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EXERCISES 3
Tillämpad Logik DV1, vt-04
2004-05-04

Exercises and Hand-in Problems 3, Tillämpad Logik DV1

Hand-in problems are

1(a), 1(d), 2, 3(c), 5(c), 5(d), 6(c), 7(a).

of the exercises below. Solutions should be handed in by Thursday, May 13, 2004, clearly written and well explained[1] in Swedish or English. (You may work in groups of three.)

1. Show that the following formulas are unprovable in IPC. This may be done by finding a suitable Heyting algebra and a valuation which give a value $\neq \top$ to the formula. Another strategy is to try to show that the formula implies (in IPC) a formula which is already known to be unprovable.
   (a) $\neg(P \to Q) \to P \wedge \neg Q$,
   (b) $(\neg Q \to \neg P) \to (P \to Q)$,
   (c) $(\neg\neg P \to \neg\neg Q) \to (P \to Q)$,
   (d) $(P \to Q \vee R) \to (P \to Q) \vee (P \to R)$.
2. Prove that in a Heyting algebra: if $a \wedge b = \bot$, $a \vee b = \top$, then $b = \neg a$. Thus every true complement is a pseudo-complement.
3. Prove that the following implications are valid in any Heyting algebra. (These are useful lemmas for the soundness theorem.)
   (a) $h \leq a \wedge b$ implies $h \leq a$ and $h \leq b$,
   (b) $h \leq a$ or $h \leq b$ implies $h \leq a \vee b$,
   (c) $h \leq (a \to b)$, $h \leq a$ implies $h \leq b$.
4. * Consider the set of open sets of the real line defined Section 10 of *Constructive Logic and Type Theory*. Compute the open sets (in terms of intervals):
   (a) $(2,3) \to (1,4)$
   (b) $(1,4) \to (2,3)$

[1] Unsatisfactorily presented solutions may be returned ungraded.

(c) $(1,3) \to (2,4)$

5. Use the tableau method to obtain either a proof or a counterexample to each of the following propositional formulas

   (a) $P \to (Q \to P)$

   (b) $(\neg P \vee \neg Q) \to (Q \wedge \neg R)$

   (c) $((P \to Q) \wedge (Q \to R)) \to (P \to R)$

   (d) $((P \to Q) \vee (Q \to R)) \to (P \to R)$

   For further exercises, see Smullyan, p. 24.

6. Prove by the tableau method each of the following formulas in predicate logic

   (a) $(\forall y)[(\forall x) P(x) \to P(y)]$

   (b) $(\forall x) P(x) \to (\exists x) P(x)$

   (c) $(\exists x)(P(x) \wedge Q(x)) \to (\exists x)P(x) \wedge (\exists x)Q(x)$

7. Determine the Skolem conjunctive normal form for each of the following formulas, and thereby its equivalent set of clauses. P and Q are predicate symbols.

   (a) $\forall u\, [\forall x \neg\exists y\, P(x,y) \to \neg\forall x\, (\exists z\, P(x,y) \to \forall y\, P(u,y))]$.

   (b) $\forall x\, (\exists y\, P(x,y) \leftrightarrow \exists z\, Q(z,x))$.
