ALGEBRAIC STRUCTURES

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Examination 2nd April 2013

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4, and 5 correspond approximately to the scores 18, 25, and 32, respectively, distributed *reasonably* evenly among the three divisions Group Theory, Ring Theory, and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. (a) Give the definition of a group G *acting* on a set X.

   (b) Consider the following set of 2×2 matrices with entries from $\mathbf{Z}_2$:

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbf{Z}_2 \ \wedge \ ad \neq bc \right\}.$$

   Shew that it is a group under matrix multiplication. Is it abelian?

   (c) How many elements does it contain? Which well-known group is it isomorphic to?

   (d) Shew that the group M acts on the set $\mathbf{Z}_2 \times \mathbf{Z}_2$ by left multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

   What are the orbits of the action?

2. (a) Define *algebraic* and *finite* extension fields. How are these two concepts related?

(b) Shew that $\mathbf{Q}[x]/(x^6 + 9x + 6)$ is a field.

(c) Let θ be such that $\theta^6 + 9\theta + 6 = 0$. Find the inverse of $1 + \theta$ in $\mathbf{Q}(\theta)$.

3. (a) What is the definition of a *normal subgroup*?

   (b) Suppose K is a normal subgroup of H, which is a normal subgroup of G. Shew, by means of an example, that K is not necessarily normal in G. (You may, for example, wish to consider the fourth dihedral group.)

4. (a) When I is an ideal of a commutative, unital ring R, define the corresponding *factor ring*.

   (b) Let F denote the commutative ring of continuous functions $\mathbf{R} \to \mathbf{R}$, under point-wise addition, subtraction, and multiplication. What is the multiplicative identity of this ring? Is F an integral domain? Is it a field?

   (c) Let C denote the set of constant functions. Is C an ideal? If so, describe the structure of the corresponding factor ring F/C.

   (d) Let Z denote the set of functions mapping 0 to 0. Is Z an ideal? If so, describe the structure of the corresponding factor ring F/Z.

5. (a) Define a *group*.

   (b) Let X be a set with n elements and let $P(X)$ denote its power set, i. e. its set of subsets. Given two sets $A, B \subseteq X$, their *symmetric difference* consists of the elements belonging to either A or B, but not both:

$$A \sqcap B = \{\, x \in X \mid x \in A \cup B \ \wedge\ x \notin A \cap B \,\}.$$

   Shew that $(P(X), \sqcap)$ is an abelian group.

   (c) Which well-known group is $P(X)$ isomorphic to? (I. e.: Classify $P(X)$ according to the Fundamental Theorem of Finitely Generated Abelian Groups.)

6. (a) What is the definition of a *field*?

   (b) A *split-quaternion* is a quantity of the form

$$a + bi + cj + dk, \quad a, b, c, d \in \mathbf{R},$$

   where i, j, k are imaginary units. Split-quaternions are added and subtracted component-wise (like complex numbers and quaternions), and multiplied according to the rules

$$i^2 = -1, \qquad j^2 = 1, \qquad k^2 = 1, \qquad ijk = 1,$$

combined with associativity and distributivity. The Split-Quaternions form a unital ring S. Find the complete 4×4 multiplication table for the elements 1, i, j, and k in S.

(c) Is S commutative? Does it have zero divisors? Is it a field?

(d) Shew that S is isomorphic to the matrix ring $\mathbf{R}^{2\times 2}$.

7. (a) Define the *Galois group* of a polynomial over a field F.

(b) Factorise $p(x) = x^6 - 64$ into irreducibles over $\mathbf{Q}$.

(c) Determine the Galois group of $p(x)$ over $\mathbf{Q}$.