Exercises 1, Applied Logic

1. Some review exercises in predicate logic to do (if necessary):

   2.1: 1,3  
   2.2: 3,4  
   2.3: 2,3,4,7,13  
   2.4: 2,6,11,12

of the textbook (Huth and Ryan 2004).

2. Solve Post’s correspondence problem for the sequence of pairs

   \((001,0), (01,011), (01,101), (10,001)\).

   (Hint: Not so easy if one gets started in the wrong way. One solution gives
a binary string of total length 154. Use a computer program if it gets too
tedious ...)

3. (Definability) Let \(M\) be a model for the language \(L\) and let \(A = |M|\) be
its universe. A subset \(S \subseteq A^n\) is \((\text{first-order})\ definable\) in \(M\) if there is an
\(L\)-formula \(\varphi\) with free variables among \(x_1, \ldots, x_n\) such that

\[
S = \{(a_1, \ldots, a_n) \in A^n : M \models \varphi \text{ and } \ell(x_1) = a_1, \ldots, \ell(x_n) = a_n\}
\]

A relation \(R \subseteq A^n\) is \(\text{definable in } M\) if the corresponding subset \(R\) is definable. A function \(f : A^n \to A\) is \(\text{definable in } M\) if its graph

\[
\text{graph } f = \{(a_1, \ldots, a_n, b) \in A^{n+1} : f(a_1, \ldots, a_n) = b\}
\]

is a definable subset in \(M\).

Show that the subsets, relations or functions in (a) – (h) below are definable
in \(\mathcal{N} = \langle \mathbb{N}; +, \cdot, 0, 1 \rangle\) using as simple formulas as seems possible.

For instance the set of even numbers is defined by

\[
\{m \in \mathbb{N} : \mathcal{N} \models (\exists x) x + x = y \text{ and } \ell(y) = m\}
\]

This also shows that the predicate \(x \text{ is even}\) is definable. The function \(f(x) = x^2\) is defined by

\[
\{(m, n) \in \mathbb{N}^2 : \mathcal{N} \models x \cdot x = y \text{ and } \ell(x) = m, \ell(y) = n\}.
\]
(a) $x$ is odd
(b) $y = x(x + 1)/2$
(c) $x \leq y$
(d) $x$ divides $y$
(e) $x$ is the sum of two prime numbers
(f) $z = \max(x, y)$
(g) ** $y = x!$. [Look up and use Gödel’s technique of the \( \beta \)-function and the Chinese remainder theorem.]
(h) $y = 2^{2^{2^2}}$

Let $L$ be a first-order language with finitely many symbols. A structure $M$ for $L$ is called *decidable*, if there is an algorithm which for every closed first order formula $\varphi$ in the language $L$ decides whether $M \models \varphi$ holds or not. A well-known example of an *undecidable* structure is the structure of natural numbers $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0, 1 \rangle$.

4. (Definability and decidability) Recall that in automata theory one studies languages as subsets of strings over a fixed alphabet. Let $\Sigma = \{a, b\}$ be an alphabet, and let $\Sigma^*$ be the set of finite strings. Thus

$$\Sigma^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Here $\varepsilon$ is the empty string. Let $\&$ denote concatenation of strings, so $baba\&bba = bababba$. We may now regard $\langle \Sigma^*; \& \rangle$ as a first-order structure with concatenation as the only operation. Find elementary propositions (formulas) over $\langle \Sigma^*; \& \rangle$ that defines the following properties (note that $=$ may be used)

(a) $x$ is a substring of $y$
(b) $x$ is an empty string (you may not mention $\varepsilon$)
(c) $x$ is a string of length 1 (you may not mention 0 or 1) (Hint: use (a) and (b). How many substrings can such a string have?)
(d) $x$ is a string of length 4.

Consider now an extended structure $\langle \Sigma^*; \&, *, a, b, \varepsilon \rangle$ where $a, b, \varepsilon$ are constants (so they may be mentioned in elementary propositions) and moreover there is a “string duplicator” $*$ that satisfies the following

$$
\begin{align*}
  u * \varepsilon &= \varepsilon &\text{(erase)} \\
  u * (a & v) &= u * v &\text{(take a pause)} \\
  u * (b & v) &= (u * v) & u &\text{(make a copy)}.
\end{align*}
$$

Thus $ab * bab = abab$ and $ab * aa = \varepsilon$. 

2
(e) Prove that the structure \( \langle \Sigma; \&, +, a, b, \varepsilon \rangle \) is undecidable, by showing that if it was decidable, then we could decide \( \langle \mathbb{N}, +, \cdot, 0, 1 \rangle \) as well, contradicting a well-known theorem.

(f)** If we remove the duplication operation from the structure in (e), does it become decidable? I.e. is the structure \( \langle \Sigma; \&, a, b, \varepsilon \rangle \) decidable?