

# ALGEBRAIC STRUCTURES

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Examination 30<sup>th</sup> August 2013

**Solutions.** Complete solutions are required for each problem.

**Marking.** Each problem is worth 6 points.

- The marks 3, 4, and 5 correspond approximately to the scores 18, 25, and 32, respectively, distributed *reasonably* evenly among the three divisions Group Theory, Ring Theory, and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 4 & 3 & 1 & 5 \end{pmatrix}.$$

- Write  $\pi$  in cycle notation.
  - Determine whether  $\pi$  is even or odd.
  - Find a permutation commuting with  $\pi$ , which is neither  $\pi$  itself nor the identity permutation.
  - Find a permutation *not* commuting with  $\pi$ .
  - Determine the order of the subgroup generated by  $\pi$ .
2. (a) Starting from the concept of an integral domain and an irreducible element, define a *unique factorisation domain*.
- (b) Consider the polynomial

$$p(x) = x^4 + x^3 + 5x^2 + 10x + 5.$$

Factorise  $p(x)$  into irreducible factors over  $\mathbf{Q}$ .

- (c) Factorise  $p(x)$  into irreducibles over  $\mathbf{Z}_3$ .
3. (a) Define the notion of a *group action*.  
 (b) Consider the set of four matrices:

$$G = \left\{ \begin{pmatrix} \pm \mathbf{I} & \mathbf{o} \\ \mathbf{o} & \pm \mathbf{I} \end{pmatrix} \right\}.$$

Shew that  $G$  is a group under matrix multiplication. Which well-known group is it isomorphic to?

- (c) Shew that  $G$  acts on the plane  $\mathbf{R}^2$  by left multiplication:  $A \cdot x = Ax$ , for  $A \in G$  and  $x \in \mathbf{R}^2$ .  
 (d) What is the orbit of a point  $P = (p, q)$  under this action? What is the stabiliser?
4. (a) Define the concept of a *ring*.  
 (b) Define *ring homomorphisms* and *ring isomorphisms*.  
 (c) Shew that the rings  $\mathbf{Q}[x]/(x^2 - 1)$  and  $\mathbf{Q} \times \mathbf{Q}$  are isomorphic.
5. (a) Define a *soluble group*.  
 (b) Is  $S_3$  soluble?  
 (c) Is  $\mathbf{Z}$  soluble?
6. (a) Define what it means for  $P$  to be a *prime ideal* of a commutative, unital ring  $R$ . What is known about the structure of the factor ring  $R/P$  when  $P$  is a prime ideal?  
 (b) Let  $R$  be a commutative, unital ring, let  $S$  be a subring, and let  $I$  be an ideal of  $R$ . Prove that, if  $S \cap I = \{\mathbf{o}\}$ , then the set

$$T = \{s + I \mid s \in S\}$$

forms a subring of  $R/I$  isomorphic to  $S$ .

7. (a) Define the *Galois group* of a polynomial over a field  $F$ .  
 (b) Factorise  $p(x) = x^4 - 4$  into irreducibles over  $\mathbf{Q}$ .  
 (c) Determine the Galois group of  $p(x)$  over  $\mathbf{Q}$ .