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Exercises 7

1. Determine the Skolem conjunctive normal form for each of the following formulas, and thereby its equivalent set of clauses. P and Q are predicate symbols.

   (a) $\forall u\, [\forall x\, \neg\exists y\, P(x,y) \to \neg\forall x\, (\exists z\, P(x,y) \to \forall y\, P(u,y))]$.

   (b) $\forall x\, (\exists y\, P(x,y) \leftrightarrow \exists z\, Q(z,x))$.

2. Deduce using resolution the empty clause ($\Box$) from the following set of clauses

   (a) $\{\neg P(x), P(x) \vee \neg Q(x), P(x) \vee \neg R(x), Q(x) \vee R(x)\}$,

   (b) $\{\neg P(x,y), P(x,y) \vee \neg Q(x,z) \vee \neg P(z,y), P(x,y) \vee \neg Q(x,y), Q(a,b), Q(b,c)\}$.

3. Deduce using resolution the empty clause ($\Box$) from the following set of clauses

   $$S = \{Q(x) \vee P(f(x)) \vee P(v), R(g(y),y) \vee \neg P(y), \neg R(g(f(z)),u), \neg Q(c)\}.$$

   For what $n \geq 0$ does $\Box \in \mathrm{Res}^n(S)$ hold? (See notation in Das.)

4. Let $\varphi(x_1,\ldots,x_m,y_1,\ldots,y_n)$ be a quantifier free formula without function symbols or the identity relation. Show that it is algorithmically decidable whether

   $$\forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n \varphi(x_1,\ldots,x_m,y_1,\ldots,y_n)$$

   has a proof or is refutable.[1] (Compare to Exercise 6.2.)

5. Show that the most general unifier of

   $$f(g(x_1,x_1), g(x_2,x_2),\ldots,g(x_n,x_n)) \text{ and } f(x_2,x_3,\ldots,x_{n+1})$$

   has 2^n occurrences of the variable x_1. (This shows that it is necessary to represent terms in some efficient way, for instance as DAGs). There is a indeed a linear time unification algorithm.[2])

6. Find all integers u such that

   $$(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})(y + x < 6 \wedge y < x + 3 \wedge 2x < 5y \to x < u \wedge y < u)$$

   using Skolem's quantifier elimination method. Check your answer using elementary geometric reasoning.

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[1] This was proven by Thoralf Skolem 1919, before the completeness theorem was formulated.

[2] A. Martelli, U. Montanari: *An efficient unification algorithm.* ACM Transactions on Programming Languages and Systems 4(1982), 258 – 281.