UPPSALA UNIVERSITET
Matematiska institutionen
Erik Palmgren

Exercises 4
Tillämpad Logik DV1, ht-00
2001-09-14

Exercises 4

1. Decide which of the following formulae are provable in intuitionistic propositional logic. For each formula give a proof in natural deduction or provide a Kripke model which shows that it is unprovable. (The symbols $\to$ and $\neg$ of Ch. 4 are here denoted $\supset$ and $\sim$.)

   (a) $(((P \supset Q) \supset Q) \supset Q) \supset (P \supset Q)$

   (b) $(P \supset Q) \vee (Q \supset P)$

   (c) $\sim\sim P \supset (P \vee \sim P)$

2. Consider a modal logic model $\mathcal{M} = (\mathbb{N}, \leq, \{V_t\}_{t \in \mathbb{N}})$ where H_t $(t = 0, 1, 2, \ldots)$ are proposition variables such that

$$V_t(H_s) = \begin{cases} 1 & \text{if } t \geq s, \\ 0 & \text{if } t < s. \end{cases}$$

   Let P be a propositional variable. Express the following using modal formulas that use a fixed number of H-variables (whose number is independent of m and n).

   (a) P holds at all time points in the interval $[m, n] = \{m, m+1, m+2, \ldots, n\}$. (I.e. find a formula A containing P such that $\mathcal{M} \models A$ iff for all $t \in [m, n]$: $V_t(P) = 1$.)

   (b) P holds at some point in time in the interval $[m, n]$.

3. Define for every $k \in \mathbb{N}$ a relation R_k on $\mathbb{N}$ such that its corresponding operator $\Box_k$ in a modal logic model $(\mathbb{N}, \{R_k\}_{k \in \mathbb{N}}, \{V_t\}_{t \in \mathbb{N}})$ behaves as

$$V_t(\Box_k A) = 1 \iff V_t(A) = V_{t+1}(A) = \cdots = V_{t+k}(A) = 1.$$

   What is the intuitive interpretation of this operator? How does $\Diamond_k$ work? Try to give some general laws for these operators and examples of what they can express.

4. Let $\mathcal{M} = (W, R, \{V_t\}_{t \in W})$ be a modal logic model. Let P be a propositional variable.

   (a)* Show that if $\mathcal{M} \models \Box P \to P$, for each possible valuation V_t, then R must be reflexive.

   (b)* Show that if $\mathcal{M} \models \Box P \to \Box\Box P$, for each possible valuation V_t, then R must be transitive.

5. Suppose that $\mathcal{M} = (W, R, \{V_t\}_{t \in W})$ is a modal logic model. Let A be an arbitrary modal formula.

   (a) Show that if R is reflexive, then $\mathcal{M} \models \Box A \to A$.

   (b) Show that if R is transitive, then $\mathcal{M} \models \Box A \to \Box\Box A$.

(c) Show that if R is symmetric, then $\mathcal{M} \models \Diamond\Box A \rightarrow A$.

(d) Show that if R is an equivalence relation, then $\Box$ satisfies the introspection axioms in $\mathcal{M}$.

(e)* Show that if R is a linear order, then $\mathcal{M} \models \Diamond\Box\Diamond A \leftrightarrow \Box\Diamond A$

6. Let $\mathcal{M}$ be a modal logic model where the accessibility relation is a linear order. Let $u \in \{\Box, \Diamond\}^*$ denote an arbitrary string of modal operators. If A is a modal formula, then $u\,A$ is a modal formula. Show using Exercise 5 that $u\,A$ is is equivalent to one of the formulae

$$A \qquad \Box A \qquad \Diamond A \qquad \Box\Diamond A \qquad \Diamond\Box A.$$

———