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Exercises 4
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Exercises 4

1. * Let E_{ab} be the equational theory of abelian groups (cf. Example 1.8, page 5, in handout "Term rewriting systems"). For variables $x_1, x_2, \ldots, x_k$ and integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ show that

$$E_{ab} \vdash_{\text{eq}} x_1^{m_1} \cdots x_k^{m_k} = x_1^{n_1} \cdots x_k^{n_k} \Leftrightarrow m_1 = n_1, \ldots, m_k = n_k.$$

(Hint: for the direction ($\Rightarrow$) use the completeness result (Corollary 1.6) and consider the additive group $\mathbb{Z}^k$ and an assignment $\rho(x_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the i:th place). Discuss how this together with the observations in Example 1.8 gives a decision procedure for E_{ab}.

2. Determine the mgu of $f(h(z, y), y)$ and $f(h(x, g(u)), g(x))$ using the algorithm of Martelli-Montanari.

3. The following term rewriting system R was noted to be confluent in Klop pp. 46 – 48. The signature is $\Sigma = \{e, I, \cdot\}$.

$$\begin{array}{rcl}
e \cdot x & \to & x \\
I(x) \cdot x & \to & e \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
I(x) \cdot (x \cdot z) & \to & z \\
y \cdot e & \to & y \\
I(I(y)) & \to & y \\
I(e) & \to & e \\
y \cdot I(y) & \to & e \\
y \cdot (I(y) \cdot x) & \to & x \\
I(x \cdot y) & \to & I(y) \cdot I(x)
\end{array}$$

Given that R is a complete term rewriting system for the equational theory E of groups. Use the term rewriting system to decide which of the following equations are provable in E. Explain why this method works!

x, y, z below are distinct variables.

(a) $x \cdot y = y \cdot x$

(b) $I(x \cdot y) \cdot x = x \cdot I(y \cdot x)$

(c) $(I(x) \cdot y) \cdot (I(x) \cdot y) = I(x \cdot x) \cdot (y \cdot y)$

(d) $I(x \cdot (I(y \cdot z) \cdot I((x \cdot y) \cdot z))) = (y \cdot (I(x) \cdot z)) \cdot (x \cdot (y \cdot z))$

4. Describe the normal forms in $\mathrm{Ter}(\Sigma)$ with respect to the term rewriting system R of Exercise 3.

5. Use the method from the proof of the completeness theorem (see Sigstam's compendium) to prove or refute the following sequents. P, Q are unary predicate symbols, R is a 3-ary predicate symbol.

   (a) $\forall x\, (P(x) \supset Q(x)) \longrightarrow \exists x\, (P(x) \wedge Q(x))$.

   (b) $\forall x \exists y \forall z\, R(x, y, z) \longrightarrow \exists x \exists y \exists z\, R(x, y, z)$.

6. * Argue that if $\varphi \longrightarrow \psi$ is a valid sequent whose both sides only contain the logical operators $\forall, \exists, \wedge$, then the sequent has a proof in the sequent calculus which only use the rules for these operators and logical axioms of the form $C, \Gamma \longrightarrow \Delta, C$. Consider an arbitrary sequent $\varphi \longrightarrow \psi$. Discuss whether there are some natural restrictions on the form of φ and ψ which guarantee that the search process in the proof of the completeness theorem stops after finitely many steps (with a proof of counter model as a result).

7. Let P be a binary predicate symbol. Let φ be the formula

$$\forall x\, \neg P(x, x) \wedge \forall xyz\, (P(x, y) \wedge P(y, z) \rightarrow P(x, z)) \wedge \forall x\, \exists y\, P(x, y).$$

   Show that φ has only infinite models.

8. A graph is a non-empty set with a reflexive and symmetric relation. The graph is connected if every pair of nodes can be connected by a finite sequence of edges. Let L be the language consisting of the binary relation R. Is there a closed L-formula φ such that for all L-structures $\mathcal{A}$:

$$\mathcal{A} \models \varphi \iff \mathcal{A} \text{ is a connected graph ?}$$

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