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EXERCISES 1
Tillämpad Logik DV1, vt-04
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Exercises and Hand-in Problems 1, Tillämpad Logik DV1

Hand-in problems are

2, 3(a), 3(b), 3(f), 4(e), 6, 7(a), 7(d)

of the exercises below. Solutions should be handed in by Friday April 2, 2004, clearly written and well explained in Swedish or English. (You may work in groups of three.)

1. Review exercises in predicate logic: 2.6.1, 2.6.2, 2.6.4, 2.6.6, 2.8.1, 2.5.1, 2.5.4, 2.5.8 (Huth and Ryan).

2. Solve Post's correspondence problem for the sequence of pairs

$$(001,0),\ (01,011),\ (01,101),\ (10,001).$$

(Hint: One solution gives a binary string of total length 154. Use a computer program if it gets too tedious ...)

3. (Definability) Let $\mathcal{M}$ be a model for the language L and let $A = |\mathcal{M}|$ be its universe. A subset $S \subseteq A^n$ is *(first-order) definable* in $\mathcal{M}$ if there is an L-formula φ with free variables among $x_1, \ldots, x_n$ such that

$$S = \{(a_1, \ldots, a_n) \in A^n : \mathcal{M} \models_\ell \varphi \text{ and } \ell(x_1) = a_1, \ldots, \ell(x_n) = a_n\}$$

A relation $R \subseteq A^n$ is *definable in $\mathcal{M}$* if the corresponding subset R is definable. A function $f : A^n \to A$ is *definable in $\mathcal{M}$* if its graph

$$\text{graph } f = \{(a_1, \ldots, a_n, b) \in A^{n+1} : f(a_1, \ldots, a_n) = b\}$$

is a definable subset in $\mathcal{M}$.

Show that the subsets, relations or functions in (a) – (h) below are definable in $\mathcal{N} = \langle \mathbb{N}; +, \cdot, 0, 1\rangle$ using as simple formulas as seems possible.

For instance the set of even numbers is defined by

$$\{m \in \mathbb{N} : \mathcal{N} \models_\ell (\exists x)\ x + x = y \text{ and } \ell(y) = m\}$$

This also shows that the predicate x *is even* is definable. The function $f(x) = x^2$ is defined by

$$\{(m,n) \in \mathbb{N}^2 : \mathcal{N} \models_\ell x \cdot x = y \text{ and } \ell(x) = m, \ell(y) = n\}.$$

(a) x is odd

(b) $y = x(x+1)/2$

(c) $x \leq y$

(d) x divides y

(e) x is the sum of two prime numbers

(f) $z = \max(x,y)$

(g) $y = x!$

(h) $y = 2^{2^{2^{2^{2}}}}$

4. (Definability and decidability) Recall that in automata theory one studies languages as subsets of strings over a fixed alfabet. Let $\Sigma = \{a,b\}$ be an alfabet, and let Σ^* be the set of finite strings. Thus

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Here ε is the empty string. Let & denote concatenation of strings, so $baba\&bba = bababba$. We may now regard $\langle \Sigma^*;\ \& \rangle$ as a first-order structure with concatenation as the only operation. Find elementary propositions (formulas) over $\langle \Sigma^*;\ \& \rangle$ that definies the following properties (note that $=$ may be used)

(a) x is a substring of y

(b) x is an empty string (you may not mention ε)

(c) x is a string of length 1 (you may not mention 0 or 1) (Hint: use (a) and (b). How many substrings can such a string have?)

(d) x is a string of length 4.

Consider now an extended structure $\langle \Sigma^*;\ \&, *, a, b, \varepsilon \rangle$ where a, b, ε are constants (so they may be mentioned in elementary propositions) and moreover there is a "string duplicator" $*$ that satisfies the following

$$\begin{array}{rcll} u * \varepsilon & = & \varepsilon & \text{(erase)} \\ u * (a\&v) & = & u * v & \text{(take a pause)} \\ u * (b\&v) & = & (u * v)\&u & \text{(make a copy).} \end{array}$$

Thus $ab * bab = abab$ and $ab * aa = \varepsilon$.

(e) Prove that the structure $\langle \Sigma;\ \&, *, a, b, \varepsilon \rangle$ is undecidable, by showing that *if* it was decidable, then we could decide $\langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ as well, contradicting a well-known theorem.

(f)** If we remove the duplication operation from the structure in (e), does it become decidable? I.e. is the structure $\langle \Sigma;\ \&, a, b, \varepsilon \rangle$ decidable?

5. (Decidability in geometry) We start out from the fact that the structure $\mathcal{R} = \langle \mathbb{R}; <, +, \cdot, 0, 1\rangle$ is decidable. The first task is to find suitable formulas over $\mathcal{R}$ that describe certain geometric objects. The second is to show that certain problems about such objects may be expressed by formulas. Then one can apply Tarski's theorem to conclude that the problem is in principle solvable by a computer. If one is lucky (and clever) the problem may actually be solvable using a system like Mathematica 4.0.

   (a) Convince yourself that 3-dimensional geometric objects like, spheres, cylinders, cubes, balls, beams and Volvo cars at various positions and angles are definable in the structure $\mathcal{R}$ as subsets of $\mathbb{R}^3$.

   (b) Suppose that $S, T \subseteq \mathbb{R}^3$ are geometric objects definable by the formulas $\psi_S(x,y,z)$ and $\psi_T(x,y,z)$ in $\mathcal{R}$. Find formulas expressing

      (i) *S and T intersects*,

      (ii) *S is contained in T*,

      (iii) *S is the complement of T*,

      (iv) *S and T are identical*,

      (v) *S is obtained from T by intersecting with some halfplane*.

   (c) First find a formula that expresses *The ball with center in* $P_1 = (x_1, y_1, z_1)$ *of radius* d_1 *is inside a sphere with center in* P_2 *and radius* d_2. Show that the following questions are solvable in principle: How many balls of radius 1 fit into a sphere of radius 2? Of radius 3? Of radius 4? Of radius 10^{100}?

6. Give an equational proof of $(x^{-1})^{-1} = x$ in the theory of Example 1.4 in (Palmgren: *Equational logic*). How many applications of transitivity are needed?

7. Let $\Sigma = \{a, f, g, h, p, q\}$ where a is a constant, f, g has arity 1, h, p has arity 2 and q has arity 3. For each of the following pair of terms compute an mgu or show that no unifier exist.

   (a) $p(f(a), g(x)),\ p(y, y)$

   (b) $p(f(x), a),\ p(y, f(w))$

   (c) $p(x, x),\ p(y, f(y))$

   (d) $q(a, x, f(g(y))),\ q(z, h(z, w), f(w))$.