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Exercises 5
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Exercises 5

1. Let E_{ab} be the equational theory of abelian groups (cf. Example 1.8, page 5, in handout "Term rewriting systems"). For variables $x_1, x_2, \ldots, x_k$ and integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ show that

$$E_{ab} \vdash_{\text{eq}} x_1^{m_1} \cdots x_k^{m_k} = x_1^{n_1} \cdots x_k^{n_k} \Leftrightarrow m_1 = n_1, \ldots, m_k = n_k.$$

   (Hint: for the direction ($\Rightarrow$) use the completeness result (Corollary 1.6) and consider the additive group $\mathbb{Z}^k$ and an assignment $\rho(x_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the i:th place). Discuss how this together with the observations in Example 1.8 gives a decision procedure for E_{ab}.

2. Determine the mgu of $f(h(z, y), y)$ and $f(h(x, g(u)), g(x))$ using the algorithm of Martelli-Montanari.

3. The following term rewriting system R was noted to be confluent in Klop pp. 46 – 48. The signature is $\Sigma = \{e, I, \cdot\}$.

$$\begin{array}{rcl}
e \cdot x & \to & x \\
I(x) \cdot x & \to & e \\
(x \cdot y) \cdot z & \to & x \cdot (y \cdot z) \\
I(x) \cdot (x \cdot z) & \to & z \\
y \cdot e & \to & y \\
I(I(y)) & \to & y \\
I(e) & \to & e \\
y \cdot I(y) & \to & e \\
y \cdot (I(y) \cdot x) & \to & x \\
I(x \cdot y) & \to & I(y) \cdot I(x)
\end{array}$$

   Prove that R is strongly normalising by using the *lexicographic path ordering* $\Rrightarrow$ (see Klop, p. 34). Replace the function symbols in Σ by different numbers and show that to each rule $s \to t$ there corresponds a reduction $s \Rrightarrow^+ t$.

4. Given that R of the previous exerecise is a complete term rewriting system for the equational theory E of groups. Use the term rewriting system to decide which of the following equations are provable in E. x, y, z are distinct variables.

(a) $x \cdot y = y \cdot x$

(b) $I(x \cdot y) \cdot x = x \cdot I(y \cdot x)$

(c) $(I(x) \cdot y) \cdot (I(x) \cdot y) = I(x \cdot x) \cdot (y \cdot y)$

(d) $I(x \cdot (I(y \cdot z) \cdot I((x \cdot y) \cdot z))) = (y \cdot (I(x) \cdot z)) \cdot (x \cdot (y \cdot z))$

5. Describe the normal forms in $\mathrm{Ter}(\Sigma)$ with respect to the term rewriting system R of Exercise 4.

6. Consider the following term rewriting system R over $\Sigma = \{0, S, A, M\}$:

$$\begin{aligned} A(x,0) &\to x \\ A(x,S(y)) &\to S(A(x,y)) \\ M(x,0) &\to 0 \\ M(x,S(y)) &\to A(M(x,y),x). \end{aligned}$$

Show that R is a complete term rewriting system.

7. Discuss how schemes of primitive recursion over data structures (natural numbers, trees) may be formulated so that the system can be shown to be complete using the methods presented in Chapter 2 of Klop. Is there some simple criterion on the arguments of a primitive recursive definition that guarantees completeness? Example of a function f defined by primitive recursion on trees:

$$\begin{aligned} f(\mathsf{leaf}, z) &\to g(z) \\ f(\mathsf{node}(x,y), z) &\to h(x,y,z,f(x,z),f(y,z)). \end{aligned}$$

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