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PROBLEM 2
Tillämpad Logik DV1, ht-01
2001-10-05

Problem set 2

The following problems are to be solved and presented by you at the black board on October 15. You may work in groups of two. At the beginning of the lesson, sign up for the problems you have solved and be prepared! Satisfactory solution of at least 2 of the problems gives 1 bonus point to be counted in the final exam.

1. Let E_{ab} be the equational theory of abelian groups (cf. Example 1.8, page 5, in handout "Term rewriting systems"). For variables $x_1, x_2, \ldots, x_k$ and integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ show that

$$E_{ab} \vdash_{\text{eq}} x_1^{m_1} \cdots x_k^{m_k} = x_1^{n_1} \cdots x_k^{n_k} \Leftrightarrow m_1 = n_1, \ldots, m_k = n_k.$$

   (Hint: for the direction ($\Rightarrow$) use the completeness result (Corollary 1.6) and consider the additive group $\mathbb{Z}^k$ and an assignment $\rho(x_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the i:th place). Discuss how this together with the observations in Example 1.8 gives a decision procedure for E_{ab}. Discuss whether there could be a complete term rewriting system for E_{ab}. (Note that commutativity can be problematic, see Exercise 2.4.3 and 2.4.17 in Klop.)

2. Show that the most general unifier of

$$f(g(x_1, x_1), g(x_2, x_2), \ldots, g(x_n, x_n)) \text{ and } f(x_2, x_3, \ldots, x_{n+1})$$

   has 2^n occurrences of some variable. Suggest effective representations of terms.

3. Argue that if $\varphi \longrightarrow \psi$ is a valid sequent whose both sides only contain the logical operators $\forall, \exists, \wedge$, then the sequent has a proof in the sequent calculus which only use the rules for these operators and logical axioms of the form $C, \Gamma \longrightarrow \Delta, C$. (Hint: watch the principal formulas and note what rules are applicable.) Consider an arbitrary sequent $\varphi \longrightarrow \psi$. Discuss whether there are some natural restrictions on the form of φ and ψ which guarantee that the search process in the proof of the completeness theorem stops after finitely many steps (with a proof of counter model as a result).

4. Prove the following formula by resolution. You need to transform the problem to a set of clauses using Skolem conjunctive normal form.

$$\forall x \forall y \forall z (R(x, y, z) \rightarrow R(y, z, x)) \wedge \forall u \exists v\, R(u, u, v) \rightarrow \forall u \exists v\, R(v, u, u).$$

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