1 Classical and intuitionstic proofs[1]

We have seen that proofs in intuitionistic logic have the remarkable property that programs may be extracted from them. Many naturally occurring proofs, in mathematical text books or journals, rely as they stand on some use of classical logic. A natural question is whether there is some mechanical method for translating classical proofs to constructive proofs. It is clear that such methods must have some limitations in view of the counter examples of the previous chapters. Kurt Gödel and Gerhard Gentzen showed that there is a method for purely logical proofs, and certain simple theories, provided the proposition proved A may be substituted by a classically equivalent proposition A^*. This substitute may not have the same meaning from a constructive point of view.

Classical predicate logic is obtained by adding the *reductio ad absurdum* (RAA) axiom scheme to intuitionistic predicate logic: i.e. adding

$$\vdash_X \neg\neg A \supset A$$

for each formula A with free variables among X. In classical logic, the logical constants (connectives and quantifiers) $\exists$ and $\vee$ are actually unnecessary, since $A \vee B$ is equivalent to $\neg(\neg A \wedge \neg B)$ and $(\exists x)C$ is equivalent to $\neg(\forall x)\neg C$. These equivalences do not hold in intuitionistic logic. We define two new logical constants $\exists^{\mathrm{c}}$, the *classical existence* quantifier, and $\vee^{\mathrm{c}}$ the *classical disjunction*.

$$A \vee^{\mathrm{c}} B \equiv \neg(\neg A \wedge \neg B) \quad (1)$$

$$(\exists^{\mathrm{c}} x)C \equiv \neg(\forall x)\neg C \quad (2)$$

A formula of predicate logic where the only logical constants used are $\perp, \wedge, \rightarrow$ and $\forall$, is called a *non-existential formula.* (We may informally think of $A_1 \vee A_2$ as a kind of existence statement, e.g. $(\exists i \in \{1,2\})A_i$.)

Define the *Gödel-Gentzen negative translation* $(\cdot)^*$ by recursion on first order formulae:

- $\perp^* = \perp$,
- $R(t_1,\ldots,t_n)^* = \neg\neg R(t_1,\ldots,t_n)$, if R is a predicate symbol (R can be $=$)
- $(A \wedge B)^* = A^* \wedge B^*$,
- $(A \vee B)^* = A^* \vee^{\mathrm{c}} B^*$,
- $(A \supset B)^* = A^* \supset B^*$,

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- $((\forall x)C)^* = (\forall x)C^*$,
- $((\exists x)C)^* = (\exists^c x)C^*$.

It should be clear that the only thing this translation does is to replace $\exists$ and $\vee$ by their classical versions and to insert two negation signs in front of every predicate symbol. Obviously, the translation A^* is provably equivalent to A in classical predicate logic.

Example 1.1 Let R be a binary predicate symbol. Consider the formula

$$A = (\forall x)(\exists y)(R(x,y) \vee R(y,x)).$$

Its Gödel-Gentzen translation is

$$A^* = (\forall x)(\exists^c y)(\neg\neg R(x,y) \vee^c \neg\neg R(y,x))$$

Spelling out the definitions of classical connectives $\exists^c$ and $\vee^c$ we get

$$(\forall x)\neg(\forall y)\neg\neg(\neg\neg\neg R(x,y) \wedge \neg\neg\neg R(y,x))$$

The translation theorem is now:

Theorem 1.2 *Let A be any formula. If A is provable in classical predicate logic, then A^* is provable in intuitionistic predicate logic.*

Proof. A formal proof goes by induction on derivations proving something seemingly stronger, namely that if $A_1, \ldots, A_n \vdash_X A$ has a proof in classical logic then $A_1^*, \ldots, A_n^* \vdash_X A^*$ has a proof in intuitionistic logic.

Since the proof rules are identical for the systems, save for RAA, one needs only to prove

$$\vdash_X \neg\neg A^* \supset A^* \tag{3}$$

in intuitionistic predicate logic, for each formula A. This is done by induction on the formula A, using the following theorems of intuitionistic logic.

$\vdash \neg\neg\bot \supset \bot$,
$\vdash \neg\neg\neg\neg B \supset \neg\neg B$,
$\neg\neg A \supset A, \neg\neg B \supset B \vdash \neg\neg(A \wedge B) \supset A \wedge B$,
$\neg\neg B \supset B \vdash \neg\neg(A \supset B) \supset (A \supset B)$,
$(\forall x)(\neg\neg A \supset A) \vdash \neg\neg(\forall x)A \supset (\forall x)A$.

We leave their proofs as exercises for the reader. □

In general the translation A^* is not intuitionistical equivalent to the original formula A. In some cases it is however. A non-existential formula A is called *negative* if every predicate symbol in A is immediately preceded by a negation. For such a formula every predicate symbol in the corresponding translation A^* will be preceded by three negations. Intuitionistically, it holds that $\neg\neg\neg B \supset\subset \neg B$. Consequently, every negative formula is equivalent to its own Gödel-Gentzen interpretation in intuitionistic logic. We have

Corollary 1.3 *Let A be a negative formula. If A is provable in classical predicate logic, then A is also provable in intuitionistic predicate logic.*

A set T of closed first-order formulae is called a *(first-order) theory.* We say that $T \vdash A$ if, for some $A_1, \ldots, A_n \in T$, we have $A_1, \ldots, A_n \vdash A$. Obviously, it is not possible to use more than finitely many assumptions in a finite proof! We have as a consequence of Theorem 1.2 that

Corollary 1.4 *For a first-order theory T: if $T \vdash A$ in classical predicate logic, then $T^* \vdash A^*$ in intuitionistic predicate logic. Here $T^* = \{A^* : A \in T\}$.*

Proof. If $T \vdash A$ in classical logic, then $A_1, \ldots, A_n \vdash A$ for some $A_1, \ldots, A_n \in T$. By Theorem 1.2, ${A_1}^*, \ldots, {A_n}^* \vdash A^*$ in intuitionistic logic. Hence $T^* \vdash A^*$ in intuitionistic logic. $\Box$

Now for $T = \emptyset$ this is just Theorem 1.2. We are interested in theories for which $T \vdash T^*$ holds in intuitionistic predicate logic. A formula is a *Horn formula* if it has the form

$$(\forall x_1) \cdots (\forall x_m) B \quad \text{or} \quad (\forall x_1) \cdots (\forall x_m)(A_1 \wedge \cdots \wedge A_n \supset B)$$

where $A_1, \ldots, A_n, B$ are atomic formulae. We leave the proof of the following lemma as an easy exercise:

Lemma 1.5 *If A is a Horn formula, then*

$$A \supset A^*$$

is provable in intuitionistic logic.

For a theory T consisting of Horn formulae (a *Horn theory*) we thus have $T \vdash A^*$ intuitionistically whenever $T \vdash A^*$ classically. Another important example is $T = PA$, the first-order theory of natural numbers, known as *Peano arithmetic*. It has the following axioms:

$$
\begin{aligned}
&(\forall x)\neg \mathsf{S}(x) = 0 \\
&(\forall x)(\forall y)[\mathsf{S}(x) = \mathsf{S}(y) \rightarrow x = y] \\
&(\forall x)\, x + 0 = x \\
&(\forall x)(\forall y)\, x + \mathsf{S}(y) = \mathsf{S}(x + y) \\
&(\forall x)\, x \cdot 0 = 0 \\
&(\forall x)(\forall y)\, x \cdot \mathsf{S}(y) = x \cdot y + x \\
&(\forall x_1, \ldots, x_n)(C[0/z] \wedge (\forall x)(C[x/z] \rightarrow C[\mathsf{S}(x)/z]) \rightarrow (\forall z)C).
\end{aligned}
$$

Here C is an arbitrary formula in the arithmetical language, where $FV(C) \subseteq \{x_1, \ldots, x_n, z\}$. Using intuitionistic logic this is just *Heyting arithmetic* which was introduced earlier.

Theorem 1.6 *If $PA \vdash A$ is provable in classical logic, then its translation A^* it is also provable in Heyting arithmetic.*

Proof. Suppose that A has been proved classically in Peano arithmetic. Then for some arithmetical axioms $A_1, \ldots, A_n$ we have a proof of

$$A_1, \ldots, A_n \vdash_X A$$

in classical logic. So by Theorem 1.2 there is a proof in intuitionistic logic of

$$A_1^*, \ldots, A_n^* \vdash_X A^*.$$

If we can find some set of axioms Γ from PA so that

$$\Gamma \vdash_X A_i^* \tag{4}$$

is provable for each $i = 1, \ldots, n$ in intuitionistic logic, then we are done. In case A_i is a Horn formula, we include A_i in Γ and get (4) by Lemma 1.5. The only axioms that are not Horn formulae are the instances of the induction scheme. Suppose

$$A_i = (\forall x_1, \ldots, x_n)(C[0/z] \wedge (\forall x)(C[x/z] \rightarrow C[\mathsf{S}(x)/z]) \rightarrow (\forall z)C).$$

Then

$$A_i{}^* = (\forall x_1, \ldots, x_n)(C^*[0/z] \wedge (\forall x)(C^*[x/z] \rightarrow C^*[\mathsf{S}(x)/z]) \rightarrow (\forall z)C^*).$$

But this is also an instance of the induction scheme, which we thus include in Γ. The so constructed Γ is a subset of PA which satisfies (4). □

The above result shows that classical arithmetic is consistent if intuitionistic arithmetic is consistent. It was an important philosophical motivation for Gödel to show that intuitionistic logic per se does not add to the safety of the foundations. The situation has however turned out to be different for stronger axiom systems than arithmetic.

It is possible to prove the following important result using further proof-theoretic techniques:

Theorem 1.7 *Let* $A = (\forall x)(\exists y)P(x,y)$ *be a formula of arithmetic where P is quantifier free. If* $PA \vdash A$ *is provable classically, then A is also provable in Heyting arithmetic.*

Proof. See Troelstra and van Dalen (1988). □

Since A in the theorem above has the format of a program specification, it is possible to use this, and similar results, to extract programs from classical proofs (see Schwichtenberg 1999).

Exercises

1. Let $P(x)$ be a predicate symbol and consider $A = (\exists x)P(x) \vee (\forall x)\neg P(x)$. Prove the Gödel-Gentzen translated formula A^* in intuitionistic logic.

2. Prove the following in intuitionistic logic.

(a) $\vdash \neg\neg\bot \supset \bot$,

(b) $\vdash \neg\neg\neg\neg B \supset \neg\neg B$,

(c) $\neg\neg A \supset A, \neg\neg B \supset B \vdash \neg\neg(A \wedge B) \supset A \wedge B$,

(d)* $\neg\neg B \supset B \vdash \neg\neg(A \supset B) \supset (A \supset B)$,

(e)* $(\forall x)(\neg\neg A \supset A) \vdash \neg\neg(\forall x)A \supset (\forall x)A$.

3. Prove Lemma 1.5 using contraposition and the following theorem of intutionistic logic

$$\vdash_X \neg\neg(A \wedge B) \supset\subset \neg\neg A \wedge \neg\neg B.$$

4.(*) This exercise gives an extension of Lemma 1.5. A closed formula

$$(\forall x_1)\cdots(\forall x_n)(A_1 \wedge \cdots \wedge A_m \supset B_1 \vee \cdots \vee B_k)$$

is a called a *clause*, if the A_i and B_j are all atomic. Show that $C \supset C^*$ is provable in intuitionistic logic for any clause.

5. Give a proof in Heyting arithmetic that $\forall xy(x = y \vee \neg x = y)$. Conclude that each non-existential aritmetical formula A is equivalent to its negative interpretation A^* in Heyting arithmetic. Hence by Theorem 1.6, we have that any non-existential formula A classically provable in Peano arithmetic, is also provable in Heyting arithmetic.