UPPSALA UNIVERSITET
Matematiska institutionen
Erik Palmgren

Exercises 2
Tillämpad Logik DV1, ht-02
2002-09-10

Exercises 2

1. Consider the modal model given by the set of week days

$$W = \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}, \}$$

where the accessibility relation $R(x, y)$ is x is no later than y. The truth-values for P, Q and R are as follows on each day.

$t =$	**M**	**Tu**	**W**	**Th**	**F**	**Sa**	**Su**
$V_t(P) =$	1	0	1	1	1	0	1
$V_t(Q) =$	0	0	1	1	0	1	1
$V_t(R) =$	1	0	1	0	0	0	0

A possible interpretation of symbols: The activities of Sergeant Snåårskog: P drinks all day, R drives a tank, Q cleans his guns. Compute the truth-value of each of the following formulas in the model just defined.

(a) $V_{\mathbf{M}}(\Diamond(Q \wedge R))$
(b) $V_{\mathbf{Th}}(\Diamond(Q \wedge R))$
(c) $V_{\mathbf{M}}(\Box(R \to P))$
(d) $V_{\mathbf{M}}(\Box(P \to \Diamond Q))$
(e) $V_{\mathbf{M}}(\Box(P \wedge \Diamond Q))$
(f) $V_{\mathbf{M}}(\Diamond\Box Q))$
(g) $V_{\mathbf{M}}(\Diamond\Box R))$
(h) Is it possible to find a formula $A = A(P, Q, R)$ which holds precisely on Thursday, i.e. such that $V_t(A) = 1$ is holds precisely when $t = \mathbf{Th}$.
(i) For which week days is it possible to find such formulas?

2. Decide which of the following formulae are provable in intuitionistic propositional logic. For each formula give a proof in natural deduction or provide a Kripke model which shows that it is unprovable. (The symbols $\to$ and $\neg$ of Ch. 4 are here denoted $\supset$ and $\sim$.)

(a) $(((P \supset Q) \supset Q) \supset Q) \supset (P \supset Q)$
(b) $(P \supset Q) \vee (Q \supset P)$
(c) $\sim\sim P \supset (P \vee \sim P)$

3. Consider a modal logic model $\mathcal{M} = (\mathbb{N}, \leq, \{V_t\}_{t\in\mathbb{N}})$ where H_t $(t = 0, 1, 2, \ldots)$ are proposition variables such that

$$V_t(H_s) = \begin{cases} 1 & \text{if } t \geq s, \\ 0 & \text{if } t < s. \end{cases}$$

Let P be a propositional variable. Express the following using modal formulas that use a fixed number of H-variables (whose number is independent of m and n).

   (a) P holds at all time points in the interval $[m, n] = \{m, m+1, m+2, \ldots, n\}$. (I.e. find a formula A containing P such that $\mathcal{M} \models A$ iff for all $t \in [m, n]$: $V_t(P) = 1$.)

   (b) P holds at some point in time in the interval $[m, n]$.

4. Define for every $k \in \mathbb{N}$ a relation R_k on $\mathbb{N}$ such that its corresponding operator $\Box_k$ in a modal logic model $(\mathbb{N}, \{R_k\}_{k\in\mathbb{N}}, \{V_t\}_{t\in\mathbb{N}})$ behaves as

$$V_t(\Box_k A) = 1 \iff V_t(A) = V_{t+1}(A) = \cdots = V_{t+k}(A) = 1.$$

What is the intuitive interpretation of this operator? How does $\Diamond_k$ work? Try to give some general laws for these operators and examples of what they can express.

5. Let $\mathcal{M} = (W, R, \{V_t\}_{t\in W})$ be a modal logic model. Let P be a propositional variable.

   (a)* Show that if $\mathcal{M} \models \Box P \to P$, for each possible valuation V_t, then R must be reflexive.

   (b)* Show that if $\mathcal{M} \models \Box P \to \Box\Box P$, for each possible valuation V_t, then R must be transitive.

6. Suppose that $\mathcal{M} = (W, R, \{V_t\}_{t\in W})$ is a modal logic model. Let A be an arbitrary modal formula.

   (a) Show that if R is reflexive, then $\mathcal{M} \models \Box A \to A$.

   (b) Show that if R is transitive, then $\mathcal{M} \models \Box A \to \Box\Box A$.

   (c) Show that if R is symmetric, then $\mathcal{M} \models \Diamond\Box A \to A$.

   (d) Show that if R is an equivalence relation, then $\Box$ satisfies the introspection axioms in $\mathcal{M}$.

   (e)* Show that if R is a linear order, then $\mathcal{M} \models \Diamond\Box\Diamond A \leftrightarrow \Box\Diamond A$

———