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EXERCISES 2
Tillämpad Logik, ht-07
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Exercises 2, Applied Logic

The first problem is an example of a non-constructive result which is often used (implicitly) in mathematical analysis.

1. (König's lemma). A finite string over the alphabet $\{l, r\}$ is regarded as describing a path in a binary tree, starting from the root. Suppose that P is an infinite set of such paths. Show that there is an infinite string

   $$d_1 d_2 d_3 \cdots$$

   such that for every n, the string $d_1 d_2 \cdots d_n$ is an initial segment of some path in P.

   Suppose that there is an algorithm which decides whether a finite path $s \in \{l, r\}^*$ is in P. Is there any hope to find an algorithm which for each index i computes the value of $d_i \in \{l, r\}$? Discuss.

2. Prove the following in intuitionistic propositional logic

   (a) $A \supset \neg\neg A$,

   (b) $\neg\neg\neg A \supset \neg A$,

   (c) $\neg\neg A \supset A, \neg\neg B \supset B \vdash \neg\neg(A \wedge B) \supset (A \wedge B)$,

   (d) $\neg\neg B \supset B \vdash \neg\neg(A \supset B) \supset (A \supset B)$,

   (e) $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$.

   Justify the judgements (a) and (e) by using Kolmogorov's interpretation.

3. Is it possible to justify the following judgement under the Kolmogorov interpretation?

   $$\neg(A \wedge B) \vdash \neg A \vee \neg B$$

4. Prove formally in Heyting arithmetic

   (a) $(\exists y)A(y) \vee (\exists y)B(y) \vdash_X (\exists y)(A(y) \vee B(y))$

   (b) $A(0), (\forall z)A(\mathsf{s}(z)) \vdash_X (\forall z)A(z)$

   (c) $(\forall x)(x + 0 = 0 + x)$
