Exercises 2, Applied Logic

The first problem is an example of a non-constructive result which is often used (implicitly) in mathematical analysis.

1. (König’s lemma). A finite string over the alphabet \( \{l, r\} \) is regarded as describing a path in a binary tree, starting from the root. Suppose that \( P \) is an infinite set of such paths. Show that there is an infinite string

\[
d_1d_2d_3\cdots
\]

such that for every \( n \), the string \( d_1d_2\cdots d_n \) is an initial segment of some path in \( P \).

Suppose that there is an algorithm which decides whether a finite path \( s \in \{l, r\}^* \) is in \( P \). Is there any hope to find an algorithm which for each index \( i \) computes the value of \( d_i \in \{l, r\} \)? Discuss.

2. Prove the following in intuitionistic propositional logic

(a) \( A \supset \neg\neg A \),

(b) \( \neg\neg\neg A \supset \neg A \),

(c) \( \neg\neg A \supset A, \neg\neg B \supset B \vdash \neg\neg (A \land B) \supset (A \land B) \),

(d) \( \neg\neg A \supset (A \lor B) \vdash (A \lor B) \supset (A \lor B) \),

(e) \( A \land (B \lor C) \vdash (A \land B) \lor (A \land C) \).

Justify the judgements (a) and (e) by using Kolmogorov’s interpretation.

3. Is it possible to justify the following judgement under the Kolmogorov interpretation?

\( \neg (A \land B) \vdash \neg A \lor \neg B \)

4. Prove formally in Heyting arithmetic

(a) \( (\exists y)A(y) \lor (\exists y)B(y) \vdash x \ (\exists y)(A(y) \lor B(y)) \)

(b) \( A(0), (\forall z)A(s(z)) \vdash x \ (\forall z)A(z) \)

(c) \( (\forall x)(x + 0 = 0 + x) \)