THE COMBINATORICS OF POLYNOMIAL FUNCTORS

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Dedalus genom sin konst och sitt snille vida beryktad
Bygde det opp; han förvirrar de ledande märken och ögat
I villfarelse för ibland skiljaktiga vägar.
Så på de Frygiska fält, man ser den klara Meandros
Leka. I tveksamt lopp han rinner och rinner tillbaka,
Möter sig ofta sjelf och skådar sin kommande bölja,
Och nu till källan vänd, nu åt obegränsade hafvet,
Rådvill öfvar sin våg. Så fyllas af Dedalus äfven
Tusen vägar med irrande svek: Knappt mäktar han sjelf att
Hitta till tröskeln igen. Så bedräglig han boningen danat.

— Ovidius, *Metamorphoses*

Argument

We propose a new description of Endofunctors of Module Categories, based upon a combinatorial category comprising finite sets and so-called *mazes*. Polynomial and numerical functors both find a natural interpretation in this frame-work.

Since strict polynomial functors, according to the work of Salomonsson, are encoded by multi-sets, the two strains of functors may be compared and contrasted through juxtaposing the respective combinatorial structures, leading to the *Polynomial Functor Theorem*, giving an effective criterion for when a numerical (polynomial) functor is strict polynomial.

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Polynomial and strict polynomial functors abound. Recent years have seen scholars vying for adequate descriptions of their different flavours, with their main focus on quadratic ones, oft-times being the only manageable case. Let us, for the moment, be contented with mentioning Baues and Pirashvili's

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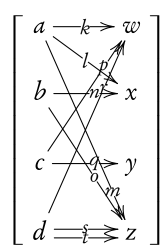
classification [2] of quadratic functors on Groups, and Hartl and Vespa's account [8] of quadratic functors from Pointed Categories to Abelian Groups. Also, it will not be superfluous to note Hartl, Pirashvili, and Vespa's recently completed classification [9] of polynomial functors from Free Algebras over set operads (which includes the cases of abelian groups and groups) to Abelian Groups.

We shall be concerned with classifying endofunctors on Module Categories, with a particular eye on polynomial ones. Work in this direction was presumably initiated by Baues, with his classification [1] of quadratic functors, and then widely expanded upon by Baues, Dreckman, Franjou, and Pirashvili in their now classical tract [3], in which polynomial endofunctors of Abelian Groups are given an accurate characterisation. To this end, the authors deploy the category of finite sets and surjections, an ingenious feat, but not without certain draw-backs. Firstly, their scheme fails to bring about a classification of *all* abelian group functors, polynomial or not. This is because, at the core of their argument, lies a structure theorem for functors on Commutative *Monoids*. Passing to polynomial functors, it will be recalled, has the curious effect of eradicating the distinction between monoids and groups. Secondly, attempts to generalise their argument to arbitrary module categories will encounter quandaries, as it is not immediately clear what category one should employ in lieu of Surjections.

We propose to resolve both of these difficulties by introducing the *Labyrinth Category* $\mathfrak{Laby}$, consisting of finite sets with *mazes* as the arrows between them. A *maze* from the set $\{a, b, c, d\}$ to the set $\{w, x, y, z\}$ (say) is something like the following diagram:



where $k, l, m, n, o, p, q, r, s, t$ are scalars. The reason for applying the name *maze* to such a contraption should not be difficult to divine.

The Labyrinth Category successfully captures the skeletal structure of the whole module category and enables the reduction of a functor to a significantly smaller collection of data. Polynomial functors find a natural interpretation in this frame-work, as do numerical functors. (The honourable reader is referred to the preliminary section for definitions.) Our main theorems along this line are as follows. The base ring is betokened by $\mathbf{B}$ and its category of right modules by $\mathfrak{Mod} = \mathfrak{Mod}_{\mathbf{B}}$. The symbol $\mathfrak{XMod}$ denotes the subcategory of finitely generated, free modules. Functors from $\mathfrak{Laby}$ are always assumed linear, while functors on the module category may be of quite an arbitrary nature.

Theorem 14: Labyrinth of Fun. — *The functor*

$$\Phi\colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Lin}(\mathfrak{Laby}, \mathfrak{Mod}),$$

where $\Phi(F)\colon \mathfrak{Laby} \to \mathfrak{Mod}$ *maps the set* X *to the cross-effect of rank* X *of* F*, evaluated on* $\mathbf{B}$*; is an equivalence of categories.*

Theorem 15. — *The module functor* F *is polynomial of degree* n *if and only if* $\Phi(F)$ *vanishes on sets with more than* n *elements.*

Theorem 16. — *Assuming a binomial base ring, the module functor* F *is numerical of degree* n *if and only if it suffices to specify the action of* $\Phi(F)$ *on* ***pure*** *mazes; i. e., mazes carrying only the label* 1*.*

Roughly speaking, mazes should be thought of as deviations, and the latter theorem substantiates the known fact that, for numerical functors, co-efficients from inside the argument may be brought out front, as testified by Theorem 10 of [14]. The quotient category encoding numerical functors will be denoted by $\mathfrak{Laby}_n$, so that: *Numerical functors, of degree n, are equivalent to linear functors from* $\mathfrak{Laby}_n$ *to Modules.*

We interpose here, referring to Theorem 13 below for the precise statement, that the labyrinth description will indeed be equivalent to Baues, Dreckmann, Franjou, and Pirashvili's approach [3] using surjections, contingent on the hypotheses that: (1) polynomiality have been duly assumed, and (2) the base ring be $\mathbf{Z}$.

The further question, as to whether strict polynomial functors admit a similar formatting, found its complete solution in Salomonsson's doctoral thesis [11]. Denoting the category of multi-sets of cardinality n by $\mathfrak{MSet}_n$, the main result that shall concern us may, slightly paraphrased, be summarised thus: *Homogeneous functors, of degree n, are equivalent to linear functors from* $\mathfrak{MSet}_n$ *to Modules.* (See Theorem 18.)

According to Theorem 11 below, there is an isomorphism

$${}_{\mathbf{B}}\mathfrak{Laby}_n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n,$$

whose heuristical significance it may not be superfluous to point out. Namely, the category of numerical functors over an *arbitrary* binomial ring will be found "identical in structure" to the category of numerical — or polynomial — functors over $\mathbf{Z}$, for

$$\begin{aligned} {}_{\mathbf{B}}\mathfrak{Num}_n &\cong \operatorname{Lin}_{\mathbf{B}}({}_{\mathbf{B}}\mathfrak{Laby}_n, \mathfrak{Mod}) \cong \operatorname{Lin}_{\mathbf{B}}(\mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n, \mathfrak{Mod}) \\ &\cong \mathbf{B} \otimes_{\mathbf{Z}} \operatorname{Lin}_{\mathbf{Z}}({}_{\mathbf{Z}}\mathfrak{Laby}_n, \mathfrak{Mod}) \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Num}_n. \end{aligned}$$

The corresponding result for homogeneous functors holds more trivially, for here it is an inherence of the construction that

$${}_{\mathbf{B}}\mathfrak{MSet}_n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{MSet}_n.$$

One obvious mode of relating the two strains of functors — numerical and strict polynomial — will be to juxtapose their combinatorics. To this end, we establish the existence of a functor

$$A_n \colon \mathfrak{Laby}_n \to \mathfrak{MSet}_n,$$

called the *Ariadne functor*, by the aid of which two principal results are procured:

THEOREM 22. — *Pre-composition with the Ariadne functor begets the forgetful functor from Homogeneous to Numerical Functors.*

THEOREM 23: THE POLYNOMIAL FUNCTOR THEOREM. — *Let F be a numerical functor of degree n, corresponding to the labyrinth module $H\colon \mathfrak{Laby}_n \to \mathfrak{Mod}$. Then F may be given a homogeneous structure of degree n if and only if the following criteria are met:*

1. *F is **quasi-homogeneous**, that is, it satisfies the equation*

$$F(r\alpha) = r^n F(\alpha),$$

   for any $r \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$ and homomorphism α (see Definition 7).

2. *H admits a factorisation through $\mathfrak{Laby}^{\oplus n}$ (see Definition 27).*

An equipollent variant is Theorem 23 of [14], stated using the language of modules.

The construction of $\mathfrak{Laby}^{\oplus n}$ should be of considerable interest, as it supplies the key to appreciating the exact obstructions for polynomial functors to be strict polynomial.

This research was carried out while a graduate student at Stockholm University under the eminent supervision of Prof. Torsten Ekedahl.

CONTENTS

Section the Noughth.
Polynomial Functors

We begin by reviewing the basic theory of Polynomial and Strict Polynomial Functors.

For the entirety of this article, **B** shall denote a fixed base ring of scalars. All modules, homomorphisms, and tensor products shall be taken over this **B**, unless other-wise stated. We let $\mathfrak{Mod} = {}_{\mathbf{B}}\mathfrak{Mod}$ betoken the category of (unital) modules over this ring, and we denote by $\mathfrak{XMod}$ be the category[1] of those modules that are finitely generated and free.

When A and B are linear categories (enriched over $\mathfrak{Mod}$), the symbols $\mathrm{Fun}(A,B)$ and $\mathrm{Lin}(A,B)$ denote the corresponding categories of functors and linear functors, respectively.

We shall also use the standard notation $[n] = \{1,\ldots,n\}$.

§1. Polynomial Functors. A **module functor** is a functor $\mathfrak{XMod} \to \mathfrak{Mod}$ — and, so as to avoid any misunderstandings, we duly emphasise that linearity will *not* be assumed. We shall be wholly content to consider such restricted functors exclusively, for a functor defined on the subcategory $\mathfrak{XMod}$ always has a canonical well-behaved extension to the whole module category $\mathfrak{Mod}$. The details are spelled out in [14].

[1] The letter X herein is intended to suggest "eXtra nice modules".

Let us recall the classical notions of polynomiality. The subsequent definitions made their first appearance in print, albeit somewhat implicitly, in Eilenberg & Mac Lane's monumental article [4], sections 8 and 9:

Definition 1. — Let $\varphi\colon M \to N$ be a map of modules. The n'th **deviation** of φ is the map

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \varphi\left(\sum_{i \in I} x_i\right)$$

of $n+1$ variables.

Definition 2. — The map $\varphi\colon M \to N$ is **polynomial** of degree n if its n'th deviation vanishes:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

for any $x_1, \ldots, x_{n+1} \in M$.

Definition 3. — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is **polynomial** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

is.

Classically, module functors have been analysed in terms of their *cross-effects*.

Definition 4. — Let F be a module functor and let k be a natural number. The **cross-effect** of rank k is the multi-functor

$$\operatorname{cr}_k F(M_1, \ldots, M_k) = \operatorname{Im} F\left(\pi_1 \diamond \cdots \diamond \pi_k\right),$$

where $\pi_i\colon \bigoplus M_i \to \bigoplus M_i$ denote the canonical projections.

Theorem 1: The Cross-Effect Decomposition ([4], Theorem 9.1).

$$F(M_1 \oplus \cdots \oplus M_k) = \bigoplus_{I \subseteq [k]} \operatorname{cr}_I F\left((M_i)_{i \in I}\right).$$

A functor is polynomial of degree n if and only if the cross-effects of rank exceeding n vanish.

§2. Numerical Functors. Next, let us suppose the base ring $\mathbf{B}$ to be *binomial*[2] in the sense of Hall ([7]); that is, commutative, unital, and in the possession of binomial co-efficients. Examples include the ring of integers, as well as all $\mathbf{Q}$-algebras.

[2]This is a *numerical ring* in the terminology [5] of Ekedahl. For the equivalence of the two notions, a proof is offered in [15].

Definition 5 ([13], Definition 5). — The map $\varphi\colon M \to N$ is **numerical** of degree (at most) n if it satisfies the following two equations:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0, \qquad x_1, \ldots, x_{n+1} \in M;$$
$$\varphi(rx) = \sum_{k=0}^{n} \binom{r}{k} \varphi\left(\Diamond_k x\right), \qquad r \in \mathbf{B},\ x \in M.$$

Definition 6 ([14], Definition 10). — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is **numerical** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is.

We denote the abelian category of numerical functors of degree n by $\mathfrak{Num}_n$.

Definition 7 ([14], Definition 11). — The numerical functor F is **quasi-homogeneous** of degree n if the extension functor[3]

$$F\colon \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{XMod} \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Mod}$$

satisfies the equation

$$F(r\alpha) = r^n F(\alpha),$$

for any $r \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$ and homomorphism α.

The category of quasi-homogeneous functors of degree n will be denoted by the symbol $\mathfrak{QHom}_n$. Being quasi-homogeneous is an obvious necessary condition for a functor to admit a homogeneous structure (see below). The Polynomial Functor Theorem below will provide a sufficient condition.

§3. The Deviations. Since the Labyrinth Category formalises deviations, it will be deemed pivotal, for a proper understanding of the theory to follow, to acquire some cognisance of the machinery of the latter. To this end, we establish the *Deviation Formula*, which seems to have made its first appearance in [12].

For (multi-)sets A and B, we shall write $K \sqsubseteq A \times B$ to indicate that $K \subseteq A \times B$ and that both canonical projections are surjective.

Lemma 1. — *Let m and n be natural numbers, and let $L \subseteq [m] \times [n]$. Then*

$$\sum_{L \subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|} = 0,$$

unless L is of the form $P \times Q$, for some $P \subseteq [m]$, $Q \subseteq [n]$.

[3] It will be recalled that binomial rings are torsion-free.

Proof. If L is not of the given form, there exists an (a,b) which is not in L, but such that some (a,j) and some (i,b) are in L. Then, for any $K \subseteq [m]\times[n]$ containing (a,b), the set K will satisfy the given inclusions if and only if $K\setminus\{(a,b)\}$ does. Because the cardinalities of these sets differ by 1, the corresponding terms in the sum will have opposite signs, and hence cancel. $\square$

LEMMA 2. — *Let m, n, p, and q be natural numbers. Then*

$$\sum_{[p]\times[q]\subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|} = (-1)^{m+n+p+q+pq}.$$

Proof. Let $Y(m,n,k)$ denote the number of sets K of cardinality k satisfying

$$[p]\times[q] \subseteq K \sqsubseteq [m]\times[n].$$

The formula is evidently true for $m=p$ and $n=q$, for then $Y(p,q,pq)=1$, and all other $Y(p,q,k)=0$.

We now proceed by recursion. Consider the pair $(m,n)\in[m]\times[n]$. The sets K containing (m,n) will fall into two classes: those where (m,n) is *mandatory* in order to satisfy $K\sqsubseteq[m]\times[n]$, and those where it is not. For a K in the latter class, removing (m,n) will yield another set counted in the sum above, but of cardinality decreased by 1. Since these two types of sets exactly pair off, with opposing signs, their contribution to the given sum is 0.

Consider then those K of which (m,n) *is* a mandatory element. They fall into three categories:

- Some $(m,j)\in K$, for $1\leqslant j\leqslant n-1$, but no $(i,n)\in K$, for $1\leqslant i\leqslant m-1$. The number of such sets is $Y(m,n-1,k-1)$.
- No $(m,j)\in K$, for $1\leqslant j\leqslant n-1$, but some $(i,n)\in K$, for $1\leqslant i\leqslant m-1$. The number of such sets is $Y(m-1,n,k-1)$.
- No $(m,j)\in K$, for $1\leqslant j\leqslant n-1$, and no $(i,n)\in K$, for $1\leqslant i\leqslant m-1$. The number of such sets is $Y(m-1,n-1,k-1)$.

Induction yields

$$\begin{aligned}
&\sum_k (-1)^k Y(m,n,k)\\
&=\sum_k (-1)^k\big(Y(m,n-1,k-1)+Y(m-1,n,k-1)+Y(m-1,n-1,k-1)\big)\\
&=-\big((-1)^{m+n-1+p+q+pq}+(-1)^{m-1+n+p+q+pq}+(-1)^{m-1+n-1+p+q+pq}\big)\\
&=(-1)^{m+n+p+q+pq},
\end{aligned}$$

as desired. $\square$

Theorem 2: The Deviation Formula. — *For a module functor F and homomorphisms $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$, the following equation holds:*

$$F(\alpha_1 \diamond \cdots \diamond \alpha_m) \circ F(\beta_1 \diamond \cdots \diamond \beta_n) = \sum_{K \sqsubseteq [m]\times[n]} F\left(\mathop{\Diamond}_{(i,j)\in K} \alpha_i\beta_j \right).$$

Proof. We have

$$\begin{aligned}
\sum_{K \sqsubseteq [m]\times[n]} F\left(\mathop{\Diamond}_{(i,j)\in K} \alpha_i\beta_j \right) &= \sum_{K \sqsubseteq [m]\times[n]} \sum_{L\subseteq K} (-1)^{|K|-|L|} F\left(\sum_{(i,j)\in L} \alpha_i\beta_j \right) \\
&= \sum_{L \sqsubseteq [m]\times[n]} \sum_{L\subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|-|L|} F\left(\sum_{(i,j)\in L} \alpha_i\beta_j \right) \\
&= \sum_{L \sqsubseteq [m]\times[n]} (-1)^{|L|} F\left(\sum_{(i,j)\in L} \alpha_i\beta_j \right) \sum_{L\subseteq K \sqsubseteq [m]\times[n]} (-1)^{|K|} \\
&= \sum_{P\times Q \sqsubseteq [m]\times[n]} (-1)^{|P||Q|} F\left(\sum_{(i,j)\in P\times Q} \alpha_i\beta_j \right) (-1)^{m+n+|P|+|Q|+|P||Q|} \\
&= \sum_{P \sqsubseteq [m]} (-1)^{m-|P|} F\left(\sum_{i\in P} \alpha_i \right) \sum_{Q \sqsubseteq [n]} (-1)^{n-|Q|} F\left(\sum_{j\in Q} \beta_j \right) \\
&= F(\alpha_1 \diamond \cdots \diamond \alpha_m) F(\beta_1 \diamond \cdots \diamond \beta_n).
\end{aligned}$$

In the fifth step the lemmata were used to evaluate the inner sum. □

§4. Strict Polynomial Functors. Let us now recall the strict polynomial maps ("lois polynomes") from the work of Roby and the strict polynomial functors introduced by Friedlander and Suslin. The base ring $\mathbf{B}$ may here be taken commutative and unital only.

Definition 8 ([10], section 1.2). — A **strict polynomial map** is a natural transformation

$$\varphi \colon M \otimes - \to N \otimes -$$

betwixt functors $\mathfrak{CAlg} \to \mathfrak{Set}$, where $\mathfrak{CAlg} = {}_{\mathbf{B}}\mathfrak{CAlg}$ designates the category of commutative, unital algebras over the ring $\mathbf{B}$, and $\mathfrak{Set}$ denotes the category of sets.

Definition 9 ([6], Definition 2.1). — The functor $F \colon \mathfrak{XMod} \to \mathfrak{Mod}$ is **strict polynomial** of degree n if the arrow maps

$$F \colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

have been given a (multiplicative) strict polynomial structure. The functor is called **homogeneous** if all its arrow maps are.

Strict polynomial maps and functors decompose as the direct sum of their homogeneous components. We shall denote by $\mathfrak{Hom}_n$ the abelian category of homogeneous functors of degree n.

For strict polynomial, or even strict analytic (see [14], §10), functors, the cross-effects may be decomposed further. Let F be a homogeneous functor. The arrow map
$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$
will factor through the universal homogeneous map
$$\operatorname{Hom}(M,N) \to \Gamma^n \operatorname{Hom}(M,N)$$
(see [10], where the full details have been expounded upon), producing a linear map
$$F\colon \Gamma^n \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N)).$$
This gives meaning to the symbol $F(\alpha^{[A]})$, when α_a, for $a \in \#A$, are homomorphisms and A a multi-set of cardinality n. (Multi-sets will be discussed below, in Section 1.)

Definition 10. — Let F be a strict polynomial functor, and let A be a multi-set whose support is $[k]$. The **multi-cross-effect** of rank A is the multi-functor
$$\operatorname{cr}_A F(M_1,\dots,M_k) = \operatorname{Im} F(\pi^{[A]}),$$
where $\pi_i\colon \bigoplus M_i \to \bigoplus M_i$ denote the canonical projections.

In particular, the meaning of the symbol $\operatorname{cr}_A F$, when A is a proper set, is unequivocal.

Theorem 3: The Multi-Cross-Effect Decomposition. — *When F is a homogeneous functor of degree n, then*
$$\operatorname{cr}_{[k]} F(M_1,\dots,M_k) = \bigoplus_{\substack{\#A=[k]\\|A|=n}} \operatorname{cr}_A F(M_1,\dots,M_k),$$
and, consequently,
$$F(M_1 \oplus \cdots \oplus M_k) = \bigoplus_{\substack{\#A\subseteq[k]\\|A|=n}} \operatorname{cr}_A F\left((M_a)_{a\in\#A}\right).$$

Proof. See, for example, [12], Theorem 10. □

SECTION THE FIRST.
MULTI-SETS

§1. Multi-Sets.

DEFINITION 11. — A **multi-set** is a pair

$$M = (\#M, \deg_M),$$

where $\#M$ is a set and

$$\deg_M \colon \#M \to \mathbf{Z}^+$$

is a function, called the **degree** or **multiplicity**. The underlying set $\#M$ is called the **support** of M.

Obviously, the support is uniquely determined by the degree function, whence it suffices to specify the latter.

The degree $\deg_M a$ counts the "number of times $a \in \#M$ occurs in M". The degree of the whole multi-set M we define to be

$$\deg M = \prod_{x \in \#M} (\deg x)!.$$

DEFINITION 12. — The **cardinality** of M is

$$|M| = \sum_{x \in M} 1 = \sum_{x \in \#M} \deg x.$$

The cardinality counts the number of elements *with* multiplicity. We tacitly assume all multi-sets under discussion to be *finite*, as these are the only ones we will ever need.

DEFINITION 13. — The following operations are defined on multi-sets.

1. The **union** $A \cup B$ of A and B is

$$\deg_{A \cup B} = \max(\deg_A, \deg_B).$$

2. The **disjoint union** $A \sqcup B$ of A and B is

$$\deg_{A \sqcup B} = \deg_A + \deg_B.$$

3. The **intersection** $A \cap B$ of A and B is

$$\deg_{A \cap B} = \min(\deg_A, \deg_B).$$

4. The **relative complement** $A \backslash B$ of B in A is

$$\deg_{A \backslash B} = \max(\deg_A - \deg_B, 0).$$

5. The **direct product** $A \times B$ of A and B is

$$\deg_{A \times B} = \deg_A \cdot \deg_B \colon \#A \times \#B \to \mathbf{Z}^+.$$

Definition 14. — A is a **sub-multi-set**[4] of B, written $A \subseteq B$, if

$$\deg_A \leqslant \deg_B$$

(element-wise inequality).

§2. Multations.

Definition 15. — Let A and B be multi-sets of equal cardinality. A **multation** $A \to B$ is a *sub-multi-set* of $A \times B$ whose multi-set of first co-ordinates equals A and whose multi-set of second co-ordinates equals B.

Informally, a multation pairs off the elements of one multi-set with those of another. The degree $\deg_\mu(a, b)$ counts the number of times $a \in A$ is paired off with $b \in B$. A multation $A \to B$ may be written as a two-row matrix, with the elements of A on top of those of B, the way permutations are usually written (indeed, multations should be thought of as generalised such).

Given a multation

$$\begin{bmatrix} a_1 & a_1 & \dots & a_2 & a_2 & \dots \\ b_1 & b_1 & \dots & b_2 & b_2 & \dots \end{bmatrix},$$

with m_j appearances of the column $\begin{bmatrix} a_j \\ b_j \end{bmatrix}$, we shall adopt the perspective of viewing it as a formal product

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^{[m_1]} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}^{[m_2]} \dots$$

of *divided powers*[5].

Example 1. — There exist two multations from the multi-set $\{a, a, b\}$ to itself, namely:

$$\begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}^{[2]} \begin{bmatrix} b \\ b \end{bmatrix} \qquad \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}.$$

△

[4] Some scholars would no doubt say *multi-subset*.

[5] The divided power $z^{[n]}$ should be thought of as $\frac{z^n}{n!}$.

§3. The Multi-Set Category. Let $\mu\colon B \to C$ and $\nu\colon A \to B$ be two multations, where $|A| = |B| = |C| = n$. Their **composition** or **product** $\mu \circ \nu$ is found by identifying the co-efficient of $x^{\mu}y^{\nu}$ in the equation

$$\left(\sum_{\substack{b\in\#B\\ c\in\#C}} x_{cb}\begin{bmatrix} b\\ c\end{bmatrix}\right)^{[n]} \circ \left(\sum_{\substack{a\in\#A\\ b\in\#B}} y_{ba}\begin{bmatrix} a\\ b\end{bmatrix}\right)^{[n]} = \left(\sum_{\substack{a\in\#A\\ b\in\#B\\ c\in\#C}} x_{cb}y_{ba}\begin{bmatrix} a\\ c\end{bmatrix}\right)^{[n]},$$

thought of as an equality of divided powers in the formal variables x and y.

The composition may also be viewed thus. The formal multiplication of columns, given by

$$\begin{bmatrix} b'\\ c\end{bmatrix} \circ \begin{bmatrix} a\\ b\end{bmatrix} = \begin{cases} \begin{bmatrix} a\\ c\end{bmatrix} & \text{if } b = b',\\ 0 & \text{if } b \neq b',\end{cases}$$

makes the free module of columns into an algebra A. Multation composition is then simply the natural multiplication

$$u^{[n]} \circ v^{[n]} = (u \circ v)^{[n]}$$

on the module $\Gamma^n(A)$ of divided n'th powers.

Example 2. — To calculate the composition

$$\begin{bmatrix} c & d & d\\ e & e & f\end{bmatrix} \circ \begin{bmatrix} a & a & b\\ c & d & d\end{bmatrix},$$

we deploy the equation

$$\begin{aligned}&\left(x_{ec}\begin{bmatrix} c\\ e\end{bmatrix} + x_{ed}\begin{bmatrix} d\\ e\end{bmatrix} + x_{fd}\begin{bmatrix} d\\ f\end{bmatrix}\right)^{[3]} \circ \left(y_{ca}\begin{bmatrix} a\\ c\end{bmatrix} + y_{da}\begin{bmatrix} a\\ d\end{bmatrix} + y_{db}\begin{bmatrix} b\\ d\end{bmatrix}\right)^{[3]}\\ &= \left(x_{ec}y_{ca}\begin{bmatrix} a\\ e\end{bmatrix} + x_{ed}y_{da}\begin{bmatrix} a\\ e\end{bmatrix} + x_{ed}y_{db}\begin{bmatrix} b\\ e\end{bmatrix} + x_{fd}y_{da}\begin{bmatrix} a\\ f\end{bmatrix} + x_{fd}y_{db}\begin{bmatrix} b\\ f\end{bmatrix}\right)^{[3]}.\end{aligned}$$

Identification of the co-efficient of $x_{ec}x_{ed}x_{fd}y_{ca}y_{da}y_{db}$ yields

$$\begin{bmatrix} c & d & d\\ e & e & f\end{bmatrix} \circ \begin{bmatrix} a & a & b\\ c & d & d\end{bmatrix} = 2\begin{bmatrix} a & a & b\\ e & e & f\end{bmatrix} + \begin{bmatrix} a & a & b\\ e & f & e\end{bmatrix}.$$

Similarly, by picking the co-efficients of $x_{ec}x_{ed}^2y_{ca}y_{da}^2$, we shall find

$$\begin{bmatrix} c & d & d\\ e & e & e\end{bmatrix} \circ \begin{bmatrix} a & a & a\\ c & d & d\end{bmatrix} = 3\begin{bmatrix} a & a & a\\ e & e & e\end{bmatrix}.$$

△

There is a simple, combinatorial rule for calculating the composition. Namely, the composition of two *ordinary* products (*not* divided powers) of columns is found by "summing over all possibilities of composing them":

$$\left(\begin{bmatrix} b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} b_n \\ c_n \end{bmatrix}\right) \circ \left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_n \\ b_n \end{bmatrix}\right) = \sum_\sigma \left(\begin{bmatrix} a_1 \\ c_{\sigma(1)} \end{bmatrix} \cdots \begin{bmatrix} a_n \\ c_{\sigma(n)} \end{bmatrix}\right),$$

where the sum is to be taken over all permutations $\sigma\colon [n] \to [n]$ such that $b_j = b_{\sigma(j)}$ for all j. We leave it to the reader to check the accuracy of this rule.

Example 3. — Computing according to this device, we find

$$\begin{aligned}\begin{bmatrix} c & d & d \\ e & e & f \end{bmatrix} \circ \begin{bmatrix} a & a & b \\ c & d & d \end{bmatrix} &= \begin{bmatrix} c \\ e \end{bmatrix}\begin{bmatrix} d \\ e \end{bmatrix}\begin{bmatrix} d \\ f \end{bmatrix} \circ \begin{bmatrix} a \\ c \end{bmatrix}\begin{bmatrix} a \\ d \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} \\ &= \begin{bmatrix} a \\ e \end{bmatrix}\begin{bmatrix} a \\ e \end{bmatrix}\begin{bmatrix} b \\ f \end{bmatrix} + \begin{bmatrix} a \\ e \end{bmatrix}\begin{bmatrix} a \\ f \end{bmatrix}\begin{bmatrix} b \\ e \end{bmatrix} = 2\begin{bmatrix} a & a & b \\ e & e & f \end{bmatrix} + \begin{bmatrix} a & a & b \\ e & f & e \end{bmatrix},\end{aligned}$$

and similarly

$$\begin{aligned}\begin{bmatrix} c & d & d \\ e & e & e \end{bmatrix} \circ \begin{bmatrix} a & a & a \\ c & d & d \end{bmatrix} &= \frac{1}{2}\begin{bmatrix} c \\ e \end{bmatrix}\begin{bmatrix} d \\ e \end{bmatrix}\begin{bmatrix} d \\ e \end{bmatrix} \circ \frac{1}{2}\begin{bmatrix} a \\ c \end{bmatrix}\begin{bmatrix} a \\ d \end{bmatrix}\begin{bmatrix} a \\ d \end{bmatrix} \\ &= \frac{1}{4}\cdot 2\begin{bmatrix} a \\ e \end{bmatrix}\begin{bmatrix} a \\ e \end{bmatrix}\begin{bmatrix} a \\ e \end{bmatrix} = 3\begin{bmatrix} a & a & a \\ e & e & e \end{bmatrix}.\end{aligned}$$

△

The **identity multation** ι_A of a multi-set A is the multation in which every element is paired off with itself. It is clear that composition is associative and that the identity multations act as identities.

Definition 16. — The n**'th multi-set category** $\mathfrak{MSet}_n$ is a linear category. Its objects are formal direct sums of multi-sets of cardinality exactly n. For two multi-sets A and B, the arrow set $\mathfrak{MSet}_n(A, B)$ is the free module generated by the multations $A \to B$.

§4. Multi-Sets on Multi-Sets. Unfortunately, later considerations shall shew the necessity of pushing abstraction up to the next level.

Definition 17. — Let M be a multi-set. A **multi-set supported in** M is a multi-set supported in the set

$$M^\# = \{\, (x,k) \in \#M \times \mathbf{Z}^+ \mid 1 \leqslant k \leqslant \deg x \,\}.$$

When speaking of multi-sets supported in a multi-set M, we will let "degree over M" stand for "degree over $M^\#$".

Example 4. — Let M be the set $\{a, b, c\}$ and N the multi-set $\{x, x, y\}$. There are three multi-sets with support M and cardinality 4:

$$\{a, a, b, c\}, \quad \{a, b, b, c\}, \quad \{a, b, c, c\}.$$

In like wise, since

$$N^{\#} = \{(x,1),(x,2),(y,1)\},$$

there are three multi-sets with support N and cardinality 4:

$$\{(x,1),(x,1),(x,2),(y,1)\},\ \{(x,1),(x,2),(x,2),(y,1)\},\ \{(x,1),(x,2),(y,1),(y,1)\}.$$

These three multi-sets all have degree 2 over N.

We shall usually, when deemed suitable, identify these multi-sets with the collection

$$\{x,x,x,y\}, \quad \{x,x,x,y\}, \quad \{x,x,y,y\}.$$

$\triangle$

Section the Second.
Mazes

§1. Mazes. Let X and Y be finite sets. A **passage** from $x \in X$ to $y \in Y$ is a (formal) arrow p from x to y, labelled with an element of **B**, denoted by $\bar{p}$. This we write as

$$p\colon x \to y \quad \text{or} \quad x \xrightarrow{\bar{p}} y\,.$$

Definition 18. — A **maze** from X to Y is a multi-set of passages from X to Y. It is required that there be at least one passage leading from every element of X and at least one passage leading to every element of Y. (We, so to speak, wish to prevent *dead ends* from forming.)

Because a maze is a multi-set, there can (and, in general, will) be multiple passages between any two given elements.

It is perfectly legal to consider the **empty maze** $\varnothing \to \varnothing$. It is the only maze into or out of $\varnothing$, and the only maze possessing no passages.

Definition 19. — We say $P\colon X \to Y$ is a **submaze** of $Q\colon X \to Y$ if $P \subseteq Q$ as multi-sets. This will be denoted by the symbol $P \leqslant Q$.

§2. The Labyrinth Category. Passages $p\colon y \to z$ and $q\colon x \to y$ are said to be **composable**, seeing that one ends where the other begins.

Definition 20. — If $P\colon Y \to Z$ and $Q\colon X \to Y$ are mazes, their **cartesian product** $P \boxtimes Q$ is the multi-set of all pairs of composable passages:

$$P \boxtimes Q = \left\{ \left(\left[z \xleftarrow{p} y \right], \left[y \xleftarrow{q} x \right] \right) \,\middle|\, \left[z \xleftarrow{p} y \right] \in P \ \wedge\ \left[y \xleftarrow{q} x \right] \in Q \right\}.$$

Recall that we, for a sub-multi-set $U \subseteq P \boxtimes Q$, write $U \sqsubseteq P \boxtimes Q$ to indicate that the projections on P and Q are both surjective. Such a set U can be naturally interpreted as a maze itself, viz.:

$$\left\{ \left[z \xleftarrow{pq} x \right] \,\middle|\, \left(\left[z \xleftarrow{p} y \right], \left[y \xleftarrow{q} x \right] \right) \in U \right\}$$

(observe the order in which p and q occur). The surjectivity condition on the projections will prevent dead ends from forming. Henceforth, this identification will be made without comment.

EXAMPLE 5. — Consider the two mazes

$$P = \left[\begin{tikzcd} x & \\ & z \arrow[ul, "a"'] \arrow[dl, "b"] \\ y & \end{tikzcd}\right], \qquad Q = \left[\begin{tikzcd} & x \arrow[dl, "c"'] \\ z & \\ & y \arrow[ul, "d"] \end{tikzcd}\right].$$

Their cartesian product is

$$\begin{aligned} P \boxtimes Q = \Big\{ &\Big(\big[\, x \xleftarrow{a} z \,\big], \big[\, z \xleftarrow{c} x \,\big]\Big), \Big(\big[\, y \xleftarrow{b} z \,\big], \big[\, z \xleftarrow{c} x \,\big]\Big), \\ &\Big(\big[\, x \xleftarrow{a} z \,\big], \big[\, z \xleftarrow{d} y \,\big]\Big), \Big(\big[\, y \xleftarrow{b} z \,\big], \big[\, z \xleftarrow{d} y \,\big]\Big) \Big\}, \end{aligned}$$

which we identify with the maze

$$\left[\begin{tikzcd} x & x \arrow[l, "ac"'] \arrow[dl, "bc"] \\ y & y \arrow[l, "bd"] \arrow[ul, "ad"'] \end{tikzcd}\right].$$

△

We now define the composition of two mazes. As for multi-sets, this composition will not in general be a maze, but rather a *sum* of mazes, and living in the free module generated by those.

DEFINITION 21. — The **composition** or **product** of the mazes P and Q is the formal sum

$$P \circ Q = \sum_{U \sqsubseteq P \boxtimes Q} U.$$

EXAMPLE 6. — Let P and Q be as above. Their composition is

$$\begin{aligned} PQ = &\left[\begin{tikzcd} x & x \arrow[l, "ac"'] \arrow[dl, "bc"] \\ y & y \arrow[l, "bd"] \arrow[ul, "ad"'] \end{tikzcd}\right] + \left[\begin{tikzcd} x & x \arrow[l, "ac"'] \\ y & y \arrow[l, "bd"] \end{tikzcd}\right] + \left[\begin{tikzcd} x & x \arrow[dl, "bc"] \\ y & y \arrow[ul, "ad"'] \end{tikzcd}\right] \\ &+ \left[\begin{tikzcd} x & x \arrow[dl, "bc"] \\ y & y \arrow[l, "bd"] \arrow[ul, "ad"'] \end{tikzcd}\right] + \left[\begin{tikzcd} x & x \arrow[l, "ac"'] \\ y & y \arrow[l, "bd"] \arrow[ul, "ad"'] \end{tikzcd}\right] + \left[\begin{tikzcd} x & x \arrow[l, "ac"'] \arrow[dl, "bc"] \\ y & y \arrow[l, "bd"] \end{tikzcd}\right] + \left[\begin{tikzcd} x & x \arrow[l, "ac"'] \arrow[dl, "bc"] \\ y & y \arrow[ul, "ad"'] \end{tikzcd}\right]. \end{aligned}$$

△

That composition is associative follows from the observation that $(PQ)R$ and $P(QR)$ both equal
$$\sum_{W \sqsubseteq P \boxtimes Q \boxtimes R} W$$
(surjective projections on all three factors). There exist **identity mazes**
$$I_X = \bigcup_{x \in X} \left\{ x \xrightarrow{1} x \right\}.$$

Definition 22. — The **Labyrinth Category** $\mathfrak{Laby}$ is a linear category. Its objects are formal direct sums of finite sets. Given two sets X and Y, the arrow set $\mathfrak{Laby}(X,Y)$ is the module generated by the mazes $X \to Y$, with the following relations imposed, for any multi-set P of passages:

I.
$$\left[P \cup \left\{ * \xrightarrow{0} * \right\} \right] = 0.$$

II.
$$\left[P \cup \left\{ * \xrightarrow{a+b} * \right\} \right] =$$
$$\left[P \cup \left\{ * \xrightarrow{a} * \right\} \right] + \left[P \cup \left\{ * \xrightarrow{b} * \right\} \right] + \left[P \cup \left\{ * \overset{a}{\underset{b}{\rightrightarrows}} * \right\} \right].$$

The second axiom may be generalised by means of mathematical induction to yield the following elementary formulæ.

Theorem 4. — *In the Labyrinth Category, the following two equations hold:*
$$\left[P \cup \left\{ * \xrightarrow{\sum_{i=1}^{n} a_i} * \right\} \right] = \sum_{\varnothing \subset I \subseteq [n]} \left[P \cup \bigcup_{i \in I} \left\{ * \xrightarrow{a_i} * \right\} \right]$$
$$\left[P \cup \bigcup_{i=1}^{n} \left\{ * \xrightarrow{a_i} * \right\} \right] = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left[P \cup \left\{ * \xrightarrow{\sum_{i \in I} a_i} * \right\} \right].$$

§3. The Quotient Labyrinth Categories. The theory developed unto this point makes sense for an arbitrary base ring. In order to construct quotient categories of $\mathfrak{Laby}$, we need the assumption that $\mathbf{B}$ be binomial.

When A is a maze, let I_A denote the maze
$$I_A = \bigcup_{[p:\, x \to y] \in A} \left\{ x \xrightarrow{1} y \right\},$$

in which all passages of A have been re-assigned the label 1.

Definition 23. — The category $\mathfrak{Laby}_n$ is the quotient category obtained from $\mathfrak{Laby}$ when imposing the following relations, for any maze P:

III.
$$P = 0, \quad \text{whenever } |P| > n.$$

IV.
$$P = \sum_{\#A = P} \prod_{p \in P} \binom{\overline{p}}{\deg_A p} I_A.$$

(The sum is in fact finite, owing to the previous axiom.)

Example 7. — When $n = 3$, an instance of the fourth axiom is the following:

$$\left[\begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & * \end{tikzcd}\right] = \binom{a}{1}\binom{b}{1}\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right] + \binom{a}{2}\binom{b}{1}\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, "1"'] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right] + \binom{a}{1}\binom{b}{2}\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, "1"] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right].$$

△

The original Labyrinth Category $\mathfrak{Laby}$ encodes arbitrary module functors. Axioms III and IV have been appended in order to encode *polynomial* and *numerical* functors, respectively. It may be shewn that, over the integers, Axiom IV is actually implied by Axiom III, comparable to how numerical and polynomial functors are equivalent in this setting.

We shall have reason to impose upon the Labyrinth Category yet another axiom. This will be for encoding *quasi-homogeneous* functors (of degree n). It will be proved presently that the category $\mathfrak{Laby}_n$ is free over $\mathbf{B}$, and so, in particular, torsion-free, leading to an inclusion of categories

$$\mathfrak{Laby}_n \subseteq \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}_n.$$

When P is a maze and a is a scalar (in a binomial base ring), denote by $a \boxdot P$ the maze obtained from P by multiplying the labels of all passages by a:

$$a \boxdot P = \left\{ \left[\, y \xleftarrow{ap} x \,\right] \,\middle|\, \left[\, y \xleftarrow{p} x \,\right] \in P \right\}.$$

Definition 24. — The category $\mathfrak{Laby}^n$ is the quotient category obtained from $\mathfrak{Laby}_n$ upon the imposition of the following axiom, for any maze P:

V.
$$a^n P = a \boxdot P, \quad a \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}.$$

Example 8. — The fifth axiom considers the ideal generated in $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}_n$, rather than $\mathfrak{Laby}_n$. The slightly sharper requirement will first make a difference in degree 4. Dividing out by the ideal *generated in* ${}_{\mathbf{Z}}\mathfrak{Laby}_4$ by elements of the form $a^4P - a \boxdot P$ makes it possible to prove

$$2\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, bend left=40, "1"] \arrow[ur, "1"] \arrow[ur, bend right=40, "1"'] \arrow[dr, "1"'] & \\ & * \end{tikzcd}\right] + 2\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, shift left, "1"] \arrow[ur, shift right, "1"'] \arrow[dr, shift left, "1"] \arrow[dr, shift right, "1"'] & \\ & * \end{tikzcd}\right] + 2\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, "1"] \arrow[dr, shift left, "1"] \arrow[dr, shift right, "1"'] & \\ & * \end{tikzcd}\right] = 12\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, "1"] \arrow[dr, "1"] & \\ & * \end{tikzcd}\right],$$

whereas we shall be needing a stronger statement. The full force of Axiom V authorises a division by 2, thus establishing

$$\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, bend left=40, "1"] \arrow[ur, "1"] \arrow[ur, bend right=40, "1"'] \arrow[dr, "1"'] & \\ & * \end{tikzcd}\right] + \left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, shift left, "1"] \arrow[ur, shift right, "1"'] \arrow[dr, shift left, "1"] \arrow[dr, shift right, "1"'] & \\ & * \end{tikzcd}\right] + \left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, "1"] \arrow[dr, shift left, "1"] \arrow[dr, shift right, "1"'] & \\ & * \end{tikzcd}\right] = 6\left[\begin{tikzcd}[row sep=small, column sep=small] & * \\ * \arrow[ur, "1"] \arrow[dr, "1"] & \\ & * \end{tikzcd}\right].$$

$\triangle$

Section the Third. The Ariadne and Theseus Functors

We propose to investigate how the Multi-Set and Labyrinth Categories are related. A functor in one direction is readily found; viz. the *Ariadne functor*

$$A_n \colon \mathfrak{Laby} \to \mathfrak{MSet}_n,$$

so called because it leads the way *out of* the labyrinth. In the case of a binomial base ring, it will factor:

$$A_n \colon \mathfrak{Laby} \to \mathfrak{Laby}_n \to \mathfrak{Laby}^n \to \mathfrak{MSet}_n.$$

A minor modification of the category $\mathfrak{Laby}^n$ will enable us to define a functor in the reverse direction. This is the *Theseus functor* T_n, going *into* the labyrinth. These two functors are inverse to each other. The modifications necessary to undertake on $\mathfrak{Laby}_n$ in order to define T_n, indicate the precise obstructions that may prevent a numerical (polynomial) functor from being strict polynomial. Confer the Polynomial Functor Theorem, Theorem 23, below.

§1. The Ariadne Functor. For the duration of this section, let n be a fixed natural number.

Let P be a maze. Consider the following sum of multations:

$$A_n(P) = \sum_{\substack{\#A=P\\|A|=n}} \left(\frac{1}{\deg_P A} \prod_{[p\colon x\to y]\in A} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sum_{\substack{\#A=P\\|A|=n}} \left(\prod_{[p\colon x\to y]\in P} \bar{p}^{\deg_A p} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_A p]} \right).$$

This will provide a functor from $\mathfrak{Laby}$ to $\mathfrak{MSet}_n$, as we now set out to prove. It is clear that $A_n(P) = 0$ if a single passage of P be labelled 0. Now to shew that

$$A_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) = \tag{1}$$
$$A_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) + A_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) + A_n\left(P \cup \left\{ u \overset{a}{\underset{b}{\rightrightarrows}} v \right\}\right).$$

Denote, when A is a multi-set of passages with support P,

$$\mu_A = \frac{1}{\deg_P A} \prod_{[p\colon x \to y] \in A} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since

$$A_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P \\ |A|=n-m}} \left(\mu_A \cdot (a+b)^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]}\right)$$
$$A_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P \\ |A|=n-m}} \left(\mu_A \cdot a^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]}\right)$$
$$A_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P \\ |A|=n-m}} \left(\mu_A \cdot b^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]}\right)$$
$$A_n\left(P \cup \left\{ u \overset{a}{\underset{b}{\rightrightarrows}} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P \\ |A|=n-m}} \sum_{\substack{i+j=m \\ i,j \geqslant 1}} \left(\mu_A \cdot a^i b^j \begin{bmatrix} u \\ v \end{bmatrix}^{[i]} \begin{bmatrix} u \\ v \end{bmatrix}^{[j]}\right),$$

the relation (1) follows from the equation

$$(x+y)^{[m]} = x^{[m]} + y^{[m]} + \sum_{\substack{i+j=m \\ i,j \geqslant 1}} x^{[i]} y^{[j]},$$

valid in every divided power algebra.

Hence A_n gives a well-defined map on the mazes of the Labyrinth Category. We now prove that it is, in fact, a functor.

Theorem 5. — *The formulæ*

$$A_n(X) = \bigoplus_{\substack{\#A=X \\ |A|=n}} A$$

$$A_n(P) = \sum_{\substack{\#A=P \\ |A|=n}} \left(\prod_{[p\colon x\to y]\in P} \bar{p}^{\deg_A p} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_A p]} \right)$$

provide a linear functor

$$A_n\colon \mathfrak{Laby} \to \mathfrak{MSet}_n.$$

Proof. Let $P\colon Y \to Z$ and $Q\colon X \to Y$ be two mazes. A straight-forward calculation gives at hand:

$$\begin{aligned}
&A_n(P) \circ A_n(Q) \\
&= \sum_{\substack{\#A=P \\ |A|=n}} \left(\prod_{[p\colon y\to z]\in P} \bar{p}^{\deg_A p} \begin{bmatrix} y \\ z \end{bmatrix}^{[\deg_A p]} \right) \circ \sum_{\substack{\#B=Q \\ |B|=n}} \left(\prod_{[q\colon x\to y]\in Q} \bar{q}^{\deg_B q} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_B q]} \right) \\
&= \sum_{\substack{\#C\sqsubseteq P\boxtimes Q \\ |C|=n}} \left(\prod_{[r\colon x\to z]\in \#C} \bar{r}^{\deg_C r} \begin{bmatrix} x \\ z \end{bmatrix}^{[\deg_C r]} \right) \\
&= \sum_{R\sqsubseteq P\boxtimes Q} \sum_{\substack{\#C=R \\ |C|=n}} \left(\prod_{[r\colon x\to z]\in R} \bar{r}^{\deg_C r} \begin{bmatrix} x \\ z \end{bmatrix}^{[\deg_C r]} \right) = A_n \left(\sum_{R\sqsubseteq P\boxtimes Q} R \right) = A_n(PQ).
\end{aligned}$$

The only possibly dubious step here is the third, which follows from the equation

$$\left(\sum_{[p\colon y\to z]\in P} \bar{p} \begin{bmatrix} y \\ z \end{bmatrix} \right)^{[n]} \circ \left(\sum_{[q\colon x\to y]\in Q} \bar{q} \begin{bmatrix} x \\ y \end{bmatrix} \right)^{[n]} = \left(\sum_{\substack{[p\colon y\to z]\in P \\ [q\colon x\to y]\in Q}} \bar{p}\bar{q} \begin{bmatrix} x \\ z \end{bmatrix} \right)^{[n]},$$

after noting that restricting attention to monomials $\bar{p}^A\bar{q}^B$ with $\#A = P$ and $\#B = Q$ in the left-hand side corresponds to considering monomials $(\bar{p}\bar{q})^C$ satisfying the three relations

$$\#C \sqsubseteq P \boxtimes Q, \qquad C_1 = A, \qquad C_2 = B$$

(canonical projections) in the right-hand side. $\square$

Definition 25. — The functor A_n is called the n'th **Ariadne functor**.

Lemma 3. — *When x is an element of a divided power algebra over a binomial ring* $\mathbf{B}$, *then*

$$a^m x^{[m]} = \sum_{k=1}^{\infty} \binom{a}{k} \sum_{\substack{g_1+\cdots+g_k=m \\ g_i\in\mathbf{Z}^+}} x^{[g_1]} \cdots x^{[g_k]}, \qquad a \in \mathbf{B},\ m \in \mathbf{Z}^+.$$

Proof. The lemma will be an easy consequence of the formula

$$a^m = \sum_{k=1}^{\infty} \binom{a}{k} \sum_{g_1+\cdots+g_k=m} \binom{m}{g_1, \ldots, g_k},$$

established by a simple combinatorial argument. When $a \in \mathbf{N}$, both sides count the number of ways to colour m objects in one of a available colours. Since both sides are polynomials, this extends to negative integers as well. For the case of an arbitrary binomial ring, invoke the Binomial Transfer Principle of [15]. □

Theorem 6. — *The Ariadne functor factors through the quotient category $\mathfrak{Laby}_n$, producing a functor*

$$A_n \colon \mathfrak{Laby}_n \to \mathfrak{MSet}_n.$$

Proof. It is clear that $A_n(P) = 0$ when $|P| > n$. In order to prove that A_n respects the relation

$$P = \sum_{\#A=P} \prod_{p\in P} \binom{\bar{p}}{\deg_A p} I_A,$$

it will be sufficient to establish that

$$A_n\left(Q \cup \left\{ u \xrightarrow{a} v \right\}\right) = \sum_{k=1}^{\infty} \binom{a}{k} A_n\left(Q \cup \bigcup_k \left\{ u \xrightarrow{1} v \right\}\right)$$

for any maze Q; one then performs induction on the passages of P. This formula follows from a suitable application of the lemma, by which

$$A_n\left(Q \cup \left\{ u \xrightarrow{a} v \right\}\right) = \sum_{m=1}^{\infty} \sum_{\substack{\#B=Q \\ |B|=n-m}} \left(\prod_{[q\colon x\to y]\in Q} \bar{q}^{\deg_B q} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_B q]} \right) a^m \begin{bmatrix} u \\ v \end{bmatrix}$$

is brought to co-incide with

$$\begin{aligned} &A_n\left(Q \cup \bigcup_k \left\{ u \xrightarrow{1} v \right\}\right) \\ &\quad = \sum_{m=1}^{\infty} \sum_{\substack{\#B=Q \\ |B|=n-m}} \sum_{\substack{g_1+\cdots+g_k=m \\ g_i\in\mathbf{Z}^+}} \left(\prod_{[q\colon x\to y]\in Q} \bar{q}^{\deg_B q} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_B q]} \right) \begin{bmatrix} u \\ v \end{bmatrix}^{[g_1]} \cdots \begin{bmatrix} u \\ v \end{bmatrix}^{[g_k]}. \end{aligned}$$

□

Theorem 7. — *The Ariadne functor factors through the quotient category $\mathfrak{Laby}^n$, producing a functor*

$$A_n \colon \mathfrak{Laby}^n \to \mathfrak{MSet}_n.$$

Proof.

$$A_n(a \boxdot P) = \sum_{\substack{\#A=P \\ |A|=n}} \left(\prod_{[p\colon x \to y] \in P} (a\bar{p})^{\deg_A p} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_A p]} \right) = a^n A_n(P).$$

□

The process of factorisation has now terminated, for passing to the category $\mathfrak{Laby}^n$ has the effect of making the Ariadne functor faithful, which is easily inferred from Theorem 10 below.

§2. Pure Mazes. The Ariadne Functor sheds light on the internal structure of the Labyrinth Categories.

Definition 26. — A maze of which all passages carry the label 1, is called a **pure** maze.

Theorem 8. — *The pure mazes are linearly independent in the category* $\mathfrak{Laby}$.

Proof. Suppose we have a relation

$$\sum_j a_j P_j = 0,$$

where the P_j are distinct pure mazes in $\mathfrak{Laby}(X, Y)$, for some $a_j \in \mathbf{B}$. Suppose all mazes have cardinality at least n. Applying the n'th Ariadne functor will kill all mazes of cardinality greater than n, and the end result will be

$$\sum_{|P_j|=n} a_j A_n(P_j) = 0.$$

Since the P_j are distinct pure mazes, the $A_n(P_j)$ will all be distinct multations, which are linearly independent in $\mathfrak{MSet}_n$. Hence all those $a_j = 0$. The claim now follows by induction. □

Theorem 9. — *The pure mazes constitute a basis for the category* $\mathfrak{Laby}_n$.

Proof. Linear independence goes through exactly as before. From the defining equations for $\mathfrak{Laby}_n$, it follows that any maze will reduce to pure ones. □

Theorem 10. — *The pure mazes with exactly n passages are linearly independent in the category* $\mathfrak{Laby}^n$ *and generate the category over* $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$.

Proof. Linear independence goes through as before. The defining equation for $\mathfrak{Laby}^n$ can be written

$$a^n P = \binom{a}{P} P + \sum_{\substack{\#A=P \\ |P|<|A|\leqslant n}} \binom{a}{A} A.$$

Since $a^n \neq \binom{a}{P}$ if $|P| < n$, such a P may be expressed in terms of mazes with more passages, provided division by integers be permissible. □

Theorem 11. — *The following categories are isomorphic:*

$$ {}_{\mathbf{B}}\mathfrak{Laby}_n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n \qquad {}_{\mathbf{B}}\mathfrak{Laby}^n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n $$

Proof. The first equation is a quick corollary of Theorem 9. Let us prove the second. By Theorem 10, $\mathfrak{Laby}^n$ is torsion-free. From the definition of $\mathfrak{Laby}^n$,

$$ \mathbf{Q} \otimes_{\mathbf{Z}} {}_{\mathbf{B}}\mathfrak{Laby}^n \cong \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n. $$

The category ${}_{\mathbf{B}}\mathfrak{Laby}^n$ (canonically) embeds in the former and $\mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n$ in the latter. □

Because the Ariadne functor embeds ${}_{\mathbf{Z}}\mathfrak{Laby}^n$ in ${}_{\mathbf{Z}}\mathfrak{MSet}_n$, which is free, it follows that ${}_{\mathbf{Z}}\mathfrak{Laby}^n$ is free as well. By the isomorphism above, ${}_{\mathbf{B}}\mathfrak{Laby}^n$ will then be free for arbitrary $\mathbf{B}$, though it seems to possess no preferred basis.

§3. The Theseus Functor. It is the purpose of the present section to adjoin objects to the category $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^n$ to the effect that certain sets split into direct sums. This will enable us to define an inverse to the Ariadne functor.

Lemma 4. — *Let M be a torsion-free module, and let $p(x) \in M \otimes \mathbf{B}[x]$. Then $p = 0$ if and only if $p(a) = 0$ for all integers a.*

Lemma 5. — *In the category $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^n$, the following equation holds, for any set X:*

$$ I_X = \sum_{\substack{\#S = I_X \\ |S| = n}} \frac{1}{\deg S} S. $$

Proof. Use Lemma 4. Identify the co-efficient of a^n in the defining equation for $\mathfrak{Laby}^n$:

$$ a^n I_X = \sum_{\substack{\#S = P \\ |S| \leqslant n}} \binom{a}{S} S. $$

□

The mazes $\frac{1}{\deg S} S$, occurring in the lemma, satisfy

$$ \frac{1}{\deg S} S \circ \frac{1}{\deg T} T = \begin{cases} 0 & \text{if } S \neq T, \\ \frac{1}{\deg S} S & \text{if } S = T. \end{cases} $$

They can thus be said to form a direct sum system, although the *objects* themselves of the system do not exist. There is a simple remedy for this: adjoin to the category $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^n$ an object $\operatorname{Im} \frac{1}{\deg S} S$ for each such S. The category $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^n$ is not, however, the minimal localisation of $\mathfrak{Laby}^n$ for which this procedure makes sense.

Let us say that passages $p\colon x \to y$ and $q\colon x \to y$, sharing starting and ending points, are **parallel**, and that a **simple** maze is one which contains no (pairs of) parallel passages. By means of the Labyrinth Axioms I and II, any maze can be written as the sum of simple mazes.

It may be verified that the mazes $\frac{1}{\deg_P A}A$, where $|A| = n$ and $\#A = P$ for some pure and simple maze P, form a basis for a subcategory C of $\mathbf{Q} \otimes \mathfrak{Laby}^n$ which contains $\mathfrak{Laby}^n$.

DEFINITION 27. — To the category C, adjoin an object $\operatorname{Im} \frac{1}{\deg S}S$ for each maze S, which, as a multi-set, is supported in some maze I_X, and denote the resulting category by $\mathfrak{Laby}^{\oplus n}$.

By Lemma 5, the set X will now split into components:

$$X = \bigoplus_{\substack{\#S = I_X \\ |S| = n}} \operatorname{Im} \frac{1}{\deg S} S.$$

EXAMPLE 9. — It is a consequence of the equation

$$\left[* \overset{1}{\underset{1}{\rightrightarrows}} * \right] = 2 \left[* \xrightarrow{1} * \right]$$

that no localisation will be required in the case $n = 2$, so that, anomalously, the categories $\mathfrak{Laby}^2$ and $\mathfrak{Laby}^{\oplus 2}$ are isomorphic. $\triangle$

EXAMPLE 10. — Consider

$$P = \begin{bmatrix} 1 \overset{1}{\underset{1}{\rightrightarrows}} 1 \\ 2 \xrightarrow[1]{} 2 \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 1 \xrightarrow{1} 1 \\ 2 \overset{1}{\underset{1}{\rightrightarrows}} 2 \end{bmatrix}.$$

In $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^3$, the equations $2I_{\{1,2\}} = P + Q$ and $PQ = QP = 0$ hold; hence, the mazes $\frac{1}{2}P$ and $\frac{1}{2}Q$ would form a direct sum system, were it not for the non-existence of the desired objects. By adjoining these, we obtain the refined category $\mathfrak{Laby}^{\oplus 3}$, in which the set $\{1, 2\}$ splits up into two:

$$\{1, 2\} = \operatorname{Im} \frac{1}{2} P \oplus \operatorname{Im} \frac{1}{2} Q.$$

$\triangle$

DEFINITION 28. — The n'th **Theseus functor**

$$T_n \colon \mathfrak{MSet}_n \to \mathfrak{Laby}^{\oplus n}$$

is given by the following formulæ:

$$A \mapsto \operatorname{Im} \frac{1}{\deg A} \bigcup_{a \in A} \left\{ a \xrightarrow{1} a \right\}$$

$$\mu \mapsto \frac{1}{\deg \mu} \bigcup_{(a,b) \in \mu} \left\{ a \xrightarrow{1} b \right\}.$$

It should be clear that this is indeed a (linear) functor, as composition in both categories is effectuated by "summing over all possibilities".

Theorem 12. — *There is an isomorphism of categories:*

$$\mathfrak{Laby}^{\oplus n} \underset{T_n}{\overset{A_n}{\rightleftarrows}} \mathfrak{MSet}_n$$

Proof. We first shew that the Ariadne functor actually factors through the category $\mathfrak{Laby}^{\oplus n}$. Suppose P is a pure and simple maze, and let A be supported in P, with $|A| = n$. Then

$$A_n(A) = \prod_{p=[1:\, x \to y] \in A} \begin{bmatrix} x \\ y \end{bmatrix} = \deg A \cdot \prod_{p=[1:\, x \to y] \in P} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_A p]},$$

and hence we may extend A_n by letting

$$\frac{1}{\deg_P A} A \mapsto \prod_{p=[1:\, x \to y] \in P} \begin{bmatrix} x \\ y \end{bmatrix}^{[\deg_A p]}.$$

Moreover, A_n maps the "virtual" biproduct system

$$I_X = \sum_{\substack{\#B = X \\ |B| = n}} \frac{1}{\deg B} \bigcup_{b \in B} \left\{ b \xrightarrow{1} b \right\}$$

in $\mathfrak{Laby}^n$ onto the "real" biproduct system

$$\sum_{\substack{\#B = X \\ |B| = n}} \iota_B$$

in $\mathfrak{MSet}_n$, and we may consequently extend A_n to $\mathfrak{Laby}^{\oplus n}$ by defining

$$A_n \left(\operatorname{Im} \frac{1}{\deg B} \bigcup_{b \in B} \left\{ b \xrightarrow{1} b \right\} \right) = B.$$

It is now easy to see that T_n and A_n are inverse to each other. □

§4. The Category of Surjections. For reference, we devote this paragraph to investigating the connexion between the Labyrinth Category and the Category of Surjections explored in [3]. For want of space, we merely sketch the relevant constructions. The reader anxious to learn the full details is referred to Section 5 of *loc. cit.*

Let C be a category possessing weak pull-backs. For two objects $X, Y \in C$, a **correspondence** or **span** from X to Y is a diagram

$$Y \longleftarrow U \longrightarrow X$$

in C, intended to be read from right to left. Construct an additive category $\widehat{C}$ in the following way. Its objects are those of C. Its arrows are formal sums of correspondences of C (identified under an obvious equivalence relation), living in the free abelian group they generate. The composition of two correspondences is found by summing over weak pull-backs. The category $\widehat{C}$ is called the **category of correspondences** on C.

It will now be observed that the category $\mathfrak{Sur}$ of finite sets and *surjections* possesses weak pull-backs. It is thus possible to build the category $\widehat{\mathfrak{Sur}}$ of surjection correspondences. We form a quotient category $\widehat{\mathfrak{Sur}}_n$ by forcing

$$\Big[\ Y \twoheadleftarrow U \twoheadrightarrow X\ \Big] = 0 \quad \text{whenever } |U| > n.$$

Theorem 13.

$$\widehat{\mathfrak{Sur}}_n \cong {}_{\mathbf{Z}}\mathfrak{Laby}_n.$$

Proof. Define the functor

$$\Xi\colon \widehat{\mathfrak{Sur}}_n \to {}_{\mathbf{Z}}\mathfrak{Laby}_n$$

to be the identity on objects. The correspondence

$$\varphi = \left[\ Y \xleftarrow{\ \varphi_*\ } U \xrightarrow{\ \varphi^*\ } X\ \right]$$

in $\widehat{\mathfrak{Sur}}_n$ shall map to the pure maze $X \to Y$ with exactly

$$\big|(\varphi^*, \varphi_*)^{-1}(x, y)\big|$$

passages going $X \ni x \to y \in Y$.

A simple and straight-forward calculation confirms that this yields a functor, which is inversible because the pure mazes form a basis. □

Curiously enough, the categories $\widehat{\mathfrak{Sur}}$ and ${}_{\mathbf{Z}}\mathfrak{Laby}$ are themselves not isomorphic. This stems from the fact that ${}_{\mathbf{Z}}\mathfrak{Laby}$ encodes functors from the category of free abelian groups, while $\widehat{\mathfrak{Sur}}$ was built to encode functors from the category of free commutative *monoids*. These functor categories are not originally equivalent, but they will be, once polynomiality is brought into effect.

Section the Fourth.
The Combinatorics of Functors

§1. Module Functors. Our first aim is to shew how general module functors are encoded by the Labyrinth Category. The base ring $\mathbf{B}$ is presumed unital only, not necessarily commutative. It will be expedient to point out that finitely generated, free modules are automatically bimodules. For homomorphisms, however, the left–right distinction is essential, and we hereby declare all maps under consideration to be *right module homomorphisms* (and hence left multiplication by a matrix of ring elements).

Let

$$\sigma_{yx}\colon \mathbf{B}^X \to \mathbf{B}^Y, \quad x \in X,\ y \in Y$$

denote the canonical *transportations*, mapping a 1 in position x to a 1 in position y, and everything else to 0.

Moreover, let $\operatorname{cr}_X F(\mathbf{B})$ denote the cross-effect of rank X of the functor F, evaluated on $|X|$ copies of the module $\mathbf{B}$. Similarly, when $\zeta\colon F \to G$ is a natural transformation, let

$$(\operatorname{cr}_X \zeta)_{\mathbf{B}}\colon\ \operatorname{cr}_X F(\mathbf{B}) \to \operatorname{cr}_X G(\mathbf{B})$$

denote the evaluation of $\operatorname{cr}_X \zeta$ on $|X|$ copies of $\mathbf{B}$.

We propose a study of the functor

$$\Phi\colon\ \operatorname{Fun}(\mathfrak{X}\mathfrak{Mod}, \mathfrak{Mod}) \to \operatorname{Lin}(\mathfrak{Laby}, \mathfrak{Mod}),$$

defined as follows. Given a module functor $F\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$, the corresponding labyrinth functor $\Phi(F)\colon \mathfrak{Laby} \to \mathfrak{Mod}$ should take:

$$X \mapsto \operatorname{cr}_X F(\mathbf{B})$$
$$[P\colon X \to Y] \mapsto \left[F\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right)\colon\ \operatorname{cr}_X F(\mathbf{B}) \to \operatorname{cr}_Y F(\mathbf{B})\right].$$

Of course, one ought to restrict the action to the appropriate cross-effects, but this turns out to be an unnecessary caution:

Lemma 6. — *The map*

$$\upsilon = F\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right)\colon F(\mathbf{B}^X) \to F(\mathbf{B}^Y)$$

is in fact a map $\operatorname{cr}_X F(\mathbf{B}) \to \operatorname{cr}_Y F(\mathbf{B})$, *in the sense that there is a commutative diagram:*

$$\begin{tikzcd}
F(\mathbf{B}^X) \arrow[r, "\upsilon"] \arrow[d] & F(\mathbf{B}^Y) \\
\operatorname{cr}_X F(\mathbf{B}) \arrow[r, "\upsilon"'] & \operatorname{cr}_Y F(\mathbf{B}) \arrow[u]
\end{tikzcd}$$

Lemma 7. — $\Phi(F)$ *is a functor* $\mathfrak{Laby} \to \mathfrak{Mod}$.

Proof. That $\Phi(F)$ respects the relations in $\mathfrak{Laby}$ follows from

$$\Phi(F)\left(P \cup \{\, x \xrightarrow{0} y \,\}\right) = F(\cdots \diamond 0) = 0$$

and

$$\begin{aligned}
&\Phi(F)\left(P \cup \{\, x \xrightarrow{a+b} y \,\}\right) = F(\cdots \diamond (a+b)\sigma_{yx})\\
&\quad = F(\cdots \diamond a\sigma_{yx}) + F(\cdots \diamond b\sigma_{yx}) + F(\cdots \diamond a\sigma_{yx} \diamond b\sigma_{yx})\\
&\quad = \Phi(F)\left(P \cup \{\, x \xrightarrow{a} y \,\}\right) + \Phi(F)\left(P \cup \{\, x \xrightarrow{b} y \,\}\right) + \Phi(F)\left(P \cup \{\, x \overset{a}{\underset{b}{\rightrightarrows}} y \,\}\right).
\end{aligned}$$

Functoriality of $\Phi(F)$ is a consequence of the Deviation Formula and the definition of maze composition. □

Let $\zeta\colon F \to G$ be a natural transformation. Define $\Phi(\zeta)\colon \Phi(F) \to \Phi(G)$ by restriction to the appropriate cross-effects:

$$\Phi(\zeta)_X = (\operatorname{cr}_X \zeta)_{\mathbf{B}}\colon\ \operatorname{cr}_X F(\mathbf{B}) \to \operatorname{cr}_X G(\mathbf{B}).$$

Lemma 8. — Φ *is a functor*

$$\operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Lin}(\mathfrak{Laby}, \mathfrak{Mod}).$$

Proof. Follows from the functoriality of cr_X. □

Lemma 9. — Φ *is fully faithful.*

Proof. The natural transformation ζ can be uniquely re-assembled from its components $\operatorname{cr}_X \zeta$. □

We now construct the inverse of Φ. Given a labyrinth functor $H\colon \mathfrak{Laby} \to \mathfrak{Mod}$, the corresponding module functor $\Phi^{-1}(H)$ should take:

$$\mathbf{B}^A \mapsto \bigoplus_{X \subseteq A} H(X)$$

$$\left[\sum_{\substack{a\in A\\ b\in B}} s_{ba}\sigma_{ba}\colon \mathbf{B}^A \to \mathbf{B}^B\right] \mapsto \sum_{K \subseteq B\times A} H\left(\bigcup_{(b,a)\in K} \left\{\ b \xleftarrow{s_{ba}} a\ \right\}\right).$$

Lemma 10. — $\Phi^{-1}(H)$ *is a module functor.*

Proof. Functoriality is established thus:

$$\begin{aligned}
&\Phi^{-1}(H)\left(\sum_{\substack{b\in B\\ c\in C}} s_{cb}\sigma_{cb}\right) \circ \Phi^{-1}(H)\left(\sum_{\substack{a\in A\\ b\in B}} t_{ba}\sigma_{ba}\right)\\
&\quad = \sum_{I\subseteq C\times B} H\left(\bigcup_{(c,b)\in I}\left\{\ c \xleftarrow{s_{cb}} b\ \right\}\right) \circ \sum_{J\subseteq B\times A} H\left(\bigcup_{(b,a)\in J}\left\{\ b \xleftarrow{t_{ba}} a\ \right\}\right)
\end{aligned}$$

$$= \sum_{I\subseteq C\times B}\sum_{J\subseteq B\times A}\sum_{K\subseteq\{((c,b),(b,a))\in I\times J\}} H\left(\bigcup_{((c,b),(b,a))\in K}\left\{\ c \xleftarrow{s_{cb}t_{ba}} a\ \right\}\right)$$

$$= \sum_{M\subseteq C\times B\times A} H\left(\bigcup_{(c,b,a)\in M}\left\{\ c \xleftarrow{s_{cb}t_{ba}} a\ \right\}\right)$$

$$= \sum_{L\subseteq C\times A} H\left(\bigcup_{(c,a)\in L}\left\{\ c \xleftarrow{\sum_{b\in B} s_{cb}t_{ba}} a\ \right\}\right) = \Phi^{-1}(H)\left(\sum_{\substack{a\in A\\ c\in C}}\left(\sum_{b\in B} s_{cb}t_{ba}\right)\sigma_{ca}\right),$$

where Theorem 4 was used in the fifth step. □

Lemma 11.

$$\Phi(\Phi^{-1}(H)) = H.$$

Proof. Let $P\colon X\to Y$ be a maze. We calculate:

$$\Phi(\Phi^{-1}(H))(P) = \Phi^{-1}(H)\left(\Diamond_{[p\colon x\to y]\in P}\bar{p}\sigma_{yx}\right) = \sum_{S\subseteq P}(-1)^{|P|-|S|}\Phi^{-1}(H)\left(\sum_{p\in S}\bar{p}\sigma_{yx}\right)$$
$$= \sum_{S\subseteq P}(-1)^{|P|-|S|}\sum_{K\subseteq S}H(K) = \sum_{K\subseteq P}H(K)\sum_{K\subseteq S\subseteq P}(-1)^{|P|-|S|} = H(P).$$

□

Assembling these results, we conclude that Φ is fully faithful and representative, and thence we obtain our main theorem.

Theorem 14: Labyrinth of Fun. — *The functor*

$$\Phi\colon\ \mathrm{Fun}(\mathfrak{XMod},\mathfrak{Mod}) \to \mathrm{Lin}(\mathfrak{Laby},\mathfrak{Mod}),$$

where $\Phi(F)\colon \mathfrak{Laby}\to\mathfrak{Mod}$ *takes*

$$X \mapsto \mathrm{cr}_X F(\mathbf{B})$$
$$[P\colon X\to Y] \mapsto \left[F\left(\Diamond_{[p\colon x\to y]\in P}\bar{p}\sigma_{yx}\right)\colon\ \mathrm{cr}_X F(\mathbf{B}) \to \mathrm{cr}_Y F(\mathbf{B})\right],$$

is an equivalence of categories.

§2. Polynomial Functors. Since mazes correspond to deviations, this very simple characterisation of polynomiality should come as no surprise.

Theorem 15. — *The module functor F is polynomial of degree n if and only if $\Phi(F)$ vanishes on sets with more than n elements; or, equivalently, on mazes with more than n passages.*

Proof. Assume first that F is polynomial of degree n. Since mazes with $n+1$ passages correspond to n'th deviations, $\Phi(F)$ will certainly vanish on mazes with more than n passages.

Suppose now, conversely, that $\Phi(F)$ annihilates mazes with more than n passages. Then each cross-effect of F of rank exceeding n will vanish, and F is polynomial of degree n. □

§3. Numerical Functors. We now investigate how to interpret numericality in the labyrinthine setting. The base ring $\mathbf{B}$ will of course be assumed binomial.

Lemma 12. — *Let $r \in \mathbf{B}$, and let $w_1, \dots, w_q$ be natural numbers. Then*

$$\prod_{j=1}^{q} \binom{r}{w_j} = \sum_{m=0}^{\infty} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}.$$

Proof. When r is an integer, both sides of the given equality count the number of ways to choose subsets $W_1, \dots, W_q \subseteq [r]$ with $|W_i| = w_i$. This is because, for a fixed subset $S \subseteq [r]$ with $|S| = m$, there are, by the Principle of Inclusion and Exclusion, exactly

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}$$

subsets $W_1, \dots, W_q \subseteq [r]$ such that $|W_i| = w_i$ and $\bigcup W_i = S$.

When r is an element of an arbitrary binomial ring, we invoke the Binomial Transfer Principle of [15]. □

Recall that, when A is a maze (hence a multi-set), the symbol I_A betokens the result of substituting 1 for the labels of all the passages of A.

Theorem 16. — *The module functor F is numerical of degree n if and only if $\Phi(F)$ vanishes on sets (or mazes) with more than n elements, and, withal, satisfies the equation*

$$\Phi(F)(P) = \sum_{\# A = P} \prod_{p \in P} \binom{\bar{p}}{\deg_A p} \Phi(F)(I_A)$$

for any maze P; so that $\Phi(F)$ factors through $\mathfrak{Laby}_n$. The functor Φ induces an equivalence of categories

$$\mathfrak{Num}_n \to \operatorname{Lin}(\mathfrak{Laby}_n, \mathfrak{Mod}).$$

Proof. By the previous theorem, $\Phi(F)$ vanishing on sets with more than n elements is equivalent to polynomiality. It is then clear from Theorem 10 of [14] that numerical functors satisfy the stated requirements.

Suppose now, conversely, that $\Phi(F)$ satisfies the conditions of the theorem. We wish to use Criterion A′ in Theorem 9 of [14] and thus seek to evaluate

$$F(r\cdot 1_{\mathbf{B}^n}) = \sum_{P\subseteq r\boxdot I_{[n]}} \Phi(F)(P) = \sum_{J\subseteq[n]} \Phi(F)\left(\bigcup_{j\in J}\left\{ j \xrightarrow{\;r\;} j \right\}\right)$$

and

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{r}{m}F\left(\Diamond_m 1_{\mathbf{B}^n}\right) &= \sum_{m=0}^{\infty}\binom{r}{m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}F(k\cdot 1_{\mathbf{B}^n})\\ &= \sum_{m=0}^{\infty}\binom{r}{m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{J\subseteq[n]}\Phi(F)\left(\bigcup_{j\in J}\left\{ j \xrightarrow{\;k\;} j \right\}\right).\end{aligned}$$

To verify the equality

$$F(r\cdot 1_{\mathbf{B}^n}) = \sum_{m=0}^{\infty}\binom{r}{m}F\left(\Diamond_m 1_{\mathbf{B}^n}\right),$$

it will then be sufficient to check that the terms corresponding to a certain fixed J are equal. There will be no loss of generality in considering the special case $J=[q]$ only. Our object will thus be to verify

$$\Phi(F)\left(\bigcup_{j\in[q]}\left\{ j \xrightarrow{\;r\;} j \right\}\right) = \sum_{m=0}^{\infty}\binom{r}{m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\Phi(F)\left(\bigcup_{j\in[q]}\left\{ j \xrightarrow{\;k\;} j \right\}\right).$$

By assumption, the left-hand side equals

$$\sum_{\substack{w_1+\dots+w_q\leqslant n\\ w_j\geqslant 1}}\prod_{j=1}^{q}\binom{r}{w_j}\Phi(F)\left(\bigcup_{j\in[q]}\bigcup_{w_j}\left\{ j \xrightarrow{\;1\;} j \right\}\right)$$

and the right-hand side

$$\sum_{m=0}^{\infty}\binom{r}{m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{\substack{w_1+\dots+w_q\leqslant n\\ w_j\geqslant 1}}\prod_{j=1}^{q}\binom{k}{w_j}\Phi(F)\left(\bigcup_{j\in[q]}\bigcup_{w_j}\left\{ j \xrightarrow{\;1\;} j \right\}\right),$$

so that the equality follows after deployment of the lemma. □

Example 11. — As a simple example, a labyrinth module H corresponding to a cubical functor will satisfy the equation

$$H\left[\begin{tikzcd} * \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & * \end{tikzcd}\right] = \binom{a}{1}\binom{b}{1}H\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right]$$

$$+\binom{a}{2}\binom{b}{1}H\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, "1"] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right]+\binom{a}{1}\binom{b}{2}H\left[\begin{tikzcd} * \arrow[r, bend left, "1"] \arrow[r, "1"] \arrow[r, bend right, "1"'] & * \end{tikzcd}\right].$$

△

The main result of Baues, Dreckmann, Franjou, and Pirashvili in [3]:

$${}_{\mathbf{Z}}\mathfrak{Num}_n \cong \mathrm{Lin}(\widehat{\mathfrak{Sur}}_n, {}_{\mathbf{Z}}\mathfrak{Mod}),$$

will now be obtained as a simple corollary from the category equivalences exhibited in Theorems 13 and 16.

§4. Quasi-Homogeneous Functors.

THEOREM 17. — *The module functor F is quasi-homogeneous of degree n if and only if $\Phi(F)$ satisfies the equation*

$$\Phi(F)(a \boxdot P) = a^n \Phi(F)(P)$$

for any maze P and scalar a; so that $\Phi(F)$ factors through $\mathfrak{Laby}^n$. The functor Φ induces an equivalence of categories

$$\mathfrak{QHom}_n \to \mathrm{Lin}(\mathfrak{Laby}^n, \mathfrak{Mod}).$$

Proof. Let F be quasi-homogeneous, and let $a \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$. For any deviation, we have

$$F(a\alpha_1 \diamond \cdots \diamond a\alpha_k) = a^n F(\alpha_1 \diamond \cdots \diamond \alpha_k),$$

and we may calculate for a pure maze P:

$$a^n\Phi(F)(P) = a^n F\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right) = F\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} a\bar{p}\sigma_{yx}\right) = \Phi(F)(a \boxdot P).$$

Conversely, assume $\Phi(F)$ factors via $\mathfrak{Laby}^n$. Then, for any $k \in \mathbf{N}$,

$$a^n F(1_{\mathbf{B}^k}) = a^n \sum_{K\subseteq[k]} \Phi(F)(I_K) = \sum_{K\subseteq[k]} \Phi(F)(a \boxdot I_K) = F(a \cdot 1_{\mathbf{B}^k}),$$

and F is quasi-homogeneous. □

§5. Strict Polynomial Functors. The following theorem was first formulated by Salomonsson in terms of Mackey functors, and later reformulated in [12] using multations.

THEOREM 18 ([11], THEOREM I.2.3; [12], THEOREM 10.9). — *The functor*

$$\Psi\colon \mathfrak{Hom}_n \to \mathrm{Lin}(\mathfrak{MSet}_n, \mathfrak{Mod}),$$

where

$$\Psi(F)\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

takes

$$A \mapsto \operatorname{Im} F(\pi^{[A]})$$
$$[\mu\colon A \to B] \mapsto \left[F(\sigma^{[\mu]})\colon\ \operatorname{cr}_A F(\mathbf{B}) \to \operatorname{cr}_B F(\mathbf{B})\right],$$

is an equivalence of categories.

It will be of interest to spell out a formula for the inverse. When $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$, the functor $\Phi^{-1}(J)\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is defined by

$$\mathbf{B}^X \mapsto \bigoplus_{\substack{\#A\subseteq X \\ |A|=n}} J(A)$$

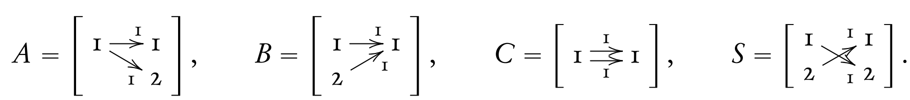
$$\left[\sum_{\substack{x\in X \\ y\in Y}} s_{yx}\sigma_{yx}\colon \mathbf{B}^X \to \mathbf{B}^Y\right] \mapsto \sum_{\substack{\#A\subseteq X,\ \#B\subseteq Y \\ |A|=|B|=n}} \sum_{\mu\colon A\to B} s^{\mu} J(\mu)$$

(where, of course, X and Y are sets, but A and B range over multi-sets).

Section the Fifth.
Numerical versus Strict Polynomial Functors

In this final section, we provide a comparison of the two strains of functors: numerical (polynomial) and strict polynomial.

§1. Quadratic Functors. We first propose to examine quadratic functors in detail, and thus seek to fathom the structure of $\mathfrak{Num}_2$. The key point lies in unravelling the structure of the category $\mathfrak{Laby}_2$. It contains three non-isomorphic objects: [0], [1], and [2]. By Theorem 8, the pure mazes form a basis. Those are identity mazes, along with the quadruple:



The (skeletal) structure of the category $\mathfrak{Laby}_2$ is thus reduced to the following, promptly suggesting the nick-name *dogegory*:

$$\begin{tikzcd} {[0]} \arrow[loop left, "I"] & {[1]} \arrow[loop above, "C"] \arrow[loop below, "I"] \arrow[r, bend left, "A"] & {[2]} \arrow[l, bend left, "B"] \arrow[loop above, "S"] \arrow[loop below, "I"] \end{tikzcd}$$

An inspection of the multiplication table, given in Table 1, reveals that the

$\circ$	A	B	C	S
A	–	$I+S$	$2A$	–
B	C	–	–	B
C	–	$2B$	$2C$	–
S	A	–	–	I

Table 1: Multiplication table for $\mathfrak{Laby}_2$.

$\circ$	α	β	σ
α	–	$\iota+\sigma$	–
β	2ι	–	β
σ	α	–	ι

Table 2: Multiplication table for $\mathfrak{MSet}_2$.

mazes A, B, C, and S are not algebraically independent, for $C = BA$ and $S = AB - I$. We thus recuperate the now classical classification of quadratic (integral) functors from [1]:

Theorem 19. — *A quadratic numerical functor is equivalent to a diagram of modules and homomorphisms as indicated, subject to the two relations:*

$$\beta\alpha\beta = 2\beta, \qquad \alpha\beta\alpha = 2\alpha.$$

$$\begin{tikzcd} K & X \arrow[r, shift left, "\alpha"] & Y \arrow[l, shift left, "\beta"] \end{tikzcd}$$

To determine $\mathfrak{Hom}_2$, we proceed similarly. The (skeletal) structure of the category $\mathfrak{MSet}_2$ is:

$$\begin{tikzcd} \{1,1\} \arrow[r, bend left, "\alpha"] & \{1,2\} \arrow[l, bend left, "\beta"] \arrow[loop right, "\sigma"] \end{tikzcd}$$

Every multation reduces to a linear combination of identity multations and the subsequent triplet, with multiplication given in Table 2:

$$\alpha = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \sigma = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Theorem 20. — *A quadratic homogeneous functor is equivalent to a diagram of modules and homomorphisms as indicated, subject to the single relation:*

$$\beta\alpha = 2.$$

$$\begin{tikzcd} X \arrow[r, shift left, "\alpha"] & Y \arrow[l, shift left, "\beta"] \end{tikzcd}$$

We then obtain the following characterisation of homogeneous quadratic functors.

Theorem 21. — *Consider the labyrinthine description of a quadratic functor F:*

$$K \qquad X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$$

The following conditions are equivalent:

A. *F is quasi-homogeneous of degree 2.*

B. *F may be (uniquely) extended to a homogeneous quadratic functor.*

C. $K = 0$ *and* $\beta\alpha = 2$.

Proof. The equivalence of B and C follows from the two preceding theorems; that of A and B is a consequence of the isomorphisms

$$\mathfrak{MSet}_2 \cong \mathfrak{Laby}^{\oplus 2} \cong \mathfrak{Laby}^2,$$

exhibited in Example 9 and Theorem 12. □

Example 12. — A wee example will serve to illustrate the theorem, and also to point out its subtlety. This same delicacy was observed in [14], Example 1. Consider, over the ring $\mathbf{Z}$, the following labyrinth module:

$$0 \qquad \mathbf{Z}/2\mathbf{Z} \underset{0}{\overset{0}{\rightleftarrows}} 0$$

Since it satisfies the conditions of the theorem, it has a unique structure of homogeneous quadratic functor — viz. the functor G given by:

$$\mathbf{Z}/2\mathbf{Z} \underset{0}{\overset{0}{\rightleftarrows}} 0$$

Yet it is possible to exhibit another strict polynomial structure F on this same underlying functor, this one linear. As a multi-set module

$$\bigoplus_{n=0}^{2} \mathfrak{MSet}_n \to \mathfrak{Mod},$$

it is given by the formulæ

$$\{\} \mapsto 0 \qquad\qquad \{1,1\} \mapsto 0$$
$$\{1\} \mapsto \mathbf{Z}/2 \qquad\qquad \{1,2\} \mapsto 0.$$

On the level of functors, $F = \mathbf{Z}/2\mathbf{Z} \otimes -$ and $G = (\mathbf{Z}/2\mathbf{Z})^{(1)} \otimes -$, the symbol ${}^{(1)}$ indicating Frobenius twist. △

§2. The Ariadne Thread. By the theory previously wrought out (Theorems 16 and 18), there are category equivalences

$$\Phi\colon \mathfrak{Num}_n \to \mathrm{Lin}(\mathfrak{Laby}_n, \mathfrak{Mod}), \qquad \Psi\colon \mathfrak{Hom}_n \to \mathrm{Lin}(\mathfrak{MSet}_n, \mathfrak{Mod}).$$

Connecting the combinatorial categories is the Ariadne functor $A_n\colon \mathfrak{Laby}_n \to \mathfrak{MSet}_n$.

Theorem 22: The Ariadne Thread. — *Pre-composition with the Ariadne functor A_n begets the forgetful functor*

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n,$$

so that

$$\Phi \circ \Psi^{-1} = (A_n)^*.$$

Proof. Let $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$ and let $P\colon X \to Y$ be a maze. We compute:

$$\begin{aligned}
\Phi\Psi^{-1}(J)(P) &= \Psi^{-1}(J)\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right) = \sum_{I\subseteq P} (-1)^{|P|-|I|}\Psi^{-1}(J)\left(\sum_{p\in I}\bar{p}\sigma_{yx}\right) \\
&= \sum_{I\subseteq P}(-1)^{|P|-|I|} \sum_{\substack{\#A\subseteq I\\ |A|=n}} J\left(\prod_{p\in\#A} \bar{p}^{\deg_A p}\begin{bmatrix} x\\ y\end{bmatrix}^{[\deg_A p]}\right) \\
&= \sum_{\substack{\#A\subseteq P\\ |A|=n}} J\left(\prod_{p\in\#A} \bar{p}^{\deg_A p}\begin{bmatrix} x\\ y\end{bmatrix}^{[\deg_A p]}\right) \sum_{\#A\subseteq I\subseteq P} (-1)^{|P|-|I|} \\
&= \sum_{\substack{\#A=P\\ |A|=n}} J\left(\prod_{p\in\#A} \bar{p}^{\deg_A p}\begin{bmatrix} x\\ y\end{bmatrix}^{[\deg_A p]}\right) = JA_n(P),
\end{aligned}$$

whence $\Phi\Psi^{-1}(J) = JA_n$, as required. □

§3. The Polynomial Functor Theorem. Let us finally record the exact obstruction for a numerical functor, of arbitrary degree, to be strict polynomial.

Theorem 23: The Polynomial Functor Theorem. — *Let F be a quasi-homogeneous functor of degree n, corresponding to the labyrinth module $H\colon \mathfrak{Laby}^n \to \mathfrak{Mod}$. Imposing the structure of homogeneous functor upon F is equivalent to exhibiting a factorisation of H through $\mathfrak{Laby}^{\oplus n}$.*

Proof. It will suffice to recall the isomorphism $\mathfrak{Laby}^{\oplus n} \cong \mathfrak{MSet}_n$ from Theorem 12. □

A word of admonishment: the existence of a factorisation of H through $\mathfrak{Laby}^{\oplus n}$ does not imply its uniqueness. As was seen in Example 12, there are, in general, many strict polynomial structures on the same functor, even of different degrees.

EXAMPLE 13. — We wish to draw the reader's attention to one particular case, when factorisation always takes place. If **B** is a **Q**-algebra, the categories

$$\mathfrak{Laby}^n = \mathfrak{Laby}^{\oplus n} \cong \mathfrak{MSet}_n$$

are isomorphic. This mirrors the well-known fact that, over a **Q**-algebra, numerical and strict polynomial functors co-incide. △

EXAMPLE 14. — For *affine* functors (degree 0 and 1), there is no discrepancy between numerical and strict polynomial functors. This will no longer be the case in higher degrees. Yet, the quadratic case will retain some regularity, in that, according to Theorem 21, any *quasi-homogeneous* functor necessarily admits a unique homogeneous structure. △

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