ALGEBRAIC STRUCTURES

Xantcha

Examination 23rd April 2014

Solutions. Complete solutions are required for each problem.

Marking. Each problem is worth 6 points.

- The marks 3, 4 and 5 correspond approximately to the scores 18, 25 and 32, respectively, distributed reasonably evenly among the three subdivisions Group Theory, Ring Theory and Field Theory.
- Also, in order to pass, a student should demonstrate some knowledge of the fundamental definitions of the course. Definitions should be written out formally, using complete sentences.

1. (a) Define what it means for a group to be *finitely generated*.
   (b) State the Fundamental Theorem for Finitely Generated Abelian Groups.
   (c) Classify the abelian groups of order 2014.
   (d) Are there any non-abelian groups of order 2014?

2. (a) Define a *subring* of a ring.
   (b) Let A be a commutative ring of characteristic 2. Consider the set of squares in A:
   $$B = \{\, a^2 \mid a \in A \,\}.$$
   Show that B is a subring of A.
   (c) Show that B is not, in general, an ideal of A, for example by considering $A = \mathbf{Z}_2[x]$.
   (d) Show that B will not necessarily be a subring if A is not required to have characteristic 2.

3. (a) Define the notion of a *group homomorphism*.
   (b) Show that the determinant

   $$\det\colon \mathrm{GL}_n(\mathbf{C}) \to \mathbf{C}^*$$

   is an homomorphism of groups, where $\mathrm{GL}_n(\mathbf{C})$ denotes the group of inversible complex $n \times n$ matrices.
   (c) Show that $\mathrm{SL}_n(\mathbf{C})$, the complex matrices of determinant 1, is a normal subgroup of $\mathrm{GL}_n(\mathbf{C})$.
   (d) Show that

   $$\mathrm{GL}_n(\mathbf{C})/\mathrm{SL}_n(\mathbf{C}) \cong \mathbf{C}^*.$$

4. (a) Define *prime* and *maximal ideals* of a commutative, unital ring.
   (b) Let $R = \mathbf{Z}_4[x]$. Show that the ideal (x) is neither prime nor maximal.
   (c) Show that the ideal $(2, x)$ is both prime and maximal in R.
   (d) Does there exist an ideal in R which is prime, but not maximal?
   (e) Does there exist an ideal in R which is maximal, but not prime?

5. (a) Define a *field*.
   (b) Compute the Galois group of $p(x) = x^8 - x^2$ over $\mathbf{Q}$.

6. (a) Suppose the group G acts on the set X. Define the *orbit* and *stabiliser* (also known as the *isotropy subgroup*) of a point $x \in X$.
   (b) Consider the set of *affine functions*

   $$A = \{ f\colon \mathbf{R} \to \mathbf{R},\ f(x) = ax + b \mid a, b \in \mathbf{R} \ \wedge\ a \neq 0 \}.$$

   Show that it is a group under function composition.
   (c) Show that the group A acts on the set $\mathbf{R}$.
   (d) What is the orbit of the point $p \in \mathbf{R}$? What is the stabiliser?

7. (a) Define the *splitting field* of a polynomial $q(x)$ over a field F.
   (b) Let $q(x)$ be a rational cubical polynomial, and let α, β and γ denote its three roots. Show that the splitting field of $q(x)$ over $\mathbf{Q}$ equals $\mathbf{Q}(\alpha, \beta)$.
   (c) Give an example where $\mathbf{Q}(\alpha)$, $\mathbf{Q}(\beta)$ and $\mathbf{Q}(\gamma)$ all equal the splitting field.
   (d) Give an example where the splitting field is neither $\mathbf{Q}(\alpha)$ nor $\mathbf{Q}(\beta)$ nor $\mathbf{Q}(\gamma)$.