ALGEBRAIC STRUCTURES

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Examination 17th December 2012

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4, and 5 correspond approximately to the scores 18, 25, and 32, respectively, distributed reasonably evenly among the three divisions Group Theory, Ring Theory, and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions ought to be written out formally, using complete sentences.

1. (a) Define what it means for a permutation to be *even or odd*.

   (b) Consider, in S_9, the permutation

   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 9 & 3 & 1 & 2 & 6 & 8 & 7 \end{pmatrix}.$$

   Write σ in cycle notation.

   (c) Determine whether σ is even or odd.

   (d) Determine whether σ is of even or odd order in S_9.

   (e) Does there exist a permutation π such that $\pi^2 = \sigma$?

2. (a) Define, for an integral domain, the notions of *irreducible* and *prime* element. How are these two concepts related?

   (b) Factorise the polynomial $p(x) = 1 + x + x^2 + x^3$ into irreducibles over $\mathbf{Z}_3$.

   (c) Construct addition and multiplication tables for the factor ring $\mathbf{Z}_3[x]/(p(x))$. Is it an integral domain? Is it a field?

3. (a) Define *abelian group*.

(b) Classify the abelian groups of order 2012 (i. e. give a complete, irredundant list of groups of order 2012).

4. (a) Define what it means for P to be a *prime ideal* of a commutative, unital ring R. What is known about the structure of the factor ring R/P when P is a prime ideal?
   (b) Let R be a commutative ring with unity. An element $x \in R$ is *nilpotent* if $x^n = 0$ for some positive integer n. Shew that x is either 0 or a zero divisor.
   (c) Shew that the nilpotent elements constitute an ideal N of R.
   (d) Shew that R/N contains no nilpotent elements apart from $0 + N$.

5. (a) Define a *subgroup* of a group.
   (b) Let G be a group and let D be the diagonal set
$$D = \{ (g, g) \mid g \in G \}.$$
   Shew that D is a subgroup of $G \times G$.
   (c) Shew that, if G is abelian, then D is a normal subgroup of $G \times G$. Describe the structure of $(G \times G)/D$.
   (d) Shew, by means of an example, that, if G is non-abelian, then D need not be normal in $G \times G$.

6. (a) What is the definition of a *unital ring*?
   (b) A *tessarine* is a quantity of the form
$$a + bi + cj + dk, \quad a, b, c, d \in \mathbf{R},$$
   where i, j, k are imaginary units. Tessarines are added and subtracted component-wise (like complex numbers and quaternions), and multiplied according to the rules
$$ij = ji = k, \qquad i^2 = -1, \qquad j^2 = 1,$$
   combined with associativity and distributivity. The Tessarines form a unital ring T. Find the complete 4×4 multiplication table for the elements $1, i, j$, and k in T.
   (c) Is T commutative? Does it have zero divisors? Is it a field?
   (d) Shew that T is isomorphic to the direct product $\mathbf{C} \times \mathbf{C}$.

7. (a) Define the *minimal polynomial* of an (algebraic) element of a field extension.
   (b) Shew that $\alpha = \sqrt{2} + i\sqrt{3}$ is algebraic over $\mathbf{Q}$ and find its minimal polynomial.
   (c) What is the dimension of the simple extension $\mathbf{Q}(\alpha)$, considered as a vector space over $\mathbf{Q}$?
   (d) Is $\mathbf{Q}(\alpha) \geqslant \mathbf{Q}$ a Galois (normal) extension? If so, find the Galois group.