \text{(take a pause)} \\ u * (b\&v) & = & (u * v)\&u & \text{(make a copy).} \end{array}$$

Thus $ab * bab = abab$ and $ab * aa = \epsilon$.

(e) Prove that the structure $\langle \Sigma;\ \&, *, a, b, \epsilon \rangle$ is undecidable, by showing that *if* it was decidable, then we could decide $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ as well, contradicting a well-known theorem.

(f)** If we remove the duplication operation from the structure in (e), does it become decidable? I.e. is the structure $\langle \Sigma;\ \&, a, b, \epsilon \rangle$ decidable?

3. (Decidability in geometry) We start out from the fact that the structure $\mathcal{R} = \langle \mathbb{R}; <, +, \cdot, 0, 1 \rangle$ is decidable. The first task is to find suitable formulas over $\mathcal{R}$ that describe certain geometric objects. The second is to show that certain problems about such objects may be expressed by formulas. Then one can apply Tarski's theorem to conclude that the problem is in principle solvable by a computer. If one is lucky (and clever) the problem may actually be solvable using a system like Mathematica 4.0.

   (a) Convince yourself that 3-dimensional geometric objects like, spheres, cylinders, cubes, balls, beams and Volvo cars at various positions and angles are definable in the structure $\mathcal{R}$ as subsets of $\mathbb{R}^3$.

   (b) Suppose that $S, T \subseteq \mathbb{R}^3$ are geometric objects definable by the formulas $\psi_S(x, y, z)$ and $\psi_T(x, y, z)$ in $\mathcal{R}$. Find formulas expressing

      (i) *S and T intersects,*

      (ii) *S is contained in T,*

      (iii) *S is the complement of T,*

      (iv) *S and T are identical,*

      (v) *S is obtained from T by intersecting with some halfplane.*

   (c) First find a formula that expresses *The ball with center in $P_1 = (x_1, y_1, z_1)$ of radius d_1 is inside a sphere with center in P_2 and radius d_2.* Show that the following questions are solvable in principle: How many balls of radius 1 fit into a sphere of radius 2? Of radius 3? Of radius 4? Of radius 10^{100}?

4. (A nonconstructive proof.) Let $S \subseteq \{0, 1\}^*$ be an infinite set of strings such that if $u = a_1 a_2 \cdots a_n \in S$, then every initial segment $a_1 a_2 \cdots a_k$ of u $(0 \leq k < n)$ also belongs to S. Show that there is an infinite sequence $b_1 b_2 b_3 \cdots$ such that all its finite initial segments belong to S.

5. [1] Construct a typed lambda term $add2$ of type $\mathbb{N} \to \mathbb{N}$ that adds 2 to each natural number. Show that $\mathsf{apply}(add2, \mathsf{S}(0)) = \mathsf{S}(\mathsf{S}(\mathsf{S}(0)))$.

[1] Exercises 5-9 courtesy of Lars Lindqvist.

6. Construct a typed lambda term *add* of type $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ that implements addition. Show that $\mathsf{apply}(\mathsf{apply}(add, \mathsf{S}(0)), \mathsf{S}(0)) = \mathsf{S}(\mathsf{S}(0))$.

   Hint: Consider the type of *add*, i.e. *add* can be considered as function that given a number n, returns a function that adds n to any number. Higher functions are required.

7. For each formula below do the following:

   - Prove the formula using the rules of intuitionistic logic.
   - Construct a witness for the formula using typed lambda calculus.

   (a) $A \to A$

   (b) $A \wedge B \to B \wedge A$

   (c) $A \vee B \to B \vee A$

   (d) $(A \to B) \wedge (B \to C) \to (A \to C)$

   (e) $(A \to \neg A) \to \neg A$

   (f) $A \to \neg\neg A$

8. In intuitionistic logic $\neg A$ is defined as $A \to \bot$. In the exercise above we saw that $A \to \neg\neg A$ has a BHK-interpretation. What would a BHK-interpretation of $\neg\neg A \to A$ be?

9. Does $\bot \to A$ have a witness? Why or why not?.

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