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EXERCISES 3
Tillämpad Logik, ht-07
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Exercises 3 — Applied Logic

1. *Lazy computation in Gödel's T.* Reduce each of terms $(\mathsf{s}\ 0) + (\mathsf{s}\ (\mathsf{s}\ 0))$ and $(\mathsf{s}\ 0) \cdot (\mathsf{s}\ (\mathsf{s}\ 0))$ to lazy normal form using the fewest possible number of steps/inferences. Use definition of addition and multiplication given in (Coquand, Dybjer, Palmgren: Type-theoretic foundations of constructive mathematics, 2007).

2. *Abstract realizability of arithmetic.* Find lambda term witnesses to the validity of the following propositions or judgements. Use Heyting Arithmetic with witnesses as decsribed on page 28-30 of (Coquand *et al.* 2007).

   (a) $A \supset \neg\neg A$,
   (b) $\neg\neg\neg A \supset \neg A$,
   (c) $A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C)$.
   (d) $A \wedge B \supset C \vdash_X A \supset (B \supset C)$.
   (e) $(\exists y)A(y) \vee (\exists y)B(y) \vdash_X (\exists y)(A(y) \vee B(y))$
   (f) $(\forall y)A(y), (\forall y)(A(y) \supset B(y)) \vdash_X (\forall y)B(y)$.

3. The *Ackermann function* $\mathsf{ack} : \mathrm{N} \to \mathrm{N} \to \mathrm{N}$ can be defined by the following recursion equations

$$\begin{aligned} \mathsf{ack}\ 0\ n &= \mathsf{s}\ n \\ \mathsf{ack}\ (\mathsf{s}\ m)\ 0 &= \mathsf{ack}\ m\ (\mathsf{s}\ 0) \\ \mathsf{ack}\ (\mathsf{s}\ m)\ (\mathsf{s}\ n) &= \mathsf{ack}\ m\ (\mathsf{ack}\ (\mathsf{s}\ m)\ n) \end{aligned}$$

   It can be shown that the Ackermann function grows faster than any primitive recursive function. Show that it nevertheless may be defined in Gödel's system T with the help of the recursion operator rec. [Hint: expand the third line of the definition.]

   Can you compute a lazy normal form of $\mathsf{ack}\ 5\ 5$ using your definition?

4. *Propositions as ordinary sets (p. 35, Coquand et al. 2007).* The proposition $(\forall n \in \mathbb{N})[n = 0 \vee (\exists m \in \mathbb{N}) n = m + 1]$ says that each natural number is 0 or a successor of another natural number. The corresponding set becomes

$$(\Pi n \in \mathbb{N})\,[E_{n,0} + (\Sigma m \in \mathbb{N}) E_{n,m+1}].$$

   Exercise: show that it contains exactly one element.