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ÖVNINGSTENTA/EXERCISE EXAM
Tillämpad logik DV1
2002-10-08

Skrivtid: 5 timmar. Tillåtna hjälpmedel:

- Fullständighetssatsen för predikatlogiken via sekventkalkyl. Kompendium av I. Sigstam, Matematiska institutionen, Uppsala universitet.
- Term rewriting systems: supplementary notes by E.Palmgren.
- Logical background to the resolution method: supplementary notes by E.Palmgren.
- 3 handwritten pages of your own notes.

5 hours are allowed for the test. No utilities except the listed items above are allowed. The maximum number of points for each problem is indicated within parentheses. For the grade "Godkänd" (passed) 18 points are required, for the grade "Väl Godkänd" (passed with distinction) 28 points will be required. The solutions may be written in English or Swedish.

1. Provide BHK-interpretations for the following formulas

   (a) $A \wedge ((A \wedge B) \rightarrow C) \rightarrow (B \rightarrow C)$

   (b) $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$

   (c) $\neg \exists x\, A(x) \rightarrow \forall x\, \neg A(x)$ (6p)

2. Is the following formula provable in intuitionistic propositional logic? Give a proof or a counter-model.

   $(P \supset Q) \supset (\sim P \vee Q)$ (3p)

3. Mention some consequences of the Gödel-Gentzen negative interpretation. (3p)

4. Consider the modal model given by the set of week days

$$W = \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}, \}$$

where the accessibility relation $R(x, y)$ is x is no later than y. The truth-values for P, Q and R are as follows on each day.

$t =$	**M**	**Tu**	**W**	**Th**	**F**	**Sa**	**Su**
$V_t(P) =$	1	1	1	1	1	0	1
$V_t(Q) =$	0	0	1	1	0	1	1
$V_t(R) =$	1	0	1	0	1	1	0

For each week day t_0 try to find a formula A mentioning only P, Q, R which is true exacty on this day, i.e. such that $V_t(A) = 1$ if, and only if $t = t_0$. Or argue that no such formula exists. (6p)

(Please turn over)

5. Translate each of the formulas in Problem 1 into type theory. Explain the role of the Σ-type in the propositions-as-types interpretation. (4p)

6. Find using the proof method for the completeness theorem a model falsifying the following sequent

$$\exists x\, R(x,x) \longrightarrow \exists x\, \forall y\, R(x,y).$$

(R is a binary relation symbol.) (4p)

6. Prove that the following recursively defined function f is well-defined (provided g and h are defined). Do this by first finding a suitable well-founded relation $\prec$ on the arguments, and then show that the arguments of the function on the right hand side is always smaller (according to this ordering) than the arguments of the functions on the left hand side. $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where

$$\begin{array}{rcl} f(0,y,0) & = & 0 \\ f(S(x),0,0) & = & S(0) \\ f(S(x),S(y),0) & = & g(f(x,S(y),S(0))) \\ f(S(x),S(y),S(z)) & = & h(f(S(x),y,0)). \end{array}$$

(4p)

7. Let L be a first-order language. Suppose that T is a consistent L-theory. Prove that T is complete if, and only if, $\mathcal{A} \equiv \mathcal{B}$ for any two models $\mathcal{A}$ and $\mathcal{B}$ of T. (Recall that $\mathcal{A} \equiv \mathcal{B}$ is defined as: for all closed L-formulas φ:

$$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.)$$

Why are complete theories T useful in connection with automatic theorem proving? (5p)

8. The relation $(\mathbb{N}^k, \leq_k)$ defined by

$$(a_1, \ldots, a_k) \leq_k (b_1, \ldots b_k) \Longleftrightarrow a_1 \leq b_1 \& \cdots \& a_k \leq b_k$$

is a well-quasi order. (You do not have to prove this.) Illustrate this order for $k = 2$ and $k = 3$. Show that if $S \subseteq \mathbb{N}^k$ is an infinite set then there exists finitely many vectors $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(m)}$ in S such that for any $\mathbf{b} \in S$ at least one of

$$\mathbf{a}_1 \leq_k \mathbf{b} \text{ or } \cdots \text{ or } \mathbf{a}_m \leq_k \mathbf{b},$$

holds. Try to illustrate this result in two dimensions. (Hint: Proof by contradiction.) (5p)

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