THE THEORY OF
POLYNOMIAL FUNCTORS

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Jag vill icke säga Licentiatens namn, men initial-bokstafven var X.

— Carl Jonas Love Almqvist, *Svensk Rättstafnings-Lära*

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Mon génie étonné tremble devant le sien.

— Jean Racine, *Brittanicus*

The epigraph contains the judgement passed by Nero on his mother Agrippina. Disregarding the peculiar circumstances under which it was uttered, the quote, as such, is rather well suited to describe the unprecedented awe and admiration we feel for our advisor, the venerable Professor Torsten Ekedahl.

It was a startling encounter, back in the summer of 2006, when we first entered his office, coyly proclaiming ourself to be his new graduate student.

“Oh yes,” he said, peering over his spectacles, “I remember you. I mean, I don’t remember you, but I recall your looking something like that. Now, what should you start with? Considering you know so little algebra[1], I was thinking you could begin with a little starter. How would you like to explore the connection between polynomial and strict polynomial functors?”

Not knowing better, we acquiesced, mainly because the word “polynomial” did not ring any alarm bells. It thus all began like an appetiser. It ended up a doctoral thesis.

The project grew under our hands, expanded in all directions, and we watched with pleasure a beautiful theory taking shape. Conducting research may be likened to exploring an unknown territory, but many a time we have felt less like an Explorer, and more like an Architect. By the point we arrived at the scene, the state of affairs was a miserable one, a malaria-infested swamp of murky waters, a jungle of buzzing mosquitoes and tangled undergrowth. But over the course of these five years, we have worked hard to clear the ground, dike the land, and cut down the trees to erect a glorious palace at their place, crowned with towers and turrets glistening in the sun and coloured banners flapping gaily in the breeze, amidst cascades of hanging gardens. Our mission has indeed been the Architect’s.

The theory, as we here present it, is beautiful; or so we feel. There remains to be seen if it can also be useful.

[1]This defect has since been remedied, we hope.

INTRODUCTION

> [. . .] but luckily Owl kept his head and told us that the Opposite of an Introduction, my dear Pooh, was a Contradiction; and, as he is very good at long words, I am sure that that's what it is.
>
> — Alan Alexander Milne, *The House at Pooh Corner*

Three questions concerning the subject at hand, polynomial functors, are begging to be answered. What are polynomial functors, where do they come from, and what are they good for?

The latter two are most easily replied to. *Polynomial* functors (the weaker notion) were introduced by Professors Eilenberg and Mac Lane in 1954, who used them to study certain homology rings ([6]). *Strict polynomial* functors were invented by Professors Friedlander and Suslin in 1997, in order to develop the theory of group schemes ([10]). Since then, the two spurious concepts have evolved side by side. Mentions of them have appeared scattered in articles, generally revolving around the themes of homotopy and homology. The stance taken is rather a pragmatic one, usually treating polynomial functors as a *means*, rather than an end in themselves.

As far as we know, no cross-fertilisation has yet taken place. This treatise is likely the first ever to actually *interrelate* the two species. That, we allege, is the ultimate end of this work: a comparison of polynomial[2] and strict polynomial functors.

What, then, is a polynomial functor? Let us consider the category ${}_{\mathbf{Z}}\mathfrak{Mod}$ of abelian groups. Two familiar functors on this category are the (co-variant) *Hom-functor*

$$\mathrm{Hom}(P, -)$$

and the *tensor functor*

$$Q \otimes -;$$

P and Q being fixed groups. They are both *additive* in the following sense:

$$\begin{aligned} \mathrm{Hom}(P, \alpha + \beta) &= (\alpha + \beta)_* = \alpha_* + \beta_* = \mathrm{Hom}(P, \alpha) + \mathrm{Hom}(P, \beta) \\ Q \otimes (\alpha + \beta) &= 1_Q \otimes (\alpha + \beta) = 1_Q \otimes \alpha + 1_Q \otimes \beta = Q \otimes \alpha + Q \otimes \beta; \end{aligned}$$

[2]Of course, as we discovered in due time, polynomial functors provide much too weak a notion. Over more general base rings than $\mathbf{Z}$, they are subsumed by *numerical* functors.

α and β denoting homomorphisms. Consider now the *tensor square* T^2, given by the equation

$$T^2(M) = M \otimes M.$$

It still maps homomorphisms to homomorphisms, but it is *itself* not additive, for

$$T^2(\alpha + \beta) = (\alpha + \beta) \otimes (\alpha + \beta) = \alpha \otimes \alpha + \alpha \otimes \beta + \beta \otimes \alpha + \beta \otimes \beta,$$

whereas

$$T^2(\alpha) + T^2(\beta) = \alpha \otimes \alpha + \beta \otimes \beta.$$

Evidently, if there be any justice in the world, this functor should belong to the quadratic family. The question is how to formalise this.

One approach is to observe that, while T^2 does not satisfy the affinity relation

$$T^2(\alpha + \beta) - T^2(\alpha) - T^2(\beta) + T^2(0) = 0$$

($T^2(0) = 0$ gives additivity), it will, however, satisfy the higher-order equation

$$\begin{aligned} T^2(\alpha + \beta + \gamma) - T^2(\alpha + \beta) - T^2(\beta + \gamma) - T^2(\gamma + \alpha) \\ + T^2(\alpha) + T^2(\beta) + T^2(\gamma) - T^2(0) = 0. \end{aligned}$$

This is what it means to be quadratic in the sense of Eilenberg and Mac Lane.

But the functor T^2 not only *behaves* like a polynomial, Friedlander and Suslin argued, it *is* in fact a polynomial. To motivate such a designation, they offered the following calculation:

$$\begin{aligned} T^2(a\alpha + b\beta) &= (a\alpha + b\beta) \otimes (a\alpha + b\beta) \\ &= a^2(\alpha \otimes \alpha) + ab(\alpha \otimes \beta + \beta \otimes \alpha) + b^2(\beta \otimes \beta). \end{aligned}$$

One would think that, in order to discuss polynomial functors, need would arise for things such as the "square of a homomorphism", but not so. It will, in fact, be sufficient that the *coefficients* a and b of the homomorphisms transform as quadratic polynomials. This is what it means for the functor to be strict quadratic.

Additive functors have been extensively studied; non-additive functors less so, and rarely for their own sake. The first real investigation of their properties was not performed until 1988, when Professor Pirashvili showed that polynomial functors are equivalent to modules over a certain ring ([17]), a result we shall build upon and generalise. A similar study was conducted on strict polynomial functors in 2003 by Dr. Salomonsson, our predecessor, in his doctoral thesis [20].

A radically different method of attack was initiated by Dr. Dreckman and Professors Baues, Franjou, and Pirashvili in the year 2000. Their approach was to *combinatorially* encode polynomial functors, for this purpose utilising the category of sets and *surjections*. Evidently inspired by this device, Dr. Salomonsson would later repeat the feat for strict polynomial functors, employing instead the category of *multi-sets*.

Such is the theory of polynomial functors as it stands today — or rather as it stood just recently. This thesis proposes the following:

1:0. To generalise the notion of polynomial functor to more general base rings than **Z**, so that it smoothly agree with the existing definition of strict polynomial functor, allowing for easy comparison. This results in the definition of *numerical functors* (Chapter 6).

2:0. To make an extensive study of numerical *maps* of modules (which will be needed so as to properly understand the *functors*), to see how they fit into Professor Roby's framework of strict polynomial maps (Chapter 5).

3:0. To conduct a survey of numerical *rings* (in order to understand the *maps*). This has, admittedly, been done before, in a somewhat different guise, but our approach will be seen to contain a few novelties (Chapter 1).

4:0. To develop the theories of numerical and strict polynomial functors (Chapter 7) so that they run (almost) in parallel (Chapter 8).

5:0. To show how also numerical functors may be interpreted as modules over a certain ring (Chapter 9).

6:0. To expound the theory of *mazes* (Chapter 3), which will be seen to vastly generalise the category of surjections employed by Professor Pirashvili et al., since they turn out to encode, not only polynomial or numerical functors, but all[3] module functors over any[4] base ring (Chapter 10).

7:0. To simplify Dr. Salomonsson's construction involving multi-sets (Chapter 2), making it more amenable to a comparison with mazes (Chapter 4).

8:0. To prove comparison theorems interrelating numerical and strict polynomial functors (Chapter 11).

9:0. And, finally, to merely *indicate* (Chapter 12) how polynomial functors may be used to extend the operad concept, a line of thought already present in Dr. Salomonsson's thesis.

[3]Fine print: *right-exact and commuting with inductive limits.*
[4]Fine print: *unital.*

POLYNOMIELLE FUNCTORER PÅ MODULE-CATEGORIER

Sedo-Lärande Tankor öfver Algebran

Wi hade litet at ſäya, om Wi allenaſt bewiſte at twå och twå äro fyra.

— Olof Dalin, *Then Swänska Argus, 1732 N:o 1*

Sedan tidernas begynnelſe har menniſkjan egnat ſig åt *Arithmetik*, hvarmed oftaſt plägar menas manipulationer af tal medelſt de fyra räkne-ſätten addition, ſubtraction, multiplication och diviſion. Detta är hvad Mathematik de fleſta menniſkjor någonſin komma i contact med, emedan det är, dels hvad de få lära ſig i ſcholan, och dels hvad ſom någorlunda eger tillämpning i hvardagslifvet.

Men med en dylik begränſad upfattning om Mathematikens väſen torde man förundra ſig ſtorliga deröfver, at det alls bedrifves *forskning* inom Mathematik. Känner man icke redan allt om de fyra räkne-ſätten, frågar ſig den mindre kunnige, och kunna des utom icke våra moderna räkne-machiner utföra deſsa operationer långt qvickare än någon menſklig hjerna?

Det är viſserligen helt rigtigt, det man icke forſkar inom Arithmetik. At correct utföra enkla räkne-operationer har menniſkjan kunnat ſedan urminnes tider. Strengt taget anſer man egentligen ej Arithmetican, för all ſin tillämplighet, vara Mathematik per ſe; ſnarare betragtas hon ſå ſom någon form af

Ämne	**Uptäckt**	**Exempel på Objecter**	**Exempel på Equationer**
Arithmetik	Tidernas begynnelſe	Tal: $0, 1, -7, \frac{1}{3}, \sqrt{2}, \pi, i, \ldots$	$(1+2)+4 = 1+(2+4)$
Elementar Algebra	1500-talet	Variabler: $x, y, z, \ldots$	$(x+y)+z = x+(y+z)$
Abſtract Algebra	1800-talet	Algebraiſka ſtructurer: Grouper, ringar, kroppar, lineara rum, moduler, …	$(x*y)*z = x*(y*z)$
Categorie-Theorie ("Abſtract Non-Sens")	1900-talet	Categorier: $\mathfrak{Grp}$, $\mathfrak{Rng}$, $\mathfrak{Fld}$, $\mathfrak{Vec}$, $\mathfrak{Mod}, \ldots$	$\begin{array}{ccc} X\otimes X\otimes X & \xrightarrow{\mu\otimes 1} & X\otimes X \\ \downarrow{\scriptstyle 1\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ X\otimes X & \xrightarrow[\mu]{} & X \end{array}$

Tabell 1: Hiſtorique öfver Algebrans Utveckling.

räkne-lära. Hon tilhandahåller endaſt de ſimplaſte af verktyg, hvilket beviſas deraf, at par exemple de gamle Ægyptierna, trots 3,000 år af obruten civiliſation, icke voro capable at löſa den quadratiſka equationen.

Detta förefaller måhända den moderne läſaren en ſmula beſynnerligt, ty at de tvenne rötterna til equationen

$$x^2 + px + q = 0$$

gifvas af

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q},$$

det torde hvar och en erinra ſig från ſin ſchol-tid. Men för Ægyptierna gick denna formul ſynbarligen ouptäckt i 3,000 år. Raiſonen härtil är icke ſvår at förſtå: utan tilgång til formler, voro de hänviſade til långa och omſtändliga beſkrifningar i ord, för at nedteckna allmänna reglor för equationers löſande. Läſaren kan ſjelf förſöka ſig på, at beſkrifva formulan ofvan i ord. Af den concentrerade formeln lär blifva en blaſkig nouvelle; och af des härledning — en hel roman! Intet under, det Ægyptierna aldrig funno den!

Ej heller de gamle Indierna kunde, för ſå vidt man vet, löſa den quadratiſka equationen. At det lyckades Babylonierna och Chineſerna, trots at äfven deſse voro hänviſade til beſkrifningar i ord, får ſes ſom en ſmärre bedrift.

Så paſserade den ſtora Revolutionen. Remarquabelt nog, timade denna ſamtidigt med öfriga culturella omhvälfningar af ſamhället, det vil ſäga, under Renaiſsancen, då man uptäckte den ſymboliſka algebran. Man fant alltſå på konſten at ſkrifva formler. Och utan formler, ingen Mathematik — då återſtår endaſt räkne-lära. Det är ſåledes ſvårt at öfverſkatta den ſymboliſka algebrans betydelſe för Mathematiken, och man ſkulle kunna likna des införande vid hjulets upfinnande.

Den Mathematiſka Wetenſkapens arbetsfält vidgades med ens, och öppnade up för hvad ſkulle kunna benämnas den *Elementara Algebrans* epoque. Det nya formel-ſpråket gjorde omedelbar ſuccès, och framgången lät icke vänta på ſig. Inom kort lyckades det Italienſka Mathematiker at löſa ſåväl den cubiſka, ſom ock den quartiſka, equationen.

Den Elementara Algebran använder ſig af *variabler* i ſtället för tal, och marquerar öfvergången från ziffer-räkning til bokſtafs-räkning. Des ſtyrka ligger ej blott deri, at den möjliggör nedſkrifvandet af equationer affedda at *löſas*; hon låter oſs ock formulera *allmängiltiga* räkne-lagar, ſanna för alla tal.

Det förhållande par exemple, at det, då tvenne tal ſkola adderas, helt ſaknar betydelſe i hvilken ordning deſsa tagas, kan då compact och öfverſkådligt ſkrifvas ſom den *Commutativa Lagen*

$$x + y = y + x,$$

giltig för alla tal x och y.

Det förhållande åter, at det, då man har at addera trenne tal (i någon gifven ordning), icke ſpelar någon rôle i hvilken ordning de tvenne additionerna utföras, utgör den *Associativa Lagen*

$$(x + y) + z = x + (y + z).$$

Mathematiker egna ſig ſåledes, tvert emot den föreſtällning folk i gemen hafva om deſse, i påfallande liten utſträckning åt ziffror. (Jemför det vulgaira uttrycket "zifferkarl"!)

Näſta abſtraction egde rum under 1800-talet, då man obſerverade, at flere af de lagar, hvilka gälla för räkning med vanliga tal, äfven ega giltighet i andra ſammanhang. Man noterade exempelvis, at den Aſsociativa Lagen för addition:

$$(x + y) + z = x + (y + z),$$

finner ſin motſvarighet för multiplication:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Det exiſterar viſserligen ockſå en aſsociativ lag för addition af vectorer:

$$(u + v) + w = u + (v + w),$$

multiplication af matricer:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C),$$

ſamt för compoſition af functioner:

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Som ſynes kunna räkne-operationerna vara af de meſt ſkilda ſlag, och de ingående ſtorheterna äro ej längre begränſade til at vara tal. Det intereſsanta är ſåledes ej längre *räkne-operationena* ſom ſådana, långt mindre hvilka objecter de verka på, utan operationernas gemenſamma egenſkap *associativitet*.

Man inſåg, det vore af ſtörſta betydelſe, at iſolera detta phenomen, och tog ſig före, at ſtudera *algebraiska structurer*. Deſsa äro mängder af Mathematiſka objecter, equiperade med en eller flera operationer, hvilka må upfylla viſsa axiomata.

Sålunda är exempelvis en *half-groupe* en mängd med en enda operation $*$, ſatiſfierande juſt den aſsociativa lagen

$$(x * y) * z = x * (y * z)$$

(för alla element x, y, z tilhörande denna mängd). *Hvad* operationen $*$ "betyder" är icke längre relevant; det kan vara addition, multiplication, compoſition; men ockſå en helt abſtract operation, utan förankring i hvardagslifvet. Det intereſsanta är egenſkapen denna poſtuleras beſitta: aſsociativitet.

Man hade nu abſtraherat up ännu en niveau och entrerat den *Abstracta Algebrans* domaine. En af den Abſtracta Algebrans tidigaſte triumpher var hennes bevis för den quintiſka equationens olöſlighet medelſt claſsiſka algebraiſka verktyg (ſå benämnde *radicaler*; deſse äro de fyra räkneſätten jemte rotutdragningar), ſå at ingen traditionel löſningsformel exiſterar i detta fall.

Studerades ſålunda ej längre “en räkne-operation i taget”, utan “alla ſamtidigt”. Vi marquerade detta ofvan genom at beteckna den godtyckliga (okända) operationen med en ſtjerna $*$, precis ſom man tidigare hade betecknat tal med bokſtäfver, för at ſtudera, icke “et tal i taget”, utan “dem alla ſamtidigt”. Detta kan ſägas vara Mathematikens väſen: ſökandet efter univerſella, allmängiltiga principer.

Et ſynſätt, ſom introducerades vid denna tid, och ſedan des vunnit laga häfd, är det, at det icke ſå mycket är de algebraiſka ſtructurerna ſjelfva, hvilka förtjena et ſtudium, ſom *afbildningarna* dem emellan. Deſsa afbildningar må icke vara af helt godtyckligt ſlag, utan dem vare det ålagdt, at bevara de ingående räkne-operationerna. En ſådan ſtructure-bevarande afbildning kallas *homomorphism*. Hafva vi exempelvis en enda operation $*$ är criterium, för at afbildningen φ vare en homomorphiſm, det, at

$$\varphi(x * y) = \varphi(x) * \varphi(y).$$

Det är ſåledes detta, der ſtuderas inom den Abſtracta Algebran: algebraiſka ſtructurer, tilſamman med deras homomorphiſmer.

Algebraiſka ſtructurer äro legio. Förutom half-grouper hafva vi, för at blott nämna de meſt frequent förekommande: grouper, ringar, kroppar, lineara rum, moduler, ... Hvar och en af deſse claſser characteriſeras af et antal operationer med tilhörande axiomata, och hvar och en egnas et ingående ſtudium inom correſponderande Mathematiſka diſcipline: Groupe-Theorie, Ring-Theorie, Kropp-Theorie, Linear Algebra, Module-Theorie, ...

Under 1900-talet begynte Mathematikerna ſyſtematiſera i denna brokiga flora (eller fauna?) af algebraiſka ſtructurer. Likaſom deras föregångare hundra år tidigare hade obſerverat likheter mellan *räkne-lagarne* för olika operationer, noterade man nu viſsa reglor och lagbundenheter, en viſs *Method in the Madness*, alla deſsa algebraiſka ſtructurer emellan. Begrepp rörande en viſs ſorts ſtructure befunnos ega motſvarigheter för en annan; viſsa theoremer för en theorie beviſtes vara ſanna äfven inom andra theorier; etc. Önſkan väcktes, at ena denna rika famille af algebraiſka ſtructurer under en och ſamma parapluie.

Vid förſta revolutionen hade man abſtraherat bort talen, i det deſsa erſattes af variabler. Under andra reſan abſtraherades ſedan vederbörligen räkne-operationerna bort, at remplaceras med godtyckliga ſådana. I focus placerades nu ſtructurerna deſsa operationer gåfvo uphof til. Under tredje vågen i Algebrans utveckling, abſtraherade man ſlutligen bort äfven ſtructurerna. Man ville ſtudera, icke *en* ſpecifique algebraiſk ſtructure — precis ſom man tidigare valt, at icke ſtudera et ſpecifique tal eller ſpecifique räkne-operation — utan dem alla ſamtidigt.

Studium inleddes af ſå kallade *categorier*, hvarmed betecknas ſamlingen af alla algebraiſka ſtructurer af et gifvet ſlag. Hafva vi ſåledes: categorien af grouper, categorien af ringar, categorien af kroppar, categorien af lineara rum, categorien af moduler, ... Man ſökte utröna, dels det inre machineriet i deſsa categorier, dels deras relationer ſins emellan (hvilken ſträfvan kan anſes ſammanfatta en Algebraikers lifsnäring: at utforſka *structure*).

Men likſom et ſtudium af algebraiſka ſtructurer vore otänkbart utan et ſamtidigt ſtudium af deras homomorphiſmer, det är: ſtructure-bevarande afbildningar; är det omöjligt at tänka ſig categorier utan ſtructure-bevarande afbildningar dem emellan. Deſsa gå här under namnet *functorer*.

Som exempel anföre vi categorien af moduler. En functor på denna categorie tager en module och producerar en annan; ſamtidigt ſom hvarje homomorphiſm transformeras til en annan homomorphiſm. Detta må då icke ſke helt godtyckligt, utan åter ſkola viſsa axiomata upfyllas.

Man fant ſtraxt, at det för ſomliga categorier vore meningsfullt, at tala om *lineare* functorer. Användningsområden för deſse äro mångfaldiga, och deras theorie följagteligen mycket väl utforſkad. Men juſt categorien af moduler har des utom viſat ſig beſitta en liten egenhet, hvars rätta betydelſe man egentligen förſt på ſednare tid inſett. Det är nämligen möjligt för functorer på denna categorie at vara (i någon mening) *polynomielle*, det vil ſäga: *quadratiske*, *cubiske*, etc. Detta är themat för föreliggande afhandling: *Polynomielle Functorer på Module-Categorier*.

Huru ſkal då begreppet "polynomialitet" lämpligen definieras? Vi lägge fram tvenne möjliga definitioner: det är begreppen *numerisk* functor, reſpective *strict polynomiel* functor. Deſsa te ſig vid förſta anblicken väſensſkilda, men vi beviſe, det de i ſjelfva verket äro nära beſlägtade. Man finner, at en ſtrict polynomiel functor par néceſsité måſte vara numeriſk, medan det omvända icke gäller — det ena begreppet är ſåledes ſvagare än det andra. Men huru mycket ſvagare? Under hvilka omſtändigheter är en numeriſk functor ſtrict polynomiel? Kan en numeriſk functor på något lämpligt vis approximeras af en ſtrict polynomiel ſådan? Det är frågor ſom deſsa vi dryfta, och de utgöra goda exempel på hvilken ſorts ſpörsmål Mathematiker egna ſig åt.

Det categoriſka tänkandet genomſyrar i dag ſtora delar af Mathematiken. Vi tacke då Categorie-Theorien icke ſå mycket för de *theoremer* hon bragt oſs, utan för hennes *philosophie*. Hon har ſkänkt oſs et communt ſpråk för Mathematiken, eller i hvart fall de delar, hvilke äro någorlunda beſlägtade med Algebran. Det måſte nu påpekas, det icke alla Mathematiker priſa Categorie-Theoriens förtjenſter ſans réſervation. Man har myntat termen *abstract non-sens* för denna gren af Mathematiken, hvars theoremer ega ſådan allmängiltighet, at de icke ſäga någon ting alls.[5]

Categorie-Theorie kan ſägas utgöra den ultimata abſtractionen — ultimata, emedan det exiſterar et ſå ſinguliert object ſom *Categorien af Categorier*. Men Categorie-Theorien är nu ökänd juſt för ſin förmåga, at ſlå knut på ſig ſjelf.

[5]Abſtraction kan löpa amok. Enligt upgift hafva äfven viſse delar af Univerſel Algebra lidit af detta. Det blef tommare och tommare.

Chapter 0

PRELIMINARIES

En tyktes wara hwaß, och grep mig an för stöld,
At jag ur böcker tog, med andras tankar jäste;
Men huru wet hon det, som aldrig nånsin läste?

— Hedwig Charlotta Nordenflycht,
Satyr emot afwundsjuka Fruentimber

§1. Set Theory

We will use standard notation for finite sets:

$$[n] = \{1, \ldots, n\}.$$

The text is pervaded by the use of multi-sets, which are given a detailed introduction in Chapter 2.

§2. Module Theory

It is not uncommon for algebraists to insist upon the existence of a multiplicative identity in each abstract ring. Where this inclination stems from is difficult to say, but the end result is that ideals are begrudged the right to be rings. We shall not follow this trend, but magnanimously allow any abelian group endowed with an associative, bilinear multiplication to call itself ring. That being said, we point out that all our rings *will* indeed possess an identity, but we shall always prefix them with the word "unital".

Since all rings under consideration will possess an identity, we shall tacitly assume all modules to be unital. Moreover, modules will be left modules, unless explicitly said otherwise.

The theory of polynomial maps and functors will be developed over a fixed base ring **B**. Consequently, we adopt the following conventions.

I. All *modules* will be unital left **B**-modules (unless otherwise stated), and $\mathfrak{Mod}$ will denote the category of these.

II. All *algebras* will be commutative and unital **B**-algebras (we shall only have reason to consider algebras in the case when **B** is commutative), and $\mathfrak{CAlg}$ will denote the category of these.

III. When **B** is numerical, all *numerical algebras* will be assumed **B**-algebras, and $\mathfrak{NAlg}$ will denote the category of such.

IV. *Linearity* will always mean **B**-linearity, and all *homomorphisms* will be **B**-module homomorphisms. (However, we will consider general *maps* of modules, and they will usually be very non-linear!)

V. *Tensor products* will be computed over **B**, unless otherwise stated.

VI. A *linear category* denotes a **B**-linear category; by which is meant a category enriched over $\mathfrak{Mod}$, so that its arrow sets are in fact **B**-modules. (In the case $\mathbf{B} = \mathbf{Z}$, this is what is known as a *pre-additive* category.)

VII. The following additional assumptions will be placed upon the base ring:

(a) When discussing *arbitrary* maps and functors, **B** can be an *arbitrary* ring (in particular, it may possibly be non-commutative).

(b) When discussing *numerical* maps and functors, **B** will of course be assumed *numerical*.

(c) When discussing *strict polynomial* maps and functors, **B** will be assumed *commutative* only.

At his[1] leisure, the reader may put $\mathbf{B} = \mathbf{Z}$, and everywhere substitute "abelian group" for "module".

§3. Category Theory

Now for some general notation concerning categories. When C is a category,

$$C^\circ$$

will denote the opposite category. The set of arrows from an object X of C to another object Y will be denoted by

$$C(X, Y).$$

There are three exceptions to this rule. Inside a module category, the set of homomorphisms between the R-modules M and N will be denoted by

$$\mathrm{Hom}_R(M, N),$$

[1]Of course, women, with their greater capability of multi-tasking, will no doubt be able to keep in mind the more general case.

and the letter R will be omitted if the ring is clear from the context (which it usually is). The set of endomorphisms of a module M will be denoted by the symbol

$$\operatorname{End} M.$$

Furthermore, in the category of categories, we will let

$$\operatorname{Fun}(A, B)$$

denote the category of functors from A to B; sometimes linear, and sometimes not, depending on context. And finally, inside a functor category, the natural transformations between two functors F and G will be denoted by

$$\operatorname{Nat}(F, G),$$

which will be shortened to

$$\operatorname{Nat} F$$

in the case $F = G$.

The following is a (not exhaustive) list of the categories we will use. Those which are not standard will be defined in the text.

$$\begin{aligned}
\mathfrak{Set} &= \text{Sets.}\\
\mathfrak{MSet} &= \text{Multi-sets.}\\
\mathfrak{Laby} &= \text{The labyrinth category.}\\
\mathfrak{Sur} &= \text{Sets with surjections.}\\
\mathfrak{CRing} &= \text{Commutative, unital rings.}\\
\mathfrak{CAlg} &= \text{Commutative, unital algebras.}\\
\mathfrak{NRing} &= \text{Numerical rings.}\\
\mathfrak{NAlg} &= \text{Numerical algebras.}\\
\mathfrak{Mod} &= \text{Modules.}\\
\mathfrak{FMod} &= \text{Free modules.}\\
\mathfrak{XMod} &= \text{Finitely generated, free modules.}\\
\mathfrak{Num} &= \text{Numerical functors.}\\
\mathfrak{QHom} &= \text{Quasi-homogeneous functors.}\\
\mathfrak{SPol} &= \text{Strict polynomial functors.}\\
\mathfrak{Hom} &= \text{Homogeneous functors.}
\end{aligned}$$

§4. Semi-Abelian Category Theory

The proposition below is (un)known to mathematicians as Delsarte's Lemma, but there seems to be no tangible way to attach Professor Delsarte's name unto it.[2] In our licenciate thesis, we gave two versions, with rather differing proofs, one for rings and one for abelian categories. We much deplored this, being firm adherents to Professor Bourbaki's infamous slogan never to prove a theorem that could be deduced as a special case of a more general theorem.

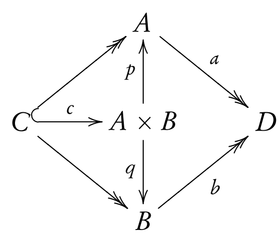
It is not without some pride that we now enunciate the following ultimate Delsarte's Lemma. For an introduction to semi-abelian categories, we refer to Professor Borceux's survey paper [2].

Theorem 1: Delsarte's Lemma.

- *Let A, B, and C be objects of a semi-abelian category such that $C \subseteq A \times B$ and the projections $C \to A$ and $C \to B$ are regularly epic. Then A and B have a common quotient object*

$$A \overset{a}{\twoheadrightarrow} D \overset{b}{\twoheadleftarrow} B \,,$$

which completes the square into a Doolittle diagram[3]:



- *Conversely, let a common quotient object*

$$A \overset{a}{\twoheadrightarrow} D \overset{b}{\twoheadleftarrow} B$$

of A and B be given, and let C be the pullback. Then C is a subobject of $A \times B$, and the projections on A and B are regularly epic.

Proof. We are grateful to Professor Ekedahl and Dr. Bergh for furnishing us with the proof, which offers an excellent opportunity to see all the classical isomorphism theorems in action. By the Fundamental Homomorphism Theorem,

$$A \cong C/\operatorname{Ker} pc, \qquad B \cong C/\operatorname{Ker} qc;$$

and so

$$D = C/(\operatorname{Ker} pc \cup \operatorname{Ker} qc)$$

is a common quotient object of A and B, where $\cup$ denotes join in the lattice of subobjects.

[2]Legend has it that the attribution was made by Professor Serre during his Collège de France lectures.

[3]Following the terminology of Professor Freyd, a *Doolittle diagram* is a square which is both a pullback and a pushout square. Professor Popescu more prosaically calls them *exact squares*.

It must be shown this yields indeed a Doolittle diagram. Denote

$$P = \operatorname{Ker} pc, \qquad Q = \operatorname{Ker} qc;$$

and consider the following diagram, whose rows by the Tower Isomorphism Theorem are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P/(P \cap Q) & \longrightarrow & C/(P \cap Q) & \longrightarrow & C/P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (P \cup Q)/Q & \longrightarrow & C/Q & \longrightarrow & C/(P \cup Q) & \longrightarrow & 0
\end{array}$$

By the Diamond Isomorphism Theorem, the left vertical arrow is an isomorphism. Write

$$K = P/(P \cap Q) = (P \cup Q)/Q,$$

and consider the diagram:

$$\begin{array}{ccccc}
K & \longrightarrow & C/(P \cap Q) & \longrightarrow & C/Q \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C/P & \longrightarrow & C/(P \cup Q)
\end{array}$$

Because a diagram of the form:

$$\begin{array}{ccc}
\operatorname{Ker} x & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle x} \\
0 & \longrightarrow & Y
\end{array}$$

is always a pullback square, the left square and the outer rectangle above are both pullback squares. It now follows, from the "unusual cancellation property of pullbacks", Theorem 2.7 of [2], that the right square is a pullback.

The pushout property is rather more trivial. Suppose the following diagram commutes:

$$\begin{array}{ccc}
C & \longrightarrow & C/P \\
\downarrow & & \downarrow \\
C/Q & \longrightarrow & T
\end{array}$$

The arrow $C \to T$ is zero on P and Q, and therefore on $P \cup Q$, which produces a unique factorisation

$$C/(P \cup Q) \to T.$$

Hence $C/(P \cup Q)$ is the pushout.

Let us finally turn to the converse of the theorem. It is clear that C, defined as the equaliser of ap and bq, is a subobject of $A \times B$. Professor Freyd's Pullback

Theorem for abelian categories (Theorem 2.54 of [9]) states that pullbacks of (regular) epimorphisms are (regular) epimorphisms, and this proposition remains valid in a semi-abelian (or exact) category. □

Because the square is a Doolittle diagram, we have in an *abelian* category the following exact sequence:

$$0 \longrightarrow C \longrightarrow A \oplus B \longrightarrow D \longrightarrow 0$$

We may then simply choose

$$D = \operatorname{Coker}(C \to A \oplus B)\,.$$

(In the general case, C may not be a normal subobject.)

§5. Abelian Category Theory

Let us say some words on abstract tensor products. Preparing for what will eventually follow, let $\mathbf{B}$ be an arbitrary ring of scalars. All categories and all functors of this section are assumed $\mathbf{B}$-linear, and all modules are $\mathbf{B}$-modules. The theory briefly accounted for below can be found in Professor Popescu's treatise [18] on abelian categories, section 3.6. (He gives the case $\mathbf{B} = \mathbf{Z}$, but the extension to an arbitrary base ring is immediate.)

Let A be a category. We recall the classical Yoneda embedding

$$\begin{aligned} \mathrm{Y}_A : A &\to \mathrm{Fun}(A^\circ, \mathfrak{Mod}) \\ X &\mapsto A(-, X). \end{aligned}$$

Suppose now that A is small[4], and let B be an abelian category with direct sums. Fix a functor $Q\colon A \to B$. According to Theorem 3.6.3 of [18], there is a unique functor

$$- \otimes_A Q\colon\ \mathrm{Fun}(A^\circ, \mathfrak{Mod}) \to B$$

having a right adjoint, and making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{Y}_A\ } & \mathrm{Fun}(A^\circ, \mathfrak{Mod}) \\ & {\scriptstyle Q}\searrow & \big\downarrow {\scriptstyle -\otimes_A Q} \\ & & B \end{array}$$

This we call the **tensor product with** Q **over** A. Its adjoint is the mediated Hom-functor

$$\begin{aligned} B(Q, -)\colon B &\to \mathrm{Fun}(A^\circ, \mathfrak{Mod}) \\ Y &\mapsto B(Q(-), Y). \end{aligned}$$

[4] When A is finite, it is spoken of in reverence as a *ring with several objects*.

Note that, when $X \in A$, the diagram above implies that

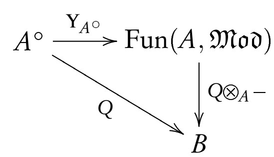
$$A(-,X) \otimes_A Q = Q(X),$$

and we may then extend by right-exactness and commutation with direct sums.

It will sometimes be convenient to have available also a reverse tensor product. Hence, if $Q\colon A^\circ \to B$ is a fixed functor, there is a unique functor

$$Q \otimes_A -\colon \ \mathrm{Fun}(A, \mathfrak{Mod}) \to B$$

having a right adjoint, and making the following diagram commute:



It will then be seen that, for functors $Q\colon A \to B$ and $M\colon A^\circ \to \mathfrak{Mod}$, the tensor products

$$Q \otimes_{A^\circ} M = M \otimes_A Q$$

are equal.

Example 1. — The special case most frequently encountered is when $A = \{*\}$ is a category with a single object, with $\mathrm{End}\, * = R$ an algebra. Then a functor $M\colon A^\circ \to \mathfrak{Mod}$ is a $\mathbf{B}$-R-bimodule, and a functor $Q\colon A \to B$ is a left R-object of B. The tensor product is usually denoted by

$$- \otimes_R Q\colon {}_{\mathbf{B}}\mathfrak{Mod}_R \to B,$$

and is uniquely specified by the equation

$$R \otimes_R Q = Q(*)$$

and the extension property.

Specialising even further, we may consider the case when also $B = \mathfrak{Mod}$. Then $Q(*)$ is simply a left R-module. Since

$$R \otimes_R Q = {}_R Q(*) = R_R \otimes_R Q(*),$$

the tensor product "of functors" coincides with the usual tensor product "of modules". △

In some situations, there is an explicit description available for the tensor product.

Theorem 2. — *Let A and B be small categories, and fix a functor*

$$Q\colon A \to \mathrm{Fun}(B, \mathfrak{Mod}).$$

Then, for $M\colon A^\circ \to \mathfrak{Mod}$ and $Y \in B$, the tensor product

$$(M \otimes_A Q)(Y)$$

is the quotient of the module

$$\bigoplus_{X \in A} M(X) \otimes Q(X)(Y)$$

by all relations

$$M(X) \otimes Q(X)(Y) \ni x \otimes Q(\alpha)_Y(y) = M(\alpha)(x) \otimes y \in F(X') \otimes Q(X')(Y),$$

for any $x \in M(X)$, $y \in Q(X')(Y)$, and $\alpha\colon X' \to X$.

The pair

$$(- \otimes_A Q, B(Q, -))$$

is, as mentioned above, an adjunction. Under certain circumstances, it is actually a category equivalence. The theorem below occurs as Corollary 3.6.4 in [18].

Theorem 3: The Morita Equivalence. — *Let C be an abelian category with direct sums, having a full subcategory P of small projective generators, with inclusion functor $J\colon P \to C$. There is a category equivalence:*

$$C \underset{-\otimes_P J}{\overset{C(J,-)}{\rightleftarrows}} \mathrm{Fun}(P^\circ, \mathfrak{Mod})$$

Example 2. — Morita equivalence is most frequently used in the situation of a single projective generator. Denoting the endomorphism ring of $P = \{*\}$ by R, we have

$$\mathrm{Fun}(P^\circ, \mathfrak{Mod}) = {}_{\mathrm{B}}\mathfrak{Mod}_R,$$

and the equivalence reduces to the familiar:

$$C \underset{-\otimes_R *}{\overset{C(*,-)}{\rightleftarrows}} {}_{\mathrm{B}}\mathfrak{Mod}_R$$

△

§6. Commutative Algebra

The subsequent (in)equality of Krull dimensions is supposedly known by "everybody" working in commutative algebra or algebraic geometry, and consequently impossible to reference. We are grateful to Professor Ekedahl for furnishing the proof.

Theorem 4: Chevalley's Dimension Argument. — *Let R be a finitely generated, non-trivial, unital ring. The (in)equality*

$$\dim R/pR = \dim \mathbf{Q} \otimes_{\mathbf{Z}} R \leqslant \dim R - 1$$

holds for all but finitely many prime numbers p.

When R is an integral domain of characteristic 0, there is in fact equality for all but finitely many primes p.

Proof. In the case of positive characteristic n, the inequality will hold trivially, for then

$$\mathbf{Q} \otimes_{\mathbf{Z}} R = 0 = R/pR,$$

except when $p \mid n$.

Consider now the case when R is an integral domain of characteristic 0. There is an embedding $\varphi\colon \mathbf{Z} \to R$, and a corresponding dominant morphism

$$\operatorname{Spec}\varphi\colon \operatorname{Spec} R \to \operatorname{Spec}\mathbf{Z}$$

of integral schemes, which is of finite type. Letting $\operatorname{Frac} P$ denote the fraction field of R/P, we may define

$$\begin{aligned} C_n &= \{P \in \operatorname{Spec}\mathbf{Z} \mid \dim(\operatorname{Spec}\varphi)^{-1}(P) = n\} \\ &= \{P \in \operatorname{Spec}\mathbf{Z} \mid \dim R \otimes_{\mathbf{Z}} \operatorname{Frac} P = n\} \\ &= \{(p) \mid \dim R/pR = n\} \cup \{(0) \mid \dim R \otimes_{\mathbf{Z}} \mathbf{Q} = n\}. \end{aligned}$$

This latter set, by Chevalley's Constructibility Theorem[5], will contain a dense, open set in $\operatorname{Spec}\mathbf{Z}$ if $n = \dim R - \dim \mathbf{Z}$. Such a set must contain (0) and (p) for all but finitely many primes p, so for those primes,

$$\dim \mathbf{Q} \otimes_{\mathbf{Z}} R = \dim R/pR = \dim R - 1.$$

Now let R be an arbitrary ring of characteristic 0. For any prime ideal Q, R/Q will be an integral domain (but not necessarily of characteristic 0!), and so we can apply the preceding to obtain

$$\dim \mathbf{Q} \otimes_{\mathbf{Z}} R/Q = \dim R/(Q + pR) \leqslant \dim R/Q - 1,$$

for all but finitely many primes p. The prime ideals of $\mathbf{Q} \otimes_{\mathbf{Z}} R$ are all of the form $\mathbf{Q} \otimes_{\mathbf{Z}} Q$, where Q is a prime ideal in R. Moreover,

$$(\mathbf{Q} \otimes_{\mathbf{Z}} R)/(\mathbf{Q} \otimes_{\mathbf{Z}} Q) = \mathbf{Q} \otimes_{\mathbf{Z}} R/Q.$$

[5] This proposition appears to belong to the folklore of algebraic geometry. An explicit reference is Théorème 2.3 of [14].

It follows that

$$\begin{aligned}\dim \mathbf{Q}\otimes_{\mathbf{Z}} R &= \max_{Q\in\operatorname{Spec} R} \dim(\mathbf{Q}\otimes_{\mathbf{Z}} R)/(\mathbf{Q}\otimes_{\mathbf{Z}} Q)\\ &= \max_{Q\in\operatorname{Spec} R} \dim \mathbf{Q}\otimes_{\mathbf{Z}} R/Q\\ &= \max_{Q\in\operatorname{Spec} R} \dim R/(Q+pR)\\ &= \max_{\overline{Q}\in\operatorname{Spec} R/pR} (R/pR)/\overline{Q} = \dim R/pR\end{aligned}$$

for all but finitely many p, because the maxima are taken over the finitely many *minimal* prime ideals only. In a similar fashion,

$$\begin{aligned}\dim \mathbf{Q}\otimes_{\mathbf{Z}} R &= \max_{Q\in\operatorname{Spec} R} \dim(\mathbf{Q}\otimes_{\mathbf{Z}} R)/(\mathbf{Q}\otimes_{\mathbf{Z}} Q)\\ &= \max_{Q\in\operatorname{Spec} R} \dim \mathbf{Q}\otimes_{\mathbf{Z}} R/Q\\ &\leqslant \max_{Q\in\operatorname{Spec} R} \dim R/Q - 1 \leqslant \dim R - 1.\end{aligned}$$

□

Chapter 1

NUMERICAL RINGS

At the age of twenty-one he wrote a treatise upon the Binomial Theorem, which has had a European vogue.

— Sherlock Holmes's description of Professor Moriarty;
Arthur Conan Doyle, *The Final Problem*

Our licentiate thesis from 2009 opened with some lavish praise on Professor Ekedahl, who purportedly "discovered"[1] numerical rings. This is only partially correct. In his article [7] from 2002, he did indeed set forth an axiomatisation of rings with binomial coefficients, and he is possibly the first to have done so, but, as we were informed of only recently, such rings had in fact been studied much earlier. Indeed, already in 1969, in connection with his work on nilpotent groups, Professor Hall ([12]) had defined a *binomial ring* as a commutative, unital ring which is torsion-free and closed under the "formation of binomial coefficients":

$$r \mapsto \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

Of course, these two approaches are radically different, and it is non-trivial that they are equivalent.

It might perhaps be argued that we ought to follow the terminology initiated by Professor Hall, the original discoverer, and use the designation *binomial*, rather than *numerical*. We have chosen to deviate from his practice, partly because we really use the axiomatic definition proposed by Professor Ekedahl, and partly because, we feel, the word *binomial ring* leads to the wrong associations: polynomial rings, and such. However, since the letter **N** is already reserved for the set of natural numbers, we will (in subsequent chapters) denote our base ring of scalars by **B** to suggest *binomial* (even though it need not necessarily be so!).

This chapter proposes to explore numerical rings for their own sake. Some of the results can no doubt be found in the literature. We cite a recent paper [8] from 2005, written by Dr. Elliott, which in particular aims to elucidate the connection between binomial rings and λ-rings.

[1] He used himself the word *introduced*, a precaution that turned out to be wise.

§1. Numerical Rings

We here present, with minor modifications, Professor Ekedahl's axioms for numerical rings. The original axioms were rather non-explicit, stated as they were in terms of three mysterious polynomials, the exact nature of which was never made precise. Our definition intends to remedy this.

Definition 1. — A **numerical ring** is a commutative ring with unity which is equipped with unary operations

$$r \mapsto \binom{r}{n}, \quad n \in \mathbf{N};$$

called **binomial coefficients** and subject to the following axioms:

I. $\displaystyle\binom{a+b}{n} = \sum_{p+q=n} \binom{a}{p}\binom{b}{q}.$

II. $\displaystyle\binom{ab}{n} = \sum_{m=0}^{n} \binom{a}{m} \sum_{\substack{q_1+\cdots+q_m=n \\ q_i \geqslant 1}} \binom{b}{q_1} \cdots \binom{b}{q_m}.$

III. $\displaystyle\binom{a}{m}\binom{a}{n} = \sum_{k=0}^{n} \binom{a}{m+k}\binom{m+k}{n}\binom{n}{k}.$

IV. $\displaystyle\binom{1}{n} = 0$ when $n \geqslant 2$.

V. $\displaystyle\binom{a}{0} = 1$ and $\displaystyle\binom{a}{1} = a.$

◇

The original definition also included a (non-explicit) formula for reducing a *composition* $\binom{\binom{a}{m}}{n}$ of binomial coefficients to simple ones. Surprisingly, this formula will be a consequence of the five axioms we have listed.

It follows easily from axioms I, IV, and V that, when the functions $\binom{-}{n}$ are evaluated on multiples of unity, we retrieve the ordinary binomial coefficients, namely

$$\binom{m \cdot 1}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!} \cdot 1, \quad m \in \mathbf{N}.$$

Since $\binom{n \cdot 1}{n} = 1$, but $\binom{0}{n} = 0$ unless $n = 0$, a numerical ring has necessarily characteristic 0.

The numerical structure on a given ring is always unique. This will be proved presently.

Example 1. — In any $\mathbf{Q}$-algebra, binomial coefficients may be defined by the usual formula:

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

△

Example 2. — For any integer m, the ring $\mathbf{Z}[m^{-1}]$ is numerical. Since it inherits the binomial coefficients from $\mathbf{Q}$, it is just a matter of verifying closure under the formation of binomial coefficients. Because

$$\binom{\frac{a}{f}}{n} = \frac{\frac{a}{f}(\frac{a}{f}-1)\cdots(\frac{a}{f}-(n-1))}{n!} = \frac{a(a-f)\cdots(a-(n-1)f)}{n!f^n},$$

it will suffice to prove that whenever $p^i \mid n!$, but $p \nmid b$, then

$$p^i \mid (a+b)(a+2b)\cdots(a+nb).$$

To this end, let

$$n = c_m p^m + \cdots + c_1 p + c_0, \qquad 0 \leqslant c_i \leqslant p-1,$$

be the base p representation of n. For fixed k and $0 \leqslant d < c_k$, the numbers

$$a + (c_m p^m + \cdots + c_{k+1} p^{k+1} + dp^k + i)b, \qquad 1 \leqslant i \leqslant p^k, \tag{1}$$

will form a set of representatives for the congruence classes modulo p^k, as will of course the numbers

$$c_m p^m + \cdots + c_{k+1} p^{k+1} + dp^k + i, \qquad 1 \leqslant i \leqslant p^k. \tag{2}$$

Note that if $x \equiv y \mod p^k$ and $j \leqslant k$, then $p^j \mid x$ iff $p^j \mid y$. Hence there are at least as many factors p among the numbers (1) as among the numbers (2). The claim now follows. △

Example 3. — As the special case $m = 1$ of the preceding example, $\mathbf{Z}$ itself is numerical. For this ring there is another, more direct, way of proving the numerical axioms. Let us indicate how they may be arrived at as solutions to problems of enumerative combinatorics.

> *Axiom I.* We have two types of balls: round balls, square[2] balls. If we have a round balls and b square balls, in how many ways may we choose n balls? Let p be the number of round balls chosen, and q the number of square balls.
>
> *Axiom II.* We have a chocolate box containing a rectangular $a \times b$ array of pralines, and we wish to eat n of these. In how many ways can this be done? Suppose the pralines we choose to feast upon

[2]This is in honour of Dr. Lars-Christer Böiers, an eminent teacher, who used an example featuring round balls and square balls during his course in discrete mathematics.

are located in m of the a rows, and let q_i be the number of chosen pralines in row number i of these m.

Axiom III. There are a mathematicians, of which m do geometry and n algebra. Naturally, there may exist people who do both or neither. How many distributions of skills are possible? Let k be the number of mathematicians who do only algebra.

Axiom IV. We are the owner of a single dog. In how many ways can we choose n of our dogs to take for a walk?

Axiom V. Snuffy the dog has a blankets. In how many ways may he choose 0 (in the summer) or 1 (in the winter) of his blankets to keep him warm in bed?

△

Example 4. — The set

$$S = \{f \in \mathbf{Q}[x] \mid f(\mathbf{Z}) \subseteq \mathbf{Z}\}$$

of **numerical maps** on $\mathbf{Z}$ is numerical. Addition and multiplication of functions are evaluated pointwise, as are binomial coefficients:

$$\binom{f}{n}(x) = \binom{f(x)}{n} = \frac{f(x)(f(x)-1)\cdots(f(x)-n+1)}{n!}.$$

Seizing the opportunity, we remind the reader that any numerical map may be written uniquely as a **numerical polynomial**

$$f(x) = \sum c_n \binom{x}{n}, \quad c_n \in \mathbf{Z}.$$

Conversely, any numerical polynomial will leave $\mathbf{Z}$ invariant. △

Example 5. — Being given by rational polynomials, the operations $r \mapsto \binom{r}{n}$ give continuous maps $\mathbf{Q}_p \to \mathbf{Q}_p$ in the p-adic topology. It should be well known that $\mathbf{Z}$ is dense in the ring $\mathbf{Z}_p$, and that $\mathbf{Z}_p$ is closed in $\mathbf{Q}_p$. Since the binomial coefficients leave $\mathbf{Z}$ invariant, the same must be true of $\mathbf{Z}_p$, which is thus numerical.

This provides an alternative proof of the fact that $\mathbf{Z}[m^{-1}]$ is closed under binomial coefficients. For this is evidently true of the localisations

$$\mathbf{Z}_{(p)} = \mathbf{Q} \cap \mathbf{Z}_p,$$

and therefore also for

$$\mathbf{Z}[m^{-1}] = \bigcap_{p \nmid m} \mathbf{Z}_{(p)}.$$

△

Example 6. — Products and tensor products of numerical rings are numerical. Also, inductive and projective limits of numerical rings are numerical. See Dr. Elliott's paper [8]. △

§2. Elementary Identities

Theorem 1. — *The following formulæ are valid in any numerical ring:*

1. $\binom{r}{n} = \dfrac{r(r-1)\cdots(r-n+1)}{n!}$ *when* $r \in \mathbf{Z}$.

2. $n!\binom{r}{n} = r(r-1)\cdots(r-n+1)$.

3. $n\binom{r}{n} = (r-n+1)\binom{r}{n-1}$.

Proof. The map

$$\varphi\colon (R,+) \to (1 + tR[[t]], \cdot), \qquad r \mapsto \sum_{n=0}^{\infty} \binom{r}{n} t^n$$

is, by axioms I and V, a group homomorphism. Therefore, when $r \in \mathbf{Z}$,

$$\varphi(r) = \varphi(1)^r = (1+t)^r,$$

which expands as usual (with ordinary binomial coefficients) by the Binomial Theorem. This proves equation 1. (An inductive proof would also work.)

To prove equations 2 and 3, we proceed differently. By axiom III,

$$\begin{aligned} r\binom{r}{n-1} &= \binom{r}{n-1}\binom{r}{1} = \sum_{k=0}^{1} \binom{r}{n-1+k}\binom{n-1+k}{1}\binom{1}{k} \\ &= \binom{r}{n-1}\binom{n-1}{1}\binom{1}{0} + \binom{r}{n}\binom{n}{1}\binom{1}{1} \\ &= (n-1)\binom{r}{n-1} + n\binom{r}{n}, \end{aligned}$$

which reduces to equation 3.

Equation 2 will then follow inductively from equation 3. □

§3. Torsion

The word *torsion* will always, here and elsewhere, refer to **Z**-*torsion*. In this section we shall prove it is absent in numerical rings. This will establish the equivalence of numerical and binomial rings, as defined by Professor Hall.

Lemma 1. — *Let m be an integer. If p is prime and $p^l \mid m$, but $p \nmid k$, then $p^l \mid \binom{m}{k}$.*

Proof. p^l divides the right-hand side of

$$k\binom{m}{k} = m\binom{m-1}{k-1},$$

and therefore also the left-hand side. But p^l is relatively prime to k, so in fact $p^l \mid \binom{m}{k}$. □

LEMMA 2. — *Let $m_1, \dots, m_k$ be natural numbers, and put*

$$m = m_1 + \cdots + m_k.$$

If

$$n = m_1 + 2m_2 + 3m_3 + \cdots + km_k$$

is prime, then

$$m \mid \binom{m}{\{m_i\}},$$

unless $m_1 = m = n$, and all other $m_i = 0$.

Proof. Let a prime power $p^l \mid m$. Because of the relation $n = \sum m_i i$, not all m_i can be divisible by p, unless we are in the exceptional case

$$m_1 = m = p = n$$

given above. Say $p \nmid m_j$; then

$$\binom{m}{\{m_i\}_i} = \binom{m}{m_j}\binom{m-m_j}{\{m_i\}_{i\neq j}}$$

is divisible by p^l according to Lemma 1. The claim follows. □

LEMMA 3. — *Consider a numerical ring R. Let $r \in R$ and $m, n \in \mathbf{N}$. If $nr = 0$, then also $mn\binom{r}{m} = 0$.*

Proof. Follows inductively, since if $nr = 0$, then

$$mn\binom{r}{m} = n(r-m+1)\binom{r}{m-1} = -n(m-1)\binom{r}{m-1}.$$

□

THEOREM 2. — *Numerical rings are torsion-free.*

Proof. Suppose $nr = 0$ in R, and, without any loss of generality, that n is prime. We calculate using the numerical axioms:

$$0 = \binom{0}{n} = \binom{nr}{n} = \sum_{m=0}^{n}\binom{r}{m}\sum_{\substack{q_1+\cdots+q_m=n\\ q_i\geqslant 1}}\binom{n}{q_1}\cdots\binom{n}{q_m}$$

$$= \sum_{m=0}^{n} \binom{r}{m} \sum_{\substack{\sum m_i = m \\ \sum m_i i = n}} \binom{m}{\{m_i\}} \prod_i \binom{n}{i}^{m_i}.$$

For given numbers q_i, we have let m_i denote the number of these that are equal to i (of course $i \geqslant 1$ and $m_i \geqslant 0$). Conversely, when the numbers m_i are given, values may be distributed to the numbers q_i in $\binom{m}{\{m_i\}}$ ways, which accounts for the multinomial coefficient above.

We claim the inner sum is divisible by mn when $m \geqslant 2$. For when $2 \leqslant m \leqslant n-1$, then $m \mid \binom{m}{\{m_i\}}$ by Lemma 2; also, there must exist some $0 < j < n$ such that $m_j > 0$, and for this j, Lemma 1 says $n \mid \binom{n}{j}^{m_j}$. In the case $m = n$, obviously all $m_i = 0$ for $i \geqslant 2$, and $m_1 = n$, so the inner sum equals $\binom{n}{1}^n$, which is divisible by $n^2 = mn$.

We can now employ Lemma 3 to kill all terms except $m = 1$. But this term is simply $\binom{r}{1} = r$, which is then equal to 0. □

This theorem is remarkable in all its simplicity. We know of no other example of a variety of algebras, of which the axioms imply lack of torsion in a non-trivial way; that is, without actually implying a **Q**-algebra structure. Not only that, the theorem is also a most crucial result in the theory of numerical rings. Over the course of the following sections, we will deduce several corollaries, seemingly without effort.

§4. Uniqueness

Theorem 3. — *There is at most one numerical ring structure on a given ring.*

Proof. We know that

$$n!\binom{r}{n} = r(r-1)\cdots(r-n+1),$$

and that $n!$ is not a zero divisor. □

§5. Embedding in Q-Algebras

Theorem 4. — *Every numerical ring can be embedded in a* **Q**-*algebra, where the binomial coefficients are given by the usual formula*

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

Consequently, a ring is numerical iff it is binomial.

Proof. Since R is torsion-free, the map $R \to \mathbf{Q} \otimes_{\mathbf{Z}} R$ is an embedding. □

§6. Iterated Binomial Coefficients

In $\mathbf{Z}$, there "exists" a formula for iterated binomial coefficients:

$$\binom{\binom{r}{m}}{n} = \sum_{k=1}^{mn} g_k \binom{r}{k}, \tag{3}$$

in the sense that there are unique integers g_k making the formula valid for every $r \in \mathbf{Z}$. Professor Golomb has examined these iterates in some detail, and his paper [11] is brought to an end with the discouraging conclusion:

> No simple reduction formulas have yet been found for the most general case of $\binom{\binom{n}{b}}{a}$.

We note, however, that (3) is a polynomial identity with rational coefficients, which means it holds in any $\mathbf{Q}$-algebra, and therefore in any numerical ring. This proves the redundancy of Professor Ekedahl's original sixth axiom:

Theorem 5. — *The formula*

$$\binom{\binom{r}{m}}{n} = \sum_{k=1}^{mn} g_k \binom{r}{k}$$

for iterated binomial coefficients is valid in every numerical ring.

§7. Homomorphisms

Definition 2. — Let R and S be numerical rings. The ring homomorphism $\varphi\colon R \to S$ is said to be **numerical** if it preserves binomial coefficients:

$$\varphi\left(\binom{r}{n}\right) = \binom{\varphi(r)}{n}.$$

S is then a **numerical algebra** over R. ◇

We denote by

$$\mathfrak{NRing}$$

the category of numerical rings, and by

$$_R\mathfrak{NAlg},$$

or simply

$$\mathfrak{NAlg},$$

the category of numerical algebras over some fixed numerical base ring R.

Theorem 6. — *Every ring homomorphism of numerical rings is numerical, so that* $\mathfrak{NRing}$ *is a full subcategory of* $\mathfrak{CRing}$.

Proof. Let R and S be numerical rings, and let $\varphi\colon R \to S$ be a ring homomorphism. Because of the absense of torsion, the equation

$$\begin{aligned} n!\varphi\left(\binom{r}{n}\right) &= \varphi\left(n!\binom{r}{n}\right) = \varphi(r(r-1)\cdots(r-n+1)) \\ &= \varphi(r)(\varphi(r)-1)\cdots(\varphi(r)-n+1) = n!\binom{\varphi(r)}{n} \end{aligned}$$

implies $\varphi\left(\binom{r}{n}\right) = \binom{\varphi(r)}{n}$, so that φ is numerical. □

§8. Free Numerical Rings

Recall from Example 4 that a *numerical polynomial* (over $\mathbf{Z}$) in the variables $x_1, \ldots, x_k$ is a formal (finite) linear combination

$$f(x) = \sum c_{n_1,\ldots,n_k}\binom{x_1}{n_1}\cdots\binom{x_k}{n_k}, \qquad c_{n_1,\ldots,n_k} \in \mathbf{Z}.$$

Also, a *numerical map* is a rational polynomial leaving $\mathbf{Z}$ invariant. These two concepts coincide.

Let X be a set, and let $E(X)$ be the *term algebra*[3] based on X. It consists of all finite words that can be formed from the alphabet

$$X \cup \left\{ +, -, \cdot, 0, 1, \binom{-}{n} \,\middle|\, n \in \mathbf{N} \right\},$$

where the symbols $+$ and $\cdot$ are binary, $-$ and $\binom{-}{n}$ are unary, and 0 and 1 nullary (constants).

Definition 3. — The free numerical ring on X is what results after the axioms of a commutative ring with unity, as well as the numerical axioms, have been imposed upon the term algebra. ◇

Of course, it need be proved that the "free" numerical ring is indeed free in the usual sense.

Theorem 7. — *There is an isomorphism*

$$\mathfrak{NRing}\left(\mathbf{Z}\binom{X}{-}, R\right) \cong \mathfrak{Set}(X, R),$$

which is functorial in the numerical ring R.

Moreover,

$$\mathbf{Z}\binom{X}{-} \cong \{f \in \mathbf{Q}[X] \mid f(\mathbf{Z}^X) \subseteq \mathbf{Z}\}.$$

[3]The term *term algebra* is taken from universal algebra; confer Definition II.10.4 of [4].

Proof. The numerical axioms, together with the formula for iterated binomial coefficients, will reduce any element of $\mathbf{Z}\binom{X}{-}$ to a numerical polynomial. The very existence of the numerical ring of numerical polynomials is enough to guarantee the uniqueness of such a representation. We have thus established

$$\mathbf{Z}\binom{X}{-} \cong \{f \in \mathbf{Q}[X] \mid f(\mathbf{Z}^X) \subseteq \mathbf{Z}\}.$$

From this isomorphism it is evident that $\mathbf{Z}\binom{X}{-}$ is free on X, for any set map $\varphi\colon X \to R$ can be uniquely extended to $\mathbf{Z}\binom{X}{-}$ by setting

$$\varphi\left(\sum c_{n_1,\ldots,n_k}\binom{x_1}{n_1}\cdots\binom{x_k}{n_k}\right) = \sum c_{n_1,\ldots,n_k}\binom{\varphi(x_1)}{n_1}\cdots\binom{\varphi(x_k)}{n_k}.$$

□

§9. Numerical Transfer

Theorem 8: The Numerical Transfer Principle. — *A numerical polynomial identity $p(x_1, \ldots, x_k) = 0$ universally valid in $\mathbf{Z}$ is valid in every numerical ring.*

Proof. From the previous section we have a canonical embedding

$$\mathbf{Z}\binom{x_1, \ldots, x_k}{-} \to \mathbf{Z}^{\mathbf{Z}^k}$$
$$p(x_1, \ldots, x_k) \mapsto (p(n_1, \ldots, n_k))_{(n_1,\ldots,n_k)\in\mathbf{Z}^k}.$$

View p as an element of $\mathbf{Z}\binom{x_1,\ldots,x_k}{-}$. It is the zero numerical map, and therefore also the zero numerical polynomial. □

Example 7. — Recall that a *pre-λ-ring* (formerly called just *λ-ring*) is a commutative ring with unity equipped with unary operations λ^n, $n \in \mathbf{N}$, satisfying the following axioms:

1. $\lambda^0(a) = 1$.
2. $\lambda^1(a) = a$.
3. $\lambda^n(a+b) = \sum_{p+q=n} \lambda^p(a)\lambda^q(b)$.

In a numerical ring, the operations $\lambda^n(a) = \binom{a}{n}$ will evidently satisfy these axioms.

The definition of a *λ-ring* (a. k. a. *special λ-ring*) involves three more axioms, which are rather cumbersome to state. The reader will believe us when we claim they are of a polynomial nature, so their verification in a numerical ring will simply consist in verifying a number of numerical polynomial identities. As these are valid in $\mathbf{Z}$ (for $\mathbf{Z}$ itself is well known to be a λ-ring), they will hold in every numerical ring by Numerical Transfer. △

§10. The Nilradical

Yet another pleasant property of numerical rings is the following.

Theorem 9: Fermat's Little Theorem. — *In a numerical ring, the congruence*

$$a^p - a \equiv 0 \mod p$$

holds for any prime number p.

Proof. Since

$$f(x) = \frac{x^p - x}{p}$$

is a numerical map, it may be written as a numerical polynomial $f(x) \in \mathbf{Z}\binom{x}{-}$. But then evidently

$$a^p - a = pf(a) \in pR.$$

□

Example 8. — The polynomial f can in fact be given explicitly. For when $a \in \mathbf{N}$, we may calculate the number of functions $[p] \to [a]$ as

$$a^p = \sum_{k=1}^{p} k! \left\{ {p \atop k} \right\} \binom{a}{k},$$

where $\left\{ {p \atop k} \right\}$ denotes a Stirling number of the second kind. Since $k!\left\{ {p \atop k} \right\}$ counts the number of onto functions $[p] \to [k]$, these numbers are all divisible by p, except in the case $k = 1$, and so

$$\frac{a^p - a}{p} = \sum_{k=2}^{p} \frac{k!}{p} \left\{ {p \atop k} \right\} \binom{a}{k}.$$

It follows from the Numerical Transfer Principle that this formula is valid in every numerical ring. △

Theorem 10. — *The nilradical of a numerical ring is divisible, and hence a vector space over* $\mathbf{Q}$.

Proof. Let p be a prime and suppose a lies in the nilradical of R. From Fermat's Little Theorem $p \mid a(a^{p-1} - 1)$, from which it inductively follows that

$$p \mid a(a^{2^m(p-1)} - 1)$$

for all $m \in \mathbf{N}$. A large enough m will kill a, and we conclude that $p \mid a$. □

§11. Numerical Ideals and Factor Rings

Let us now make a short survey of numerical ideals and factor rings.

Theorem 11. — *Let I be an ideal of the numerical ring R. The equation*

$$\binom{r+I}{n} = \binom{r}{n} + I$$

will yield a numerical structure on R/I iff

$$\binom{e}{n} \in I$$

for every $e \in I$ and $n > 0$.

Proof. The condition is clearly necessary. To show sufficiency, note that, when $r \in R$, $e \in I$, and the condition is satisfied, then

$$\binom{r+e}{n} = \sum_{p+q=n} \binom{r}{p}\binom{e}{q} \equiv \binom{r}{n}\binom{e}{0} = \binom{r}{n} \mod I.$$

The numerical axioms in R/I follow immediately from those in R. □

Definition 4. — An ideal of a numerical ring satisfying the condition of the previous theorem will be called a **numerical ideal**. ⋄

Example 9. — $\mathbf{Z}$ does not possess any non-trivial numerical ideals, because all its non-trivial factor rings have torsion. Neither do the rings $\mathbf{Z}[m^{-1}]$. △

Theorem 12. — *Let R be a (commutative, unital) ring, and let I be an ideal. Suppose I is a vector space over $\mathbf{Q}$, and that R/I is numerical. Then R itself is numerical, and I is a numerical ideal.*

Proof. Since I and R/I are both torsion-free, so is R, and there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} I & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} R & \longrightarrow & \mathbf{Q} \otimes_{\mathbf{Z}} R/I & \longrightarrow & 0
\end{array}$$

It will suffice to show that R is closed under the formation of binomial coefficients in $\mathbf{Q} \otimes_{\mathbf{Z}} R$. Let $r \in R$. Calculating in in the ring $\mathbf{Q} \otimes_{\mathbf{Z}} R/I$ yields

$$\frac{r(r-1)\cdots(r-n+1)}{n!} + I = \binom{r+I}{n}.$$

Since $\binom{r+I}{n}$ in fact lies in R/I, it must be that

$$\frac{r(r-1)\cdots(r-n+1)}{n!} \in R,$$

and we are finished.

That I is numerical follows from the fact that it is a $\mathbf{Q}$-vector space. □

The quotient map $R \to R/I$ will automatically be a numerical ring homomorphism.

§12. Finitely Generated Numerical Rings

Lemma 4. — *If a ring R is torsion-free and finitely generated as an abelian group, its fraction ring is $\mathbf{Q} \otimes_{\mathbf{Z}} R$.*

Proof. By the Structure Theorem for Finitely Generated Abelian Groups, R is isomorphic to some $\mathbf{Z}^n$ as an abelian group. Let $a \in \mathbf{Z}^n$. Multiplication by a is a linear transformation on $\mathbf{Z}^n$, and so may be represented by an integer matrix A. The condition that a not be a zero divisor corresponds to A being non-singular. It will then have an inverse A^{-1} with *rational* entries. The inverse of a is given by

$$a^{-1} = A^{-1}1 \in \mathbf{Q}^n = \mathbf{Q} \otimes_{\mathbf{Z}} R,$$

where 1 denotes the multiplicative identity of R, considered as a column vector. □

Lemma 5. — *Let A denote the algebraic integers in the field $K \supseteq \mathbf{Q}$. If K is finitely generated over $\mathbf{Q}$, A is finitely generated over $\mathbf{Z}$.*

The following theorem, together with its proof, is due to Professor Ekedahl. It classifies completely those numerical rings which are finitely generated as *rings* (forgetting the numerical structure).

Before we enter the very technical proof, let us recall from Example 2 that $\mathbf{Z}[m^{-1}]$ inherits a numerical structure from $\mathbf{Q}$. Recall also that products of numerical rings are numerical, with componentwise evaluation of binomial coefficients.

Theorem 13: The Structure Theorem for Finitely Generated Numerical Rings. — *Let R be a numerical ring which is finitely generated as a ring. There exist unique positive, square-free*[4] *integers $m_1, \ldots, m_k$ such that*

$$R \cong \mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}].$$

[4] A *square-free*, or *simply composite*, number is a positive integer that is a (possibly empty) product of distinct primes.

Proof. Case A: R is finitely generated as an abelian group. We first impose the stronger hypothesis that R be finitely generated as an abelian group.

If $r^n = 0$, then, because of Fermat's Little Theorem, r is divisible by p for all primes $p > n$. But in $\mathbf{Z}^n$ this can only be if $r = 0$; hence R is reduced. By the lemma above, the fraction ring of R is $\mathbf{Q} \otimes_{\mathbf{Z}} R$. As this is reduced and artinian, being finite-dimensional over $\mathbf{Q}$, it splits up into a product of fields of characteristic 0.

Case A1: The fraction ring of R is a field. Let us first consider the special case when the fraction ring $\mathbf{Q} \otimes_{\mathbf{Z}} R$ is a field, whose ring of algebraic integers we denote by A. We examine the subgroup $A \cap R$ of A. Since $A \subseteq \mathbf{Q} \otimes_{\mathbf{Z}} R$, an arbitrary element of A will have an integer multiple lying in R. This means $A/(A \cap R)$ is a torsion group. Also, the fraction ring $\mathbf{Q} \otimes_{\mathbf{Z}} R$ is finitely generated over $\mathbf{Q}$, so from the lemma above, we deduce that A is finitely generated over $\mathbf{Z}$. Because the factor group $A/(A \cap R)$ is both finitely generated and torsion, it is killed by a single integer N, so that

$$N(A/(A \cap R)) = 0,$$

and as a consequence

$$(A \cap R)[N^{-1}] = A[N^{-1}].$$

Now let $z \in A$ and let p be a prime. The element

$$z \in A[N^{-1}] = (A \cap R)[N^{-1}]$$

can be written $z = \frac{a}{N^k}$, where $a \in A \cap R$ and $k \in \mathbf{N}$. Using Fermat's Little Theorem(s),

$$(N^k)^p = N^k + pn$$
$$a^p = a + pb$$

for some $n \in \mathbf{Z}$ and $b \in R$. Observe that pb belongs to $A \cap R$, hence to $A[N^{-1}]$, so that $b \in A$, as long as p does not divide N. We then have

$$\begin{aligned} z^p - z &= \frac{a^p}{N^{kp}} - \frac{a}{N^k} = \frac{a + pb}{N^k + pn} - \frac{a}{N^k} \\ &= \frac{(a + pb)N^k - a(N^k + pn)}{(N^k + pn)N^k} = p\frac{N^k b - na}{(N^k + pn)N^k} = p\frac{N^k b - na}{N^{(p+1)k}}, \end{aligned}$$

and hence

$$pu = z^p - z \in A$$

for some $u \in A[N^{-1}]$, assuming $p \nmid N$. But then in fact $u \in A$.

Consequently, for all $z \in A$ and all sufficiently large primes p, the relation $z^p - z \in pA$ holds, so that $z^p = z$ in A/pA. Being reduced and artinian, A/pA may be written as a product of fields, and because of the equation $z^p = z$, these fields must all equal $\mathbf{Z}/p$, which means all sufficiently large primes split completely in A. It is then a consequence of Chebotarev's Density Theorem[5] that $\mathbf{Q}\otimes_{\mathbf{Z}} R = \mathbf{Q}$. Since we are working under the assumption that R is finitely generated as an abelian group, we infer that $R = \mathbf{Z}$.

Case A2: The fraction ring of R is a product of fields. If the fraction ring of R is a product $\prod K_j$ of fields, the projections R_j of R on the factors K_j will each be numerical. Hence $R \subseteq \prod R_j$, with each R_j being isomorphic to $\mathbf{Z}$, according to the above argument. But $\mathbf{Z}$ possesses no non-trivial numerical ideals, so by Delsarte's Lemma, R must equal the whole product

$$R = \prod R_j = \prod \mathbf{Z}.$$

Case B: R is not finitely generated as an abelian group. Finally, we drop the assumption that R be finitely generated as a group, and assume it finitely generated as a ring only. Because of the relation $p \mid r^p - r$, R/pR will be a finitely generated torsion group for each prime p. It will then have Krull dimension 0, and it follows from Chevalley's Dimension Argument that $\dim \mathbf{Q}\otimes_{\mathbf{Z}} R = 0$, so that $\mathbf{Q}\otimes_{\mathbf{Z}} R$ is a finite-dimensional vector space over $\mathbf{Q}$. Only finitely many denominators are employed in a basis, so there exists an integer M for which $R[M^{-1}]$ is finitely generated over $\mathbf{Z}[M^{-1}]$.

We can now more or less repeat the previous argument. $R[M^{-1}]$ will still be reduced, and as before, $\mathbf{Q}\otimes_{\mathbf{Z}} R[M^{-1}]$ will be finite-dimensional, hence a product of fields, and we may reduce to the case when $\mathbf{Q}\otimes_{\mathbf{Z}} R[M^{-1}]$ is a field. Letting A denote the algebraic integers in $\mathbf{Q}\otimes_{\mathbf{Z}} R[M^{-1}]$, the factor group $A/R[M^{-1}]$ will be finitely generated and torsion, and hence killed by some integer, so that again we are lead to $R[N^{-1}] = A[N^{-1}]$. As before, we may draw the conclusion that $\mathbf{Q}\otimes_{\mathbf{Z}} R = \mathbf{Q}$, and consequently that $R = \mathbf{Z}[N^{-1}]$. This concludes the proof. □

§13. Modules

A most elegant application of the Structure Theorem is the classification of torsion-free modules.

Lemma 6. — *Consider a ring homomorphism $\varphi\colon R \to S$. If R is numerical and S is torsion-free, then* $\operatorname{Ker}\varphi$ *will be a numerical ideal.*

Proof.

$$n!\varphi\left(\binom{r}{n}\right) = \varphi\left(n!\binom{r}{n}\right) = \varphi(r(r-1)\cdots(r-n+1)) = 0,$$

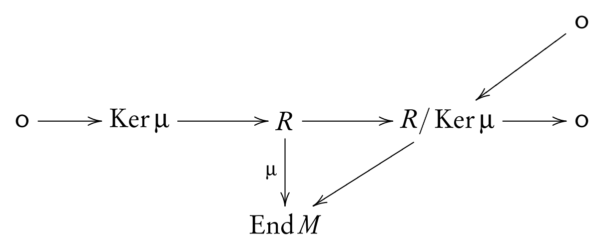
if $r \in \operatorname{Ker}\varphi$ and $n > 0$. Thus $\binom{r}{n} \in \operatorname{Ker}\varphi$, which is then numerical. □

[5](A special case of) Chebotarev's Density Theorem states the following: The density of the primes that split completely in a number field K equals $\frac{1}{|\mathrm{Gal}(K/\mathbf{Q})|}$. In our case, this set has density 1.

Let M be a torsion-free module over the numerical ring R, with module structure given by the group homomorphism

$$\mu\colon R \to \operatorname{End} M.$$

We have the following commutative diagram:



The group $\operatorname{End} M$ is torsion-free, so, by the lemma, $\operatorname{Ker}\mu$ is a numerical ideal. Therefore M will in fact be a module over the numerical ring $R/\operatorname{Ker}\mu$.

Let us now also assume that $\operatorname{End} M$ is finitely generated (as a module) over $\mathbf{Z}[n^{-1}]$ for some integer n. Because $\mathbf{Z}[n^{-1}]$ is a noetherian ring, $\operatorname{End} M$ is a noetherian module. The submodule $R/\operatorname{Ker}\mu$ is finitely generated as a module over $\mathbf{Z}[n^{-1}]$, and therefore also as a ring, so by the Structure Theorem,

$$R/\operatorname{Ker}\mu \cong \mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}],$$

for square-free, positive integers m_j. The module M will split up as a direct sum

$$M = M_1 \oplus \cdots \oplus M_k,$$

with each M_j a torsion-free module over $\mathbf{Z}[m_j^{-1}]$. Because these rings are principal, the modules M_j are in fact free, and we have proved:

Theorem 14. — *Consider a module M over a numerical ring. Suppose M is torsion-free and finitely generated over $\mathbf{Z}[n^{-1}]$ for some integer n. There exist positive integers m_j, r_j such that*

$$M \cong \mathbf{Z}[m_1^{-1}]^{r_1} \oplus \cdots \oplus \mathbf{Z}[m_k^{-1}]^{r_k}$$

as a module over

$$\mathbf{Z}[m_1^{-1}] \times \cdots \times \mathbf{Z}[m_k^{-1}].$$

§14. Exponentiation

When A is a ring, the symbol

$$\sqrt[A]{0}$$

shall denote its nilradical.

Let R be a numerical ring, and consider a (commutative, unital) algebra A over it. There is an induced *exponentiation* on $1 + \sqrt[A]{0}$, given by the (finite) binomial expansion:

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n.$$

The numerical axioms imply the following properties.

I. $(1+x)^r(1+x)^s = (1+x)^{r+s}$.

II. $\big((1+x)^r\big)^s = (1+x)^{rs}$.

III. $(1+x)^r(1+y)^r = \big((1+x)(1+y)\big)^r$.

IV. $(1+x)^1 = 1+x$.

V. $(1+x)^r \equiv 1 + rx \mod(\sqrt{0})^2$.

Exponentiation will thus make the abelian group $(1+\sqrt[A]{0}, \cdot)$ into an R-module. Indeed, property III shows that exponentiation by r gives an endomorphism $\varepsilon(r)$ of the group, and properties I, II, and IV show that

$$\varepsilon_A : R \to \mathrm{End}(1 + \sqrt[A]{0}, \cdot)$$

is a unital ring homomorphism.

The module structure is *natural* in the following sense. Given two algebras A and B and an algebra homomorphism $\varphi\colon A \to B$, the subsequent diagram commutes for any $r \in R$:

$$\begin{array}{ccc} 1+\sqrt[A]{0} & \xrightarrow{\varepsilon_A(r)} & 1+\sqrt[A]{0} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ 1+\sqrt[B]{0} & \xrightarrow[\varepsilon_B(r)]{} & 1+\sqrt[B]{0} \end{array}$$

Let us now reverse the procedure.

Theorem 15: The Binomial Theorem. *Let R be a commutative, unital ring.*

- *If R is numerical, the equation*

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \tag{4}$$

defines a module structure on $(1+\sqrt[A]{0}, \cdot)$, *which is natural in R-algebras A, and in addition satisfies*

$$(1+x)^r \equiv 1 + rx \mod(\sqrt{0})^2. \tag{5}$$

- *Conversely, if there is a natural module structure on* $(1+\sqrt[A]{0},\cdot)$ *for all R-algebras A, satisfying (5), then there is a (necessarily unique) numerical ring structure on R, fulfilling the equation (4).*

Proof. There remains to establish the second part. Let a natural module structure be given, and consider

$$\varepsilon_A : R \to \mathrm{End}(1+\sqrt[A]{0},\cdot),$$

where $A = R[t]/(t^{n+1})$, and n is some (large) number. We have

$$\varepsilon(r)(1+t) = (1+t)^r = a_0 + a_1 t + \cdots + a_n t^n,$$

and clearly the coefficients a_k are independent of n. Therefore, we may without ambiguity define $\binom{r}{k} = a_k$. This will make the identity (4) hold in A, and then it will hold everywhere by naturality.

The axioms for a numerical ring should now be immediate, as they are simply direct translations of the module axioms. For example, identification of the coefficients of t^n in

$$\sum_{i=0}^{\infty}\binom{r}{i}t^i \sum_{j=0}^{\infty}\binom{s}{j}t^j = (1+t)^r(1+t)^s = (1+t)^{r+s} = \sum_{n=0}^{\infty}\binom{r+s}{n}t^n$$

proves axiom I. (Proving III will of course involve the polynomial ring in two variables.) □

And this little “treatise upon the Binomial Theorem” closes the chapter on numerical rings.

Chapter 2

MULTI-SETS

Är Du en Enhet eller delar?
Jag bäfvar, mod och sansning felar,
  Min fråga giör mig stel och stum.

— Hedwig Charlotta Nordenflycht,
Ode i Anledning af Exod. XXXIII: Cap. v. 18. 20.
och XXXIV: Cap. v. 5. 6.

The text will be pervaded by the use of multi-sets, and we develop here their theory from scratch. This we do partly to fix notation, and partly because some concepts we need are possibly not standard. We certainly lay no claims of originality upon the theory explored in this chapter. It is conceivable, and even very likely, that all the results of this chapter can be found somewhere in the literature.

After giving the basic definitions, we propose to answer the following question: What would be the natural arrows of a category of multi-sets? The non-existence of a definite answer does not depend on a lack of suggestions. Dr. Salomonsson's thesis [20] presents a plethora: multijections, multi-maps, maps, and bimultijections. He finally decides to build the multi-set category with *multijections* as the basic arrows, a choice we believe is less suited to our purposes. We have settled on the latter kind, *bimultijections*, as giving the most natural theory, and in the process renamed them *multations*. Our conviction that this is the correct choice stems from the tight connection that is seen to exist between multations and divided powers.

One advantage of using multijections is that they allow for a unique composition. We have found it expedient to drop that requirement. The very nature of multi-sets, with their repeated elements, seems to exclude unambiguous composition in the usual sense. The solution we have accepted (which is so natural, it might almost be termed canonical) is to "sum over all possible compositions".

§1. Multi-Sets

Definition 1. — A **multi-set** is a pair

$$M = (\#M, \deg_M),$$

where $\#M$ is a set and

$$\deg_M \colon \#M \to \mathbf{Z}^+$$

is a function, called the **degree**, or **multiplicity**. The underlying set $\#M$ is called the **support** of M. ◇

We call

$$\deg_M a$$

the **degree** or **multiplicity** of an object $a \in \#M$; it counts the "number of times a occurs in M". The degree of the whole multi-set M we define to be

$$\deg M = \prod_{x \in \#M} (\deg x)!.$$

It might conceivably be convenient to have the degree function defined on the whole set-theoretic universe. A multi-set may then equivalently be defined as a function

$$M = \deg_M \colon \mathfrak{Set} \to \mathbf{N}$$

vanishing outside some set. The support is given by

$$\#M = \{x \mid \deg_M(x) \neq 0\}.$$

In order to give a multi-set, it suffices to specify such a degree function.

Definition 2. — The **cardinality** of M is

$$|M| = \sum_{x \in M} 1 = \sum_{x \in \#M} \deg x.$$

◇

The cardinality counts the number of elements with multiplicity. We tacitly assume all multi-sets under discussion to be *finite*, as these are the only ones we will ever need.

Example 1. — The multi-set $\{a, a, b\}$ has cardinality 3 and support $\{a, b\}$. We have $\deg a = 2$, $\deg b = 1$, and $\deg c = 0$. △

It is now an easy matter to generalise the elementary set operations to multi-sets.

Definition 3. — The **union** $A \cup B$ of A and B is

$$\deg_{A \cup B} = \max(\deg_A, \deg_B).$$

◇

Definition 4. — The **disjoint union**[1] $A \sqcup B$ of A and B is

$$\deg_{A \sqcup B} = \deg_A + \deg_B .$$

◇

[1] Please note that [20] employs a different definition of disjoint union, and calls this the *sum* of A and B.

Definition 5. — The **intersection** $A \cap B$ of A and B is

$$\deg_{A \cap B} = \min(\deg_A, \deg_B).$$

◇

Definition 6. — The **relative complement** $A \backslash B$ of B in A is

$$\deg_{A \backslash B} = \max(\deg_A - \deg_B, 0).$$

◇

Definition 7. — The **direct product** $A \times B$ of A and B is

$$\deg_{A \times B} = \deg_A \cdot \deg_B \colon \#A \times \#B \to \mathbf{Z}^+.$$

◇

Definition 8. — A is a **sub-multi-set**[2] of B, written $A \subseteq B$, if

$$\deg_A \leqslant \deg_B$$

(element-wise inequality). ◇

Definition 9. — The **power multi-set** of A is

$$2^A = \{B \mid B \subseteq A\},$$

where every sub-multi-set of A is counted "according to multiplicity". ◇

In other words, the classical formula $|2^A| = 2^{|A|}$ will still be valid.

Example 2. — Given $A = \{x, x, y\}$ and $B = \{x, y, z\}$, we have

$$\begin{aligned}
A \cup B &= \{x, x, y, z\} \\
A \sqcup B &= \{x, x, x, y, y, z\} \\
A \cap B &= \{x, y\} \\
A \backslash B &= \{x\} \\
B \backslash A &= \{z\} \\
A \times B &= \{(x,x), (x,x), (y,x), (x,y), (x,y), (y,y), (x,z), (x,z), (y,z)\} \\
2^A &= \{\emptyset, \{x\}, \{x\}, \{y\}, \{x,x\}, \{x,y\}, \{x,y\}, \{x,x,y\}\}.
\end{aligned}$$

△

Recall that the Principle of Inclusion and Exclusion, in one form, states the following: *If f and g are functions such that*

$$\sum_{X \subseteq Y} f(X) = g(Y),$$

[2] Some people would say *multi-subset*.

then

$$f(Y) = \sum_{X \subseteq Y} (-1)^{|Y|-|X|} g(X).$$

Of course, X and Y are here limited to be sets, but a generalisation to multi-sets is immediate.

Theorem 1: The Principle of Inclusion and Exclusion. — *Let S be a set with n elements. Consider functions f defined on multi-sets with support included in S and cardinality n; and functions g defined on the power set of S. If*

$$\sum_{\substack{\#A \subseteq X \\ |A|=n}} f(A) = g(X),$$

then

$$\sum_{\substack{\#A = Y \\ |A|=n}} f(A) = \sum_{X \subseteq Y} (-1)^{|Y|-|X|} g(X).$$

Proof. Calculate

$$\begin{aligned}\sum_{X \subseteq Y} (-1)^{|Y|-|X|} g(X) &= \sum_{X \subseteq Y} (-1)^{|Y|-|X|} \sum_{\substack{\#A \subseteq X \\ |A|=n}} f(A) \\ &= \sum_{\substack{\#A \subseteq Y \\ |A|=n}} (-1)^{|Y|} f(A) \sum_{\#A \subseteq X \subseteq Y} (-1)^{|X|},\end{aligned}$$

and note that the inner sum vanishes, unless $\#A = Y$. □

§2. Multations

Let A and B be multi-sets of equal cardinality. A **multation** $\mu \colon A \to B$ is a pairing of their elements. We shall write multations as two-row matrices, with the elements of A on top of those of B, the way ordinary permutations are usually written:

$$\mu = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix}$$

The order of the columns is of course irrelevant.

Observe that μ is not a "function" from A to B, since identical copies of some element of A may very well be paired off with distinct elements of B. It will, however, be a *sub-multi-set* of $A \times B$, with the property that every element of A occurs exactly once as the first component of a pair in μ, and each element of B exactly once as a second component. The degree $\deg_\mu(a, b)$ counts the number of times $a \in A$ is paired off with $b \in B$.

Given a multation

$$\begin{bmatrix} a_1 & a_1 & \dots & a_2 & a_2 & \dots \\ b_1 & b_1 & \dots & b_2 & b_2 & \dots \end{bmatrix},$$

with m_j appearances of the column $\begin{bmatrix} a_j \\ b_j \end{bmatrix}$, we shall adopt the perspective of viewing it as a formal product

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}^{[m_1]} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}^{[m_2]} \dots$$

of *divided powers*[3].

Example 3. — There exist two multations from the multi-set $\{a, a, b\}$ to itself, namely:

$$\begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}^{[2]} \begin{bmatrix} b \\ b \end{bmatrix} \qquad \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}.$$

The degree of (a, b) is 0 with respect to the first of these, and 1 with respect to the second. △

§3. Confluent Products

Some heavy notation will inevitably come into play when writing a thesis. We here describe some shorthand, which will be used extensively for the remainder of the text.

Let A be a multi-set, and let M be a module. Consider elements $x_a \in M$ indexed over the support of A. Define the **confluent product** over A as

$$\bigodot_{a\in A} x_a = \prod_{a\in \# A} x_a^{[\deg a]} \in \Gamma(M).$$

For example, we have

$$x \odot x \odot y = x^{[2]} y.$$

We may further abbreviate

$$x^{[A]} = \bigodot_{a\in A} x_a = \prod_{a\in \# A} x_a^{[\deg_A a]},$$

and similarly

$$x^{A} = \prod_{a\in A} x_a = \prod_{a\in \# A} x_a^{\deg_A a}.$$

[3]A divided power $x^{[n]}$ should be thought of as $\frac{x^n}{n!}$. We shall later discuss divided power algebras, but some previous familiarity with these structures will be assumed of the reader.

This latter product is defined in any algebra.

A special case of this practice arises when $\mu\colon A \to B$ is a multation, and the variables x_{ba} have been doubly indexed over the supports of B and A simultaneously. We then have

$$x^{[\mu]} = \bigodot_{(a,b)\in\mu} x_{ba} = \prod_{\substack{a\in\#A\\ b\in\#B}} x_{ba}^{[\deg_\mu(a,b)]},$$

and

$$x^{\mu} = \prod_{(a,b)\in\mu} x_{ba} = \prod_{\substack{a\in\#A\\ b\in\#B}} x_{ba}^{\deg_\mu(a,b)}.$$

As a special case of *that* practice, we have, for quantities x_a and y_b indexed by $\#A$ and $\#B$, respectively,

$$(xy)^{[\mu]} = \bigodot_{(a,b)\in\mu} x_a y_b = \prod_{\substack{a\in\#A\\ b\in\#B}} (x_a y_b)^{[\deg_\mu(a,b)]}.$$

Finally, consider a multation $\mu\colon A \to B$ and variables $x_b \in M$ indexed by the support of B. We then let

$$x^{\otimes[\mu]} = \bigotimes_{a\in\#A} \bigodot_{(a,b)\in\mu} x_b \in \bigotimes_{a\in\#A} \Gamma^{\deg_A a}(M) = \Gamma^A(M).$$

So for example, if

$$\mu = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 3 \end{bmatrix},$$

then

$$x^{\otimes[\mu]} = x_1^{[2]} x_2 \otimes x_2 x_3.$$

§4. The Multi-Set Category

Let $\mu\colon B \to C$ and $\nu\colon A \to B$ be two multations, where $|A| = |B| = |C| = n$. Their **composition** $\mu \circ \nu$ is found by identifying the coefficient of $x^\mu y^\nu$ in the equation

$$\left(\sum_{\substack{b\in\#B\\ c\in\#C}} x_{cb} \begin{bmatrix} b \\ c \end{bmatrix}\right)^{[n]} \circ \left(\sum_{\substack{a\in\#A\\ b\in\#B}} y_{ba} \begin{bmatrix} a \\ b \end{bmatrix}\right)^{[n]} = \left(\sum_{\substack{a\in\#A\\ b\in\#B\\ c\in\#C}} x_{cb} y_{ba} \begin{bmatrix} a \\ c \end{bmatrix}\right)^{[n]}.$$

In somewhat fancier language, we have defined a formal multiplication of columns:

$$\begin{bmatrix} b' \\ c \end{bmatrix} \circ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \begin{bmatrix} a \\ c \end{bmatrix} & \text{if } b = b', \\ 0 & \text{if } b \neq b'. \end{cases}$$

This makes the free module of columns into an algebra A. Multation composition is then given by the formula

$$u^{[n]} \star v^{[n]} = (uv)^{[n]},$$

an operation which we shall later have occasion to baptise the *product multiplication* on $\Gamma^n(A)$.

EXAMPLE 4. — For example, to find the composition

$$\begin{bmatrix} c & d & d \\ a & a & b \end{bmatrix} \circ \begin{bmatrix} a & a & b \\ c & d & d \end{bmatrix},$$

we use the equation

$$\left(x_{ac} \begin{bmatrix} c \\ a \end{bmatrix} + x_{ad} \begin{bmatrix} d \\ a \end{bmatrix} + x_{bd} \begin{bmatrix} d \\ b \end{bmatrix} \right)^{[3]} \circ \left(y_{ca} \begin{bmatrix} a \\ c \end{bmatrix} + y_{da} \begin{bmatrix} a \\ d \end{bmatrix} + y_{db} \begin{bmatrix} b \\ d \end{bmatrix} \right)^{[3]}$$
$$= \left(x_{ac}y_{ca} \begin{bmatrix} a \\ a \end{bmatrix} + x_{ad}y_{da} \begin{bmatrix} a \\ a \end{bmatrix} + x_{ad}y_{db} \begin{bmatrix} b \\ a \end{bmatrix} + x_{bd}y_{da} \begin{bmatrix} a \\ b \end{bmatrix} + x_{bd}y_{db} \begin{bmatrix} b \\ b \end{bmatrix} \right)^{[3]}.$$

Identification of the coefficients of

$$x_{ac}x_{ad}x_{bd}y_{ca}y_{da}y_{db}$$

yields

$$\begin{bmatrix} c & d & d \\ a & a & b \end{bmatrix} \circ \begin{bmatrix} a & a & b \\ c & d & d \end{bmatrix} = 2 \begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix} + \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix}.$$

Similarly, by picking the coefficients of

$$x_{ac}x_{ad}^2y_{ca}y_{da}^2,$$

we obtain

$$\begin{bmatrix} c & d & d \\ a & a & a \end{bmatrix} \circ \begin{bmatrix} a & a & a \\ c & d & d \end{bmatrix} = 3 \begin{bmatrix} a & a & a \\ a & a & a \end{bmatrix}.$$

△

There is a simpler, combinatorial rule for calculating the composition. Namely, the composition of two divided power *products* is found by "summing over all possibilities of composing them":

$$\left(\begin{bmatrix} b_1 \\ c_1 \end{bmatrix} \cdots \begin{bmatrix} b_n \\ c_n \end{bmatrix} \right) \circ \left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \cdots \begin{bmatrix} a_n \\ b_n \end{bmatrix} \right) = \sum_{\sigma} \left(\begin{bmatrix} a_1 \\ c_{\sigma(1)} \end{bmatrix} \cdots \begin{bmatrix} a_n \\ c_{\sigma(n)} \end{bmatrix} \right),$$

where the sum is to be taken over all permutations $\sigma\colon [n] \to [n]$ such that $b_j = b_{\sigma(j)}$ for all j. We leave it to the reader to check the accuracy of this rule.

Example 5. — Computing according to this device, we have

$$\begin{aligned}
\begin{bmatrix} c & d & d \\ a & a & b \end{bmatrix} \circ \begin{bmatrix} a & a & b \\ c & d & d \end{bmatrix} &= \begin{bmatrix} c \\ a \end{bmatrix} \begin{bmatrix} d \\ a \end{bmatrix} \begin{bmatrix} d \\ b \end{bmatrix} \circ \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} \\
&= \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix} + \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\
&= 2 \begin{bmatrix} a \\ a \end{bmatrix}^{[2]} \begin{bmatrix} b \\ b \end{bmatrix} + \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \\
&= 2 \begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix} + \begin{bmatrix} a & a & b \\ a & b & a \end{bmatrix},
\end{aligned}$$

and similarly

$$\begin{aligned}
\begin{bmatrix} c & d & d \\ a & a & a \end{bmatrix} \circ \begin{bmatrix} a & a & a \\ c & d & d \end{bmatrix} &= \begin{bmatrix} c \\ a \end{bmatrix} \begin{bmatrix} d \\ a \end{bmatrix}^{[2]} \circ \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix}^{[2]} \\
&= \frac{1}{2} \begin{bmatrix} c \\ a \end{bmatrix} \begin{bmatrix} d \\ a \end{bmatrix} \begin{bmatrix} d \\ a \end{bmatrix} \circ \frac{1}{2} \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} \\
&= \frac{1}{4} \cdot 2 \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} \\
&= 3 \begin{bmatrix} a \\ a \end{bmatrix}^{[3]} = 3 \begin{bmatrix} a & a & a \\ a & a & a \end{bmatrix}.
\end{aligned}$$

△

The **identity multation** ("identitation") ι_A of a multi-set A is the multation in which every element is paired off with itself. It is clear that composition is associative and that the identity multations act as identities.

Definition 10. — The n**th multi-set category** has as objects the multi-sets of cardinality exactly n. Given two multi-sets A and B, the arrow set $\mathfrak{MSet}_n(A, B)$ is the free module generated by the multations $A \to B$. ◇

§5. Multi-Sets on Multi-Sets

Unfortunately, we shall have to push things one step further, and deal with multi-sets supported by multi-sets, which may sound like rather a baroque consideration. But the situation is not nearly as unpleasant as it was originally, before Professor Franjou graciously helped us tidy things up a bit, for which we are humbly grateful.

Definition 11. — Let M be a multi-set. A **multi-set supported in** M is a multiset supported in the set

$$M^{\#} = \left\{ (x, k) \in \#M \times \mathbf{Z}^{+} \;\middle|\; 1 \leqslant k \leqslant \deg x \right\}.$$

◇

Example 6. — Let M be the set $\{a, b, c\}$ and N the multi-set $\{x, x, y\}$. There are three multi-sets with support M and cardinality 4:

$$\{a, a, b, c\}, \quad \{a, b, b, c\}, \quad \{a, b, c, c\}.$$

Likewise, since

$$N^{\#} = \{(x, 1), (x, 2), (y, 1)\},$$

there are three multi-sets with support N and cardinality 4:

$$\{(x,1),(x,1),(x,2),(y,1)\},\ \{(x,1),(x,2),(x,2),(y,1)\},\ \{(x,1),(x,2),(y,1),(y,1)\}.$$

△

When speaking of multi-sets supported in a multi-set M, we will let "degree over M" stand for "degree over $M^{\#}$".

Example 7. — The three multi-sets above supported in N have all degree 2 over N. △

§6. Partitions and Compositions

Partitions and compositions of multi-sets will come into play when we investigate operads. The generalisation from the case of sets should be straightforward.

Definition 12. — Let X be a multi-set. The multi-set E constitutes a **partition** of X if

$$\bigsqcup_{Y \in E} Y = X.$$

We let

$$\operatorname{Par} X$$

denote the set of partitions of X. ⋄

Example 8. — The four partitions of $\{1, 1, 2\}$ are

$$\{1, 1, 2\} = \{1, 1\} \sqcup \{2\} = \{1\} \sqcup \{1, 2\} = \{1\} \sqcup \{1\} \sqcup \{2\}.$$

△

Definition 13. — Let X be a multi-set. The multation $\omega \colon A \to B$ constitutes an A-**composition** of X if

$$\bigsqcup_{a \in A} \omega(a) = X.$$

We let

$$\operatorname{Com}_A X$$

denote the set of A-compositions of X. ⋄

Example 9. — The ten $\{a, a, b\}$-compositions of $\{1, 1, 2\}$ are

$$\begin{bmatrix} a & a & b \\ \{1,1,2\} & \emptyset & \emptyset \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \emptyset & \emptyset & \{1,1,2\} \end{bmatrix},$$
$$\begin{bmatrix} a & a & b \\ \{1,1\} & \{2\} & \emptyset \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \{1,1\} & \emptyset & \{2\} \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \emptyset & \{2\} & \{1,1\} \end{bmatrix},$$
$$\begin{bmatrix} a & a & b \\ \{1,2\} & \{1\} & \emptyset \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \{1,2\} & \emptyset & \{1\} \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \emptyset & \{1\} & \{1,2\} \end{bmatrix},$$
$$\begin{bmatrix} a & a & b \\ \{1\} & \{1\} & \{2\} \end{bmatrix}, \quad \begin{bmatrix} a & a & b \\ \{2\} & \{1\} & \{1\} \end{bmatrix}.$$

(Each row corresponds to a partition above.) △

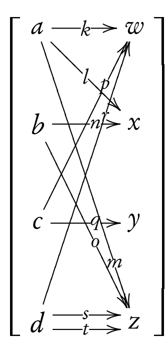
Chapter 3

MAZES

Dedalus genom sin konst och sitt snille vida beryktad
Bygde det opp; han förvirrar de ledande märken och ögat
I villfarelse för ibland skiljaktiga vägar.
Så på de Frygiska fält, man ser den klara Meandros
Leka. I tveksamt lopp han rinner och rinner tillbaka,
Möter sig ofta sjelf och skådar sin kommande bölja,
Och nu till källan vänd, nu åt obegränsade hafvet,
Rådvill öfvar sin våg. Så fyllas af Dedalus äfven
Tusen vägar med irrande svek: Knappt mäktar han sjelf att
Hitta till tröskeln igen. Så bedräglig han boningen danat.

— Ovidius, *Metamorphoses*

Once upon a time[1], we introduced[2] the concept of mazes. A *maze* from the set $\{a, b, c, d\}$ to the set $\{w, x, y, z\}$ (say) is something like the following diagram:



where $k, l, m, n, o, p, q, r, s, t$ are scalars. The reason for applying the name *maze* to such a contraption should be apparent from the picture. This is not to say that other names have not been suggested. "Why not *quiver*," we were once asked, "for something that obviously contains a lot of arrows?" Because quivers have already been invented.

The *labyrinth category* is the category with finite sets as objects, and mazes as arrows. It might at first seem a puzzling object. Why would anyone be

[1]This was back in 2009, in our licenciate thesis.
[2]Syn.: *discovered*, *invented*.

interested in a category with such strange arrows (and an even stranger law of composing them)? The answer is that the labyrinth category provides a very natural, one might even say *canonical*, means of encoding endofunctors of the category $\mathfrak{Mod}$.

When defining the labyrinth category and exploring its properties, we remark that the base ring **B** need not be commutative. The existence of a unity is required, as always, but otherwise it may be of quite an arbitrary nature.

§1. Mazes

Let X and Y be finite sets. A **passage** from $x \in X$ to $y \in Y$ is a (formal) arrow p from x to y, labelled with an element of **B**, denoted by $\bar{p}$. This we write as

$$p\colon x \to y,$$

or

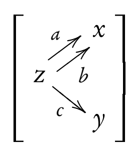
$$x \xrightarrow{\bar{p}} y\ ,$$

though we shall frequently forget the bar over p when no confusion is likely to arise.

Definition 1. — A **maze** from X to Y is a multi-set of passages from X to Y. It is required that there be at least one passage leading from every element of X, and at least one passage leading to every element of Y. (We, so to speak, wish to prevent *dead ends* from forming.) ◇

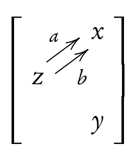
Because a maze is a multi-set, there can (and, in general, will) be multiple passages between any two given elements.

Example 1. — The following is a maze from $\{z\}$ to $\{x, y\}$:

$Q =$ 

,

whereas the following is not:



.

△

Example 2. — It is perfectly legal to consider the **empty maze** $\emptyset \to \emptyset$. It is the only maze into or out of $\emptyset$, and is the only maze having no passages. △

Definition 2. — We say $P\colon X \to Y$ is a **submaze** of $Q\colon X \to Y$ if $P \subseteq Q$ as multi-sets. ◇

Example 3. — The following is a submaze of Q above:

$$P = \left[\begin{array}{ccc} & \overset{a}{\nearrow} & x \\ z & & \\ & \underset{c}{\searrow} & y \end{array}\right].$$

△

Definition 3. — Let $P\colon X \to Y$ be a maze, and consider subsets $X' \subseteq X$ and $Y' \subseteq Y$. The **restriction** of P to $X' \to Y'$ is the maze (if indeed it is one) containing only those passages of P that begin in X' and end in Y'. It will be denoted by

$$P\big|_{X' \to Y'}.$$

◇

Note that $P\big|_{X' \to Y'}$ is not a submaze of P (unless $X' = X$ and $Y' = Y$).

Example 4. — With Q as above, we have

$$Q\big|_{\{z\} \to \{x\}} = \left[\begin{array}{ccc} & \overset{a}{\nearrow} & x \\ z & \underset{b}{\nearrow} & \\ & & \end{array}\right].$$

△

Restrictions may not always exist, as in the following example.

Example 5. — If

$$R = \left[\begin{array}{c} x \xrightarrow{a} x \\ y \xrightarrow[b]{} y \end{array}\right],$$

then the restriction $R\big|_{\{x\} \to \{y\}}$ does not exist (it is not a maze). In such a case, it will be convenient to define the symbol

$$R\big|_{\{x\} \to \{y\}} = 0.$$

△

§2. The Labyrinth Category

Passages $p\colon y \to z$ and $q\colon x \to y$ are said to be **composable**, because one ends where the other begins.

Definition 4. — If $P\colon Y \to Z$ and $Q\colon X \to Y$ are mazes, we define the **cartesian product** $P \boxtimes Q$ to be the multi-set of all pairs of composable passages:

$$P \boxtimes Q = \left\{ \left(\left[z \xleftarrow{p} y \right], \left[y \xleftarrow{q} x \right] \right) \,\middle|\, \left[z \xleftarrow{p} y \right] \in P \ \wedge \ \left[y \xleftarrow{q} x \right] \in Q \right\}.$$

◇

For a sub-multi-set $U \subseteq P \boxtimes Q$, we shall write

$$U \sqsubseteq P \boxtimes Q$$

to indicate that the projections on P and Q are both onto.

Note that such a set U can be naturally interpreted as a maze itself, namely

$$\left\{ \left[z \xleftarrow{pq} x \right] \middle| \left(\left[z \xleftarrow{p} y \right], \left[y \xleftarrow{q} x \right] \right) \in U \right\}.$$

Observe carefully the order in which p and q occur. The surjectivity condition on the projections will prevent dead ends from forming.

When writing $P \boxtimes Q$, we will sometimes refer to the cartesian product, and sometimes the maze which thus is naturally associated therewith, and hope the circumstances will make clear which is meant.

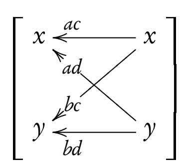
Example 6. — Consider the two mazes

$$P = \begin{bmatrix} x & & \\ & \overset{a}{\nwarrow} & \\ & & z \\ & \underset{b}{\swarrow} & \\ y & & \end{bmatrix}, \qquad Q = \begin{bmatrix} & & x \\ & \overset{c}{\swarrow} & \\ z & & \\ & \underset{d}{\nwarrow} & \\ & & y \end{bmatrix}.$$

Their cartesian product is

$$P \boxtimes Q = \Big\{ \left(\left[x \xleftarrow{a} z \right], \left[z \xleftarrow{c} x \right] \right), \left(\left[y \xleftarrow{b} z \right], \left[z \xleftarrow{c} x \right] \right),$$
$$\left(\left[x \xleftarrow{a} z \right], \left[z \xleftarrow{d} y \right] \right), \left(\left[y \xleftarrow{b} z \right], \left[z \xleftarrow{d} y \right] \right) \Big\},$$

which we identify with the maze



.

△

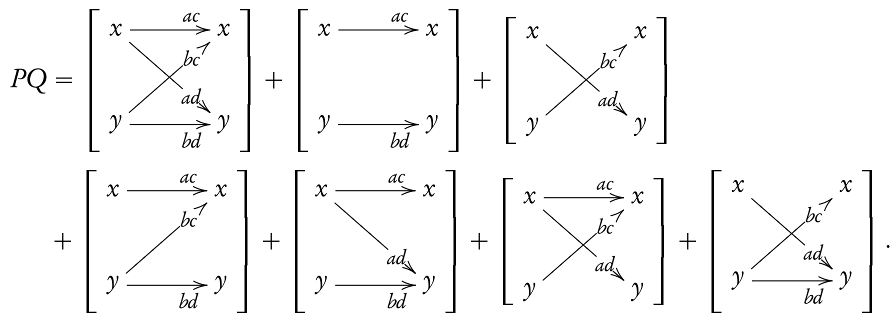
We now define the composition of two mazes. As for multi-sets, this composition will not in general be a maze, but rather a *sum* of mazes, and living in the free module generated by those.

Definition 5. — The **composition** or **product** of the mazes P and Q is the formal sum

$$PQ = \sum_{U \sqsubseteq P \boxtimes Q} U.$$

◇

Example 7. — Let P and Q be as above. Their composition is



△

That composition is associative follows from the observation that $(PQ)R$ and $P(QR)$ both equal

$$\sum_{W \sqsubseteq P \boxtimes Q \boxtimes R} W.$$

There exist **identity mazes**

$$I_X = \bigcup_{x \in X} \left\{ x \xrightarrow{1} x \right\}.$$

Definition 6. — The **labyrinth category** $\mathfrak{Laby}$ has as objects the finite sets. Given two sets X and Y, the arrow set $\mathfrak{Laby}(X, Y)$ is the module generated by the mazes $X \to Y$, with the following relations imposed.

I.

$$\left[P \cup \left\{ * \xrightarrow{0} * \right\} \right] = 0,$$

for any multi-set P of passages.

II.

$$\left[P \cup \left\{ * \xrightarrow{a+b} * \right\} \right] =$$
$$\left[P \cup \left\{ * \xrightarrow{a} * \right\} \right] + \left[P \cup \left\{ * \xrightarrow{b} * \right\} \right] + \left[P \cup \left\{ * \overset{a}{\underset{b}{\rightrightarrows}} * \right\} \right],$$

for any multi-set P of passages.

◇

The second axiom may be generalised by means of mathematical induction to yield the following elementary formulæ.

THEOREM 1. — *In the labyrinth category, the following two equations hold:*

$$\left[P \cup \left\{ * \xrightarrow{\sum_{i=1}^{n} a_i} * \right\}\right] = \sum_{\emptyset \subset I \subseteq [n]} \left[P \cup \bigcup_{i \in I} \left\{ * \xrightarrow{a_i} * \right\}\right]$$

$$\left[P \cup \bigcup_{i=1}^{n} \left\{ * \xrightarrow{a_i} * \right\}\right] = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left[P \cup \left\{ * \xrightarrow{\sum_{i \in I} a_i} * \right\}\right].$$

§3. OPERATIONS ON MAZES

Passages $p\colon x \to y$ and $q\colon x \to y$ sharing starting and ending points, will be said to be **parallel**.

DEFINITION 7. — The maze P is called **simple**, if it contains no (pair of) parallel passages. ◇

By means of the labyrinth axioms I and II, any maze can be written as the sum of simple mazes.

DEFINITION 8. — The mazes $P, Q\colon X \to Y$ are **similar**, if, for any $x \in X$ and $y \in Y$, there are as many passages $x \to y$ in P as there are in Q. ◇

EXAMPLE 8. — The following two mazes are similar:

$$\left[\begin{array}{ccc} & \overset{a}{\nearrow}\!\!\nearrow & x \\ z & b & \\ & \underset{c}{\searrow} & y \end{array}\right], \qquad \left[\begin{array}{ccc} & \overset{d}{\nearrow}\!\!\nearrow & x \\ z & e & \\ & \underset{f}{\searrow} & y \end{array}\right].$$

△

DEFINITION 9. — Let $P\colon Y \to Z$ and $Q\colon X \to Y$ be mazes. Their **functional product** is

$$P \boxdot Q = \left\{ \left[z \xleftarrow{\sum pq} x \right] \middle| z \in Z, x \in X \right\},$$

where the sum is taken over all pairs $\left[z \xleftarrow{p} y\right] \in P$ and $\left[y \xleftarrow{q} x\right] \in Q$ of composable passages. ◇

The functional product is always a simple maze.

EXAMPLE 9. — Let

$$P = \left[\begin{array}{cc} x \overset{a}{\nwarrow} & \\ & z \\ y \underset{b}{\swarrow} & \end{array}\right], \qquad Q = \left[\begin{array}{cc} & \overset{c}{\swarrow} x \\ z & \\ & \underset{d}{\nwarrow} y \end{array}\right].$$

Then the functional product $P \boxcircle Q$ coincides with the cartesian product $P \boxtimes Q$, computed earlier. The functional product

$$Q \boxcircle P = \left[\ z \xleftarrow{ca+db} z\ \right],$$

however, differs from the cartesian product

$$Q \boxtimes P = \left[\ z \underset{db}{\overset{ca}{\leftleftarrows}} z\ \right].$$

△

The following formula relates the cartesian and functional products.

Theorem 2.

$$\sum_{V \subseteq P \boxtimes Q} V = \sum_{W \subseteq P \boxcircle Q} W.$$

Proof. Consider a submaze $W \subseteq P \boxcircle Q$. It has at most one passage running between any two given elements. We then consider the set

$$E(W) = \{ V \subseteq P \boxtimes Q \mid \exists\, [x \to z] \in V \ \leftrightarrow\ \exists\, [x \to z] \in W \}.$$

Apply the first formula of Theorem 1 to each passage of $E(W)$ to show

$$\sum_{V \in E(W)} V = W.$$

This proves the theorem. □

Definition 10. — Let P be a maze, and let a be a scalar. The **functional scalar multiplication** of P by a is the maze $a \boxdot P$ obtained from P by multiplying the labels of all passages by a:

$$a \boxdot P = \left\{ \left[\ y \xleftarrow{ap} x\ \right] \,\middle|\, \left[\ y \xleftarrow{p} x\ \right] \in P \right\}.$$

◇

Naturally, one should distinguish between left and right multiplication, but we shall only employ this operation in the case of a commutative base ring.

Definition 11. — Let $P, Q \colon X \to Y$ be similar, simple mazes. Their **functional sum** is

$$P \boxplus Q = \left\{ \left[\ x \xrightarrow{p+q} y\ \right] \,\middle|\, \left[\ x \xrightarrow{p} y\ \right] \in P, \left[\ x \xrightarrow{q} y\ \right] \in Q \right\}.$$

◇

Example 10. — Plainly,

$$\left[\begin{array}{ccc} & {\scriptstyle a}\nearrow & x \\ z & & \\ & {\scriptstyle b}\searrow & y \end{array}\right] \boxplus \left[\begin{array}{ccc} & {\scriptstyle c}\nearrow & x \\ z & & \\ & {\scriptstyle d}\searrow & y \end{array}\right] = \left[\begin{array}{ccc} & {\scriptstyle a+c}\nearrow & x \\ z & & \\ & {\scriptstyle b+d}\searrow & y \end{array}\right].$$

△

Applying some induction to Theorem 1 yields the following formula for the functional sum.

Theorem 3. — *Let $P_1, \ldots, P_n$ be similar, simple mazes, and let the passages of P_i be $p_{i1}, \ldots, p_{im}$ (where it is understood that the passages $p_{1j}, \ldots, p_{nj}$ all run in parallel). Then*

$$P_1 \boxplus \cdots \boxplus P_n = \sum_K \{p_{ij} \mid (i,j) \in K\},$$

where the sum is taken over all $K \subseteq [n] \times [m]$ such that projection on the second component is onto.

These functional operations will later be shown to have very natural interpretations, which will account for the epithet "functional".

Definition 12. — Let $P\colon X \to Y$ be a maze in $\mathfrak{Laby}(\mathbf{B})$. The **dual** of P is the maze

$$P^\circ\colon Y \to X$$

in $\mathfrak{Laby}(\mathbf{B}^\circ)$, obtained by reversing all passages in P. ◇

Of course, the dual will also belong to $\mathfrak{Laby}(\mathbf{B})$, but it is more natural to let it lie in $\mathfrak{Laby}(\mathbf{B}^\circ)$. Letting $*$ denote the multiplication of $\mathbf{B}^\circ$ (the opposite multiplication of $\mathbf{B}$), and also the corresponding induced multiplication on the labyrinth category $\mathfrak{Laby}(\mathbf{B}^\circ)$, we have the equation

$$(PQ)^\circ = Q^\circ * P^\circ.$$

The dual thus provides an isomorphism of categories

$$\mathfrak{Laby}(\mathbf{B})^\circ \cong \mathfrak{Laby}(\mathbf{B}^\circ).$$

§4. The Quotient Labyrinth Categories

Everything up to this point makes sense for an arbitrary base ring. In order to construct quotient categories of $\mathfrak{Laby}$, we need the assumption that $\mathbf{B}$ be numerical.

Let $P\colon X \to Y$ be a maze. We shall consider multi-sets A supported in P (recall this notion from Chapter 2). This is intended to capture the informal notion that $A\colon X \to Y$ is the maze that results after certain passages of P have been multiplied.

When A is a maze, we let E_A denote the maze

$$E_A = \bigcup_{[p\colon x \to y] \in A} \left\{ x \xrightarrow{1} y \right\},$$

in which all passages of A have been reassigned the label 1.

While the full labyrinth category encodes arbitrary module functors, the following quotient category will be seen to encode *numerical functors* (of degree n).

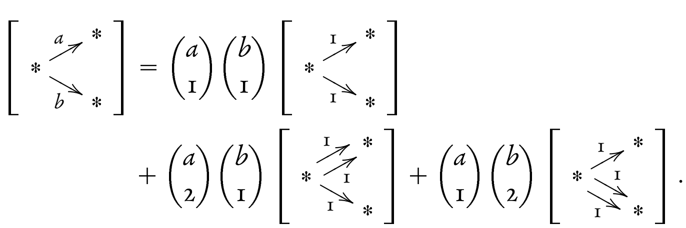
Definition 13. — The category $\mathfrak{Laby}_n$ is the quotient category obtained from $\mathfrak{Laby}$ when the following relations are imposed:

III.
$$P = 0,$$
whenever P contains more than n passages.

IV.
$$P = \sum_{\substack{\#A = P \\ |A| \leqslant n}} \prod_{p \in P} \binom{\bar{p}}{\deg_A p} E_A,$$
for all mazes P.

◇

Example 11. — An instance of the fourth axiom is the following:



△

It may be proved that, over the integers, Axiom IV is redundant, as it follows from Axiom III.

Definition 14. — A maze of which all passages carry the label 1 is called a **pure** maze. ◇

The pure mazes evidently generate the category $\mathfrak{Laby}_n$. Not as evident, but equally true, is that they actually consitute a basis. This will be proved presently.

We shall have reason to impose upon the labyrinth category yet another axiom. This will be for encoding *quasi-homogeneous* functors (of degree n).

Definition 15. — The category $\mathfrak{Laby}^n$ is the quotient category

$$\mathfrak{Laby}_n/(L \cap \mathfrak{Laby}_n),$$

where L is the ideal of $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}_n$ generated by the following elements:

V.
$$a^n P - a \boxdot P,$$
for any pure maze P and $a \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$.

◇

Using the fourth axiom, this fifth axiom can be equivalently written

$$a^n P = \sum_{\substack{\#A = P \\ |A| \leqslant n}} \binom{a}{A} A.$$

Implicit in the fifth axiom is the existence of an inclusion of categories

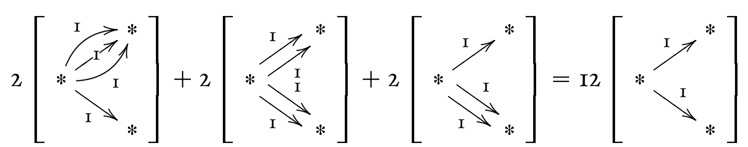
$$\mathfrak{Laby}_n \subseteq \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}_n,$$

which is not quite justified at this point, unless we take it on faith (which we do) that the category $\mathfrak{Laby}_n$ is free, and hence torsion-free.

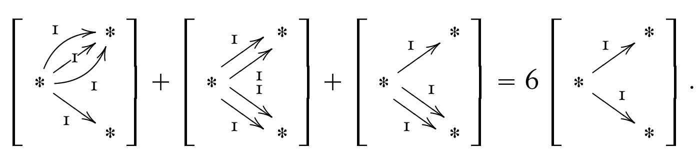
Example 12. — The fifth axiom considers the ideal generated in $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}_n$, rather than just $\mathfrak{Laby}_n$. The slightly sharper requirement will first make a difference in degree 4. Consider, for example, ${}_{\mathbf{Z}}\mathfrak{Laby}_4$. Dividing out by the ideal *generated in* $\mathfrak{Laby}_4$ by elements of the form

$$a^4 P - a \boxdot P$$

makes it possible to prove



whereas we shall be needing a stronger statement. By allowing the full force of Axiom V, we make it possible to divide by 2, and thus establish



△

Chapter 4

MULTI-SETS VERSUS MAZES

I leave to the various futures (not to all) my garden of forking paths.

— Jorge Luis Borges, *The Garden of Forking Paths*

We propose to investigate how the multi-set and labyrinth categories are related. A functor in one direction is readily found; viz. the *Ariadne functor*

$$A_n \colon \mathfrak{Laby} \to \mathfrak{MSet}_n,$$

so called because it leads the way *out of* the labyrinth. In the case of a numerical base ring, it turns out to factorise, yielding a functor

$$A_n \colon \mathfrak{Laby} \to \mathfrak{Laby}_n \to \mathfrak{Laby}^n \to \mathfrak{MSet}_n.$$

We will later see that multi-sets encode strict polynomial functors, while mazes encode numerical ones. The Ariadne functor will then correspond to the forgetful functor from the former functor category to the latter.

Does there exist a functor in the other direction? No, at least not in general. It is, however, possible to slightly tweak the labyrinth category in some rather non-invasive way, which will enable defining a functor in the reverse direction. This is the *Theseus functor T_n*, going *into* the labyrinth.

For the purposes of this chapter, we shall modify slightly the definitions of the categories of multi-sets and mazes by *adjoining to them direct sums*, thus making them additive. This can be done in a way that is not only well-known, but universal (left adjoint to a forgetful functor, and so on); hence we omit the details. It will make no difference for linear functors out of these categories, as we shall later have occasion to consider.

One of the reasons for keeping the original definition is that, sometimes, it will be nice to know that the categories $\mathfrak{Laby}_n$ and $\mathfrak{MSet}_n$ possess finite skeleta. Not actually helpful, perhaps, but still nice to know.

Finally, we pay tribute to Dr. Dreckman and Professors Baues, Franjou, and Pirashvili, as their article [1] has provided a wonderful source of inspiration for our work. Their main object of study $\mathfrak{Sur}$, the category of sets and surjections, may rightly be called the early ancestor of the labyrinth category.

§1. The Ariadne Functor

For the duration of this section, let n be a fixed natural number.

Let P be a maze. Consider the following sum of multations:

$$A_n(P) = \sum_{\substack{\#A=P\\|A|=n}} \left(\bigodot_{[p:\,x\to y]\in A} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \sum_{\substack{\#A=P\\|A|=n}} \left(\frac{1}{\deg_A P} \prod_{[p:\,x\to y]\in A} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix} \right).$$

This will provide a functor from $\mathfrak{Laby}$ to $\mathfrak{MSet}_n$, which we now set out to prove. It is clear that $A_n(P) = 0$ if a single passage of P is labelled 0. Now to show that

$$A_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) =$$

$$A_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) + A_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) + A_n\left(P \cup \left\{ u \overset{a}{\underset{b}{\rightrightarrows}} v \right\}\right).$$

Denote

$$\mu = \bigodot_{[p:\,x\to y]\in A} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since

$$A_n\left(P \cup \left\{ u \xrightarrow{a+b} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P\\|A|=n-m}} \left(\mu \odot (a+b)^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]} \right)$$

$$A_n\left(P \cup \left\{ u \xrightarrow{a} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P\\|A|=n-m}} \left(\mu \odot a^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]} \right)$$

$$A_n\left(P \cup \left\{ u \xrightarrow{b} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{\#A=P\\|A|=n-m}} \left(\mu \odot b^m \begin{bmatrix} u \\ v \end{bmatrix}^{[m]} \right)$$

$$A_n\left(P \cup \left\{ u \overset{a}{\underset{b}{\rightrightarrows}} v \right\}\right) = \sum_{m=1}^{n} \sum_{\substack{i+j=m\\i,j\geqslant 1}} \sum_{\substack{\#A=P\\|A|=n-m}} \left(\mu \odot a^i b^j \begin{bmatrix} u \\ v \end{bmatrix}^{[i]} \begin{bmatrix} u \\ v \end{bmatrix}^{[j]} \right),$$

this relation follows from the equation

$$(x+y)^{[m]} = x^{[m]} + y^{[m]} + \sum_{\substack{i+j=m\\i,j\geqslant 1}} x^{[i]} y^{[j]}.$$

Hence A_n gives a well-defined map on the mazes of the labyrinth category. We now prove that it is in fact a functor.

Theorem 1. — *The formulæ*

$$A_n(X) = \bigoplus_{\substack{\#A=X\\|A|=n}} A$$

$$A_n(P) = \sum_{\substack{\#A=P\\|A|=n}} \left(\bigodot_{[p\colon x\to y]\in A} \bar{p}\begin{bmatrix} x\\ y\end{bmatrix}\right)$$

provide a linear functor

$$A_n\colon \mathfrak{Laby} \to \mathfrak{MSet}_n.$$

Proof. Let $P\colon Y \to Z$ and $Q\colon X \to Y$ be two mazes. We wish to show that

$$\begin{aligned}
A_n(P)\circ A_n(Q) &= \sum_{\substack{\#A=P\\|A|=n}} \left(\bigodot_{[p\colon y\to z]\in A} \bar{p}\begin{bmatrix} y\\ z\end{bmatrix}\right) \circ \sum_{\substack{\#B=Q\\|B|=n}} \left(\bigodot_{[q\colon x\to y]\in B} \bar{q}\begin{bmatrix} x\\ y\end{bmatrix}\right)\\
&= \sum_{\substack{\#C\sqsubseteq P\boxtimes Q\\|C|=n}} \left(\bigodot_{[r\colon x\to z]\in C} \bar{r}\begin{bmatrix} x\\ z\end{bmatrix}\right)\\
&= \sum_{R\sqsubseteq P\boxtimes Q} \sum_{\substack{\#C=R\\|C|=n}} \left(\bigodot_{[r\colon x\to z]\in C} \bar{r}\begin{bmatrix} x\\ z\end{bmatrix}\right)\\
&= A_n\left(\sum_{R\sqsubseteq P\boxtimes Q} R\right) = A_n(PQ).
\end{aligned}$$

The dubious step here is the third one, which follows from the equation

$$\left(\sum_{[p\colon y\to z]\in P} \bar{p}\begin{bmatrix} y\\ z\end{bmatrix}\right)^{[n]} \circ \left(\sum_{[q\colon x\to y]\in Q} \bar{q}\begin{bmatrix} x\\ y\end{bmatrix}\right)^{[n]} = \left(\sum_{\substack{[p\colon y\to z]\in P\\ [q\colon x\to y]\in Q}} \overline{pq}\begin{bmatrix} x\\ z\end{bmatrix}\right)^{[n]},$$

after noting that restricting attention to monomials $\bar{p}^A\bar{q}^B$ with $\#A = P$ and $\#B = Q$ in the left-hand side corresponds to considering $(\overline{pq})^C$ with $C \subseteq P\boxtimes Q$ in the right-hand side. □

Definition 1. — This functor is called the ***n*th Ariadne functor**. ◇

Theorem 2. — *The Ariadne functor factors through the quotient category* $\mathfrak{Laby}_n$*, producing a functor*

$$A_n\colon \mathfrak{Laby}_n \to \mathfrak{MSet}_n.$$

Proof. It is clear that $A_n(P) = 0$ when $|P| > n$. We now prove that A_n respects the relation

$$P = \sum_{\substack{\#A=P \\ |A|\leqslant n}} \prod_{p\in P} \binom{\bar{p}}{\deg_A p} E_A.$$

Calculate:

$$\begin{aligned} A_n &\left(\sum_{\substack{\#A=P \\ |A|\leqslant n}} \prod_{p\in P} \binom{\bar{p}}{\deg_A p} E_A \right) \\ &= \sum_{\substack{\#A=P \\ |A|\leqslant n}} \prod_{p\in P} \binom{\bar{p}}{\deg_A p} \sum_{\substack{\#S=A \\ |S|=n}} \left(\bigodot_{[p:\, x\to y]\in S} \begin{bmatrix} x \\ y \end{bmatrix} \right) \\ &= \sum_{\substack{\#A=P \\ |A|\leqslant n}} \sum_{\substack{\#S=A \\ |S|=n}} \left(\frac{\deg_P S}{\deg_A S} \prod_{p\in P} \binom{\bar{p}}{\deg_A p} \right) \left(\frac{1}{\deg_P S} \prod_{[p:\, x\to y]\in S} \begin{bmatrix} x \\ y \end{bmatrix} \right) \end{aligned}$$

The quantity

$$\prod_{p\in P} \binom{\bar{p}}{\deg_A p}$$

counts the number of ways to colour the passages of A (or E_A) in distinct colours, with a selection of $\bar{p}$ colours available for (copies of) passage p. There are exactly

$$\frac{\deg_P S}{\deg_A S}$$

distinct ways in which the passages of S can inherit the colours from A. Hence the coefficient of

$$\frac{1}{\deg_P S} \prod_{[p:\, x\to y]\in S} \begin{bmatrix} x \\ y \end{bmatrix}$$

in the horrendous sum above is the number of ways to colour the passages of S arbitrarily, with $\bar{p}$ colours to choose from for passage p. Consequently, the sum equals

$$\begin{aligned} \sum_{\substack{\#S=P \\ |S|=n}} \left(\prod_{p\in S} \bar{p} \right) \left(\frac{1}{\deg_P S} \prod_{[p:\, x\to y]\in S} \begin{bmatrix} x \\ y \end{bmatrix} \right) &= \sum_{\substack{\#S=P \\ |S|=n}} \left(\bigodot_{[p:\, x\to y]\in S} \bar{p} \begin{bmatrix} x \\ y \end{bmatrix} \right) \\ &= A_n(P). \end{aligned}$$

□

Theorem 3. — *The Ariadne functor factors through the quotient category* $\mathfrak{Laby}^n$, *producing a functor*

$$A_n\colon \mathfrak{Laby}^n \to \mathfrak{MSet}_n.$$

Proof. When P is pure,

$$\begin{aligned} A_n(a \boxdot P) &= \sum_{\substack{\#A=P \\ |A|=n}} \left(\bigodot_{[x\to y]\in A} a \begin{bmatrix} x \\ y \end{bmatrix} \right) \\ &= a^n \sum_{\substack{\#A=P \\ |A|=n}} \left(\bigodot_{[x\to y]\in A} \begin{bmatrix} x \\ y \end{bmatrix} \right) = a^n A_n(P). \end{aligned}$$

□

There will be no more factorisation after this, for passing to the category $\mathfrak{Laby}^n$ has the effect of making the Ariadne functor faithful. This is easy to see from the theorem below, asserting that the pure mazes with exactly n passages generate the category $\mathfrak{Laby}^n$ over $\mathbf{Q}\otimes_{\mathbf{Z}}\mathbf{B}$. We shall even manufacture an inverse of sorts to A_n.

§2. Pure Mazes

The Ariadne functor can be used to shed important light on the internal structure of the labyrinth categories.

Theorem 4. — *The pure mazes are linearly independent in the category* $\mathfrak{Laby}$.

Proof. Let $P_{m,j}$ denote a pure maze of cardinality m, where all $P_{m,j}$ are assumed distinct. Suppose we have a relation

$$\sum_j a_{n,j} P_{n,j} + \sum_j a_{n+1,j} P_{n+1,j} + \cdots = 0$$

in $\mathfrak{Laby}(X, Y)$, for some $a_{m,j} \in \mathbf{B}$. The nth Ariadne functor will kill all mazes of cardinality greater than n, and the end result after application will be

$$\sum_j a_{n,j} A_n(P_{n,j}) = 0.$$

Since the $P_{n,j}$ are distinct pure mazes, the $A_n(P_{n,j})$ will all denote distinct multations. Hence all $a_{n,j} = 0$. The claim now follows by induction. □

Theorem 5. — *The pure mazes constitute a basis for the category* $\mathfrak{Laby}_n$.

Proof. The proof for linear independence goes through exactly as before, because the Ariadne functor factors through $\mathfrak{Laby}_n$. From the defining equations for $\mathfrak{Laby}_n$, we see that any maze will reduce to pure ones. □

Theorem 6. — *The pure mazes with exactly n passages are linearly independent in the category $\mathfrak{Laby}^n$, and generate the category over $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$.*

Proof. Linear independence goes through as before. The defining equation for $\mathfrak{Laby}^n$ can be written

$$a^n P = \binom{a}{P} P + \sum_{\substack{\#A = P \\ |P| < |A| \leqslant n}} \binom{a}{A} A,$$

and since $a^n \neq \binom{a}{P}$ if $|P| < n$, such a P may be expressed in terms of mazes with more passages, provided division by integers is permissible. □

As can be seen from the proof, it is in fact not necessary to have available inverses for every integer. It suffices to invert the positive integers up to n.

Theorem 7. — *The following categories are isomorphic:*

$$\begin{aligned} {}_{\mathbf{B}}\mathfrak{Laby}_n &\cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n \\ {}_{\mathbf{B}}\mathfrak{Laby}^n &\cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n. \end{aligned}$$

Proof. The first equation is an immediate corollary of the pure mazes constituting a basis for $\mathfrak{Laby}_n$. Let us prove the second one. By the theorem above, $\mathfrak{Laby}^n$ is torsion-free. From the definition of $\mathfrak{Laby}^n$,

$$\mathbf{Q} \otimes_{\mathbf{Z}} {}_{\mathbf{B}}\mathfrak{Laby}^n \cong \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n.$$

${}_{\mathbf{B}}\mathfrak{Laby}^n$ will then embed in the former of those, and $\mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n$ in the latter. □

Because the Ariadne functor embeds ${}_{\mathbf{Z}}\mathfrak{Laby}^n$ in ${}_{\mathbf{Z}}\mathfrak{MSet}_n$, which is free, it follows that ${}_{\mathbf{Z}}\mathfrak{Laby}^n$ is free as well. By the isomorphism above, ${}_{\mathbf{B}}\mathfrak{Laby}^n$ will then be free for arbitrary $\mathbf{B}$, though it does not, in general, seem to possess a preferred basis.

§3. The Theseus Functor

Example 1. — Let

$$P = \begin{bmatrix} 1 \overset{1}{\underset{1}{\rightrightarrows}} 1 \\ 2 \xrightarrow[1]{} 2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 \xrightarrow{1} 1 \\ 2 \overset{1}{\underset{1}{\rightrightarrows}} 2 \end{bmatrix}.$$

In $\mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{Laby}^3$, as may be checked, the following equation holds:

$$2I_{\{1,2\}} = \begin{bmatrix} 1 \xrightarrow{1} 1 \\ 2 \xrightarrow[1]{} 2 \end{bmatrix} = P + Q.$$

Also, evidently $PQ = QP = 0$; hence the mazes $\frac{1}{2}P$ and $\frac{1}{2}Q$ would form a direct sum system, splitting the set $\{1, 2\}$ in two. But no such *sets* exist. △

It is the purpose of this section to adjoin objects to the category $\mathbf{Q}\otimes_{\mathbf{Z}}\mathfrak{Lady}^n$ so that some sets (like $\{1,2\}$ in the example) actually split as direct sums. This will enable us to define an inverse to the Ariadne functor.

Lemma 1. — *Let M be a torsion-free module, and let*

$$p(x) \in M \otimes \mathbf{B}[x].$$

Then $p = 0$ iff $p(a) = 0$ for all integers a.

Proof. Induction on the degree of p. □

Lemma 2. — *In the category $\mathbf{Q}\otimes_{\mathbf{Z}}\mathfrak{Lady}^n$, the following equation holds, for any set X:*

$$I_X = \sum_{\substack{\#S=I_X\\ |S|=n}} \frac{1}{\deg S} S.$$

Proof. Use the previous lemma. Identify the coefficient of a^n in the defining equation for $\mathfrak{Lady}^n$:

$$a^n I_X = \sum_{\substack{\#S=P\\ |S|\leqslant n}} \binom{a}{S} S.$$

□

Evidently, the mazes $\frac{1}{\deg S}S$, with S supported in I_X, satisfy

$$\frac{1}{\deg S}E_S \circ \frac{1}{\deg T}E_T = \begin{cases} 0 & \text{if } S \neq T,\\ \frac{1}{\deg S}S & \text{if } S = T. \end{cases}$$

Hence they form a direct sum system, though the *objects* themselves of the system do not exist. There is a simple remedy for this: adjoin to the category $\mathbf{Q}\otimes_{\mathbf{Z}}\mathfrak{Lady}^n$ an object

$$\operatorname{Im}\frac{1}{\deg S}S$$

for each such S.

However, we first observe that the category $\mathbf{Q}\otimes_{\mathbf{Z}}\mathfrak{Lady}^n$ is not the minimal localisation of $\mathfrak{Lady}^n$, for which this procedure makes sense. It may be verified that the mazes

$$\frac{1}{\deg A}A,$$

where $|A| = n$, and $\#A = P$ for some pure and simple maze P, form a basis for a subcategory of $\mathbf{Q}\otimes\mathfrak{Lady}^n$ which contains $\mathfrak{Lady}^n$.

To this category we adjoin an object

$$\operatorname{Im}\frac{1}{\deg S}S$$

for each multi-set S supported in some I_X, and denote the resulting category by

$$\mathfrak{Laby}^{\oplus n}.$$

The set X will now split into components:

$$X = \bigoplus_{\substack{\#S = I_X \\ |S| = n}} \operatorname{Im} \frac{1}{\deg S} S.$$

Example 2. — The category $\mathfrak{Laby}^{\oplus 3}$ contains exactly five (isomorphism classes of) objects. The set $\{1, 2\}$ has been split up in two:

$$\{1, 2\} = \operatorname{Im} \frac{1}{2} P \oplus \operatorname{Im} \frac{1}{2} Q.$$

△

Definition 2. — The n**th Theseus functor**

$$T_n \colon \mathfrak{MSet}_n \to \mathfrak{Laby}^{\oplus n}$$

is given by the following formulæ:

$$A \mapsto \operatorname{Im} \frac{1}{\deg A} \bigcup_{a \in A} \left\{ a \xrightarrow{1} a \right\}$$

$$\prod_{k=1}^{n} \begin{bmatrix} a_k \\ b_k \end{bmatrix} \mapsto \bigcup_{k=1}^{n} \left\{ a_k \xrightarrow{1} b_k \right\}.$$

◇

It should be rather clear that this is indeed a (linear) functor, as composition in both categories is effectuated by "summing over all possibilities". An arbitrary multation transforms as

$$\mu \mapsto \frac{1}{\deg \mu} \bigcup_{(a,b) \in \mu} \left\{ a \xrightarrow{1} b \right\}.$$

Theorem 8. — *There is an isomorphism of categories:*

$$\mathfrak{Laby}^{\oplus n} \underset{T_n}{\overset{A_n}{\rightleftarrows}} \mathfrak{MSet}_n$$

Proof. We first show that the Ariadne functor actually factors through the category $\mathfrak{Laby}^{\oplus n}$. Let A be a multi-set with $|A| = n$ and $\#A = P$ a pure maze. Then

$$A_n(A) = \prod_{\left[x \xrightarrow{1} y\right] \in A} \begin{bmatrix} x \\ y \end{bmatrix} = \deg A \cdot \bigodot_{\left[x \xrightarrow{1} y\right] \in A} \begin{bmatrix} x \\ y \end{bmatrix},$$

and hence we may extend A_n by letting

$$\frac{1}{\deg A} A \mapsto \bigodot_{\left[x \xrightarrow{1} y\right] \in A} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Moreover, A_n maps the "virtual" biproduct system

$$I_X = \sum_{\substack{\#B = X \\ |B| = n}} \frac{1}{\deg B} \bigcup_{b \in B} \left\{ b \xrightarrow{1} b \right\}$$

in $\mathfrak{Laby}^n$ onto the "real" biproduct system

$$\sum_{\substack{\#B = X \\ |B| = n}} \iota_B$$

in $\mathfrak{MSet}_n$, and we may consequently extend A_n to $\mathfrak{Laby}^{\oplus n}$ by defining

$$A_n \left(\operatorname{Im} \frac{1}{\deg B} \bigcup_{b \in B} \left\{ b \xrightarrow{1} b \right\} \right) = B.$$

It is now easy to see that T_n and A_n are inverse to each other. □

§4. The Category of Correspondences

For reference, we devote this last section to investigating the connection between the labyrinth category and the category of surjections explored by the quartet Baues, Dreckmann, Franjou & Pirashvili in [1].

Let C be a category possessing *weak pullbacks*; that is, a finite number of universal ways to complete an incomplete pullback square. For two objects $X, Y \in C$, a **correspondence**[1] from X to Y is a diagram

$$Y \leftarrow U \rightarrow X,$$

to be read from right to left. Suppose there exists a commutative diagram of the following shape, with the middle column an isomorphism:

$$\begin{array}{ccccc} Y & \longleftarrow & U & \longrightarrow & X \\ \| & & \updownarrow & & \| \\ Y & \longleftarrow & V & \longrightarrow & X \end{array}$$

We then identify the two correspondences

$$\left[Y \leftarrow U \rightarrow X \right] = \left[Y \leftarrow V \rightarrow X \right].$$

[1] Professor Pirashvili and his colleagues employ the word *flèche*, a rather unfortunate choice, as this may also denote a *single* arrow.

Construct a category $\hat{C}$ in the following way. Its objects are those of C. Its arrows are formal sums of correspondences of C (identified under the just described equivalence relation), in the free abelian group they generate. The composition of correspondences is defined as:

$$\Big[\, Z \leftarrow V \rightarrow Y \,\Big] \circ \big[\, Y \leftarrow U \rightarrow X \,\big] = \sum \Big[\, Z \leftarrow W \rightarrow X \,\Big],$$

where the sum is taken over all weak pullbacks:

$$\begin{array}{ccccc} & & W & & \\ & \swarrow & & \searrow & \\ & V & & U & \\ \swarrow & & \searrow\ \swarrow & & \searrow \\ Z & & Y & & X \end{array}$$

(If C does indeed possess pullbacks, there will be no need to bother to these formal sums, and composition can be defined simply as the pullback.) The category $\hat{C}$ is pre-additive, and is called the **category of correspondences** in C.

It will now be observed that the category $\mathfrak{Sur}$ of finite sets and *surjections* possesses weak pullbacks. Namely, the condition

$$W \sqsubseteq A \times_P B = \{(a,b) \in A \times B \mid \alpha(a) = \beta(b)\}$$

(that the projections of W on A and B both be onto) is necessary and sufficient for the following to be a weak pullback square:

$$\begin{array}{ccc} W & \cdots\rightarrow & B \\ \vdots & & \downarrow \beta \\ A & \xrightarrow[\alpha]{} & P \end{array}$$

We call $A \times_P B$ (the pullback in $\mathfrak{Set}$) the **principal pullback**.

It is thus possible to build the category $\widehat{\mathfrak{Sur}}$ of surjection correspondences. We form a quotient category $\widehat{\mathfrak{Sur}}_n$ by forcing

$$\big[\, Y \leftarrow U \rightarrow X \,\big] = 0$$

whenever $|U| > n$. This category, so vividly dissected in [1], turns out to be an old acquaintance of ours.

Theorem 9.

$$\widehat{\mathfrak{Sur}}_n \cong {}_{\mathrm{Z}}\mathfrak{Laby}_n.$$

Proof. Define a functor

$$\Xi\colon \widehat{\mathfrak{Sur}}_n \to {}_{\mathrm{Z}}\mathfrak{Laby}_n$$

by $\Xi(X) = X$ for any finite set X. The correspondence

$$\varphi = \left[\; Y \xleftarrow{\varphi_*} U \xrightarrow{\varphi^*} X \;\right]$$

in $\widehat{\mathfrak{Sur}}_n$ will map to the pure maze $X \to Y$, of which the passages $x \to y$ number exactly

$$\left|(\varphi^*, \varphi_*)^{-1}(x, y)\right|$$

(the cardinality of the fibre above $(x, y) \in X \times Y$).

We now explain why this gives a functor. Let

$$\psi = \left[\; Z \xleftarrow{\psi_*} V \xrightarrow{\psi^*} Y \;\right], \qquad \varphi = \left[\; Y \xleftarrow{\varphi_*} U \xrightarrow{\varphi^*} X \;\right]$$

be two wedges, corresponding to the mazes $Q\colon Y \to Z$ and $P\colon X \to Y$, respectively. The number of passages $x \to y$ in P equals

$$|(\varphi^*, \varphi_*)^{-1}(x, y)|,$$

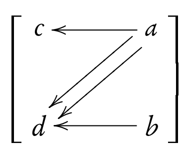
and the number of passages $y \to z$ in Q equals

$$|(\psi^*, \psi_*)^{-1}(y, z)|.$$

The functoriality of Ξ follows from the observation that $U \times_Y V$ may be naturally identified with $Q \boxtimes P$, and subsets $W \sqsubseteq U \times_Y V$ with submazes $R \subseteq Q \boxtimes P$.

Since the pure mazes form a basis, this functor is invertible. □

Example 3. — An example will clarify the idea. The pure maze



should be thought of as a correspondence

$$\{c, d\} \longleftarrow \{(c, a), (d, a)_1, (d, a)_2, (d, b)\} \longrightarrow \{a, b\}\ ,$$

where the maps are the obvious projections. △

Curiously enough, even though

$$\widehat{\mathfrak{Sur}}_n \cong {}_{\mathbf{Z}}\mathfrak{Laby}_n$$

for all n, the categories $\widehat{\mathfrak{Sur}}$ and ${}_{\mathbf{Z}}\mathfrak{Laby}$ are themselves not isomorphic. This stems from the fact that ${}_{\mathbf{Z}}\mathfrak{Laby}$ encodes functors from the category of free abelian groups, while $\widehat{\mathfrak{Sur}}$ was built to encode functors from the category of free commutative *monoids*. These functor categories are not originally equivalent, but they will be, once polynomiality is assumed.

Chapter 5

POLYNOMIAL MAPS

> And God said unto the animals: "Be fruitful and multiply."
>
> But the snake answered: "How could I? I am an adder!"[1]

Before embarking on the study of module *functors*, it seems worthwhile to gain a reasonable understanding of module *maps*. The words of Professor Roby [19] may serve as an opening vignette:

> L'idée qui nous fit entreprendre le travail que nous présentons ici a son origine dans le souci d'élargir l'arsenal des applications dont on dispose communément en algèbre. L'analyse classique dans les espaces $\mathbf{R}^n$ ou $\mathbf{C}^n$ utilise une grande diversité de fonctions ou d'applications (continues, différentiables, analytiques, holomorphes, etc.) ; il en est de même, par exemple, dans l'étude des variétés pour lesquelles $\mathbf{R}^n$ et $\mathbf{C}^n$ servent de modèle local. Mais dès qu'on considère des espaces vectoriels plus généraux, et pis encore s'il s'agit de modules, on ne dispose plus guère que d'applications et de formes linéaires.

To us, a *map* of modules shall always denote an *arbitrary* map — in general non-linear. On those rare occasions when a linear map is actually under consideration, we shall try to proclaim it a "homomorphism" quite loudly.

All modules will of course be **B**-modules, and our module maps will be generalisations of ordinary polynomial maps (as defined on fields). The problem is then, naturally, how to form "polynomials" on modules, where no multiplication is at hand.

One possible approach is to study maps that are (let us phrase it carefully) *polynomial-like*[2], in the sense that they satisfy certain equations somehow thought to characterise polynomials. For example, a map $\varphi\colon \mathbf{C}^k \to \mathbf{C}^k$ is *quadratic* (polynomial of degree 2) iff, as may be checked, it satisfies the two equations

$$\begin{aligned}\varphi(x+y+z) - \varphi(x+y) - \varphi(y+z) - \varphi(z+x) \\ +\varphi(x) + \varphi(y) + \varphi(z) - \varphi(0) = 0\end{aligned}$$

[1] In some retellings of this myth, it is said that God constructed a wooden table for the snakes to crawl upon, since even adders can multiply on a log table. God is not assumed to be familiar with tensor products.

[2] På swensko: *polynom-agtiga*.

and

$$\varphi(ax) = \binom{a}{0}\varphi(0) + \binom{a}{1}\big(\varphi(x) - \varphi(0)\big) + \binom{a}{2}\big(\varphi(2x) - 2\varphi(x) + \varphi(0)\big),$$

for any $x, y, z \in \mathbf{C}^k$ and $a \in \mathbf{C}$. We observe the binomial coefficients, which explains why numerical rings will inevitably enter the theory.

It is perhaps surprising that the importance of the second condition was not discovered until now. (We presume we are the very first to explore its consequences.) The explanation for this could possibly be that (1) numerical rings are not very much in use, and (2) people have only been interested in maps of abelian groups, and never cared for the case of a general module. But we digress.

The quantity

$$\begin{aligned}\varphi(x \diamond y \diamond z) = \varphi(x + y + z) &- \varphi(x + y) - \varphi(y + z) - \varphi(z + x)\\ &+ \varphi(x) + \varphi(y) + \varphi(z) - \varphi(0)\end{aligned}$$

is called the *(second) deviation* of φ. From the formula it should be clear how to form higher-order deviations. In general, a map $\varphi\colon M \to N$ of modules is said to be *numerical* of degree n if it satisfies the two equations

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

and

$$\varphi(ax) = \binom{a}{0}\varphi(\diamond) + \binom{a}{1}\varphi(\diamond x) + \binom{a}{2}\varphi(x \diamond x) + \cdots,$$

for any $a \in \mathbf{B}$ and $x_1, \dots, x_{n+1}, x \in M$.

Apart from requiring a numerical base ring, such an approach has the seeming disadvantage of producing maps that merely *behave* as polynomials, without actually *being* polynomials. On the other hand, Roby's *strict polynomial maps* (he called them himself *lois polynomes*) actually look like polynomials. He develops their theory in a journal article [19] spanning a hundred and thirty pages, from which we quote:

> [...] la généralisation en vue devrait conduire à associer, à «quelque chose» qui s'écrirait : $x_1T_1 + \cdots x_pT_p$, une «autre chose» qui s'écrirait
>
> $$\sum_{i=1}^{q} y_i Q_i(T_1, \dots, T_p),$$
>
> les Q_i étant cette fois des polynomes. Manifestement s'introduisent ici les modules produits tensoriels [...].

The definition Roby hints at is the following. A *strict polynomial map* $\varphi\colon M \to N$ is a natural transformation

$$\varphi\colon M \otimes - \to N \otimes -$$

of functors $\mathfrak{CAlg} \to \mathfrak{Set}$. This may look even less like a polynomial than the numerical maps defined above, but because of naturality, it is easily seen that the following holds: *for any $u_1, \dots, u_k \in M$ there exist unique elements $v_X \in N$ (only finitely many of which are non-zero), X varying over multi-sets supported in $[k]$, such that*

$$\varphi(u_1 \otimes x_1 + \cdots + u_k \otimes x_k) = \sum_X v_X \otimes x^X,$$

for all x_j in all algebras.

It turns out that a similar property holds for numerical maps, namely: *for any $u_1, \dots, u_k \in M$ there exist unique elements $v_X \in N$ (only finitely many of which are non-zero), X varying over multi-sets supported in $[k]$, such that*

$$\varphi(x_1 u_1 + \cdots + x_k u_k) = \sum_X \binom{x}{X} \otimes v_X,$$

for all x_j in all numerical algebras..

A remarkable fact then ensues, namely, that the numerical maps admit a description similar in spirit to Roby's framework. The resulting theorem is that the map $\varphi\colon M \to N$ is numerical (essentially) iff it can be extended to a natural transformation

$$\varphi\colon M \otimes - \to N \otimes -$$

of functors $\mathfrak{NAlg} \to \mathfrak{Set}$! This provides a beautiful and unexpected unification of the two notions.

§1. Polynomiality

Let us begin by making an extremely general discussion of *polynomiality*, and then identify the two notions we will actually make use of.

Let $D \subseteq \mathfrak{Mod}$ be a *finitary algebraic category*, by which is simply meant an equational class in the sense of universal algebra. Since D is a subcategory of $\mathfrak{Mod}$, the objects of D are first of all **B**-modules, possibly equipped with some extra structure.

For a set of variables V, let

$$\langle V \rangle_D$$

denote the free algebra on V in D. That the free algebra exists is a basic fact of universal algebra; see for example [4].

Definition 1. — Let M be a module, not necessarily in D. An element of

$$M \otimes \langle x_1, \dots, x_k \rangle_D$$

is called a D-**polynomial** over M in the variables $x_1, \dots, x_k$.

A **linear form** over M in these variables is a polynomial of the form

$$\sum u_j \otimes x_j,$$

for some $u_j \in M$. ◇

Theorem 1: Ekedahl's Esoteric Polynomiality Principle. — *Let two modules M and N be given, and a family of maps*

$$\varphi_A : M \otimes A \to N \otimes A, \quad A \in D.$$

The following statements are equivalent:

A. *For every D-polynomial $p(x) = p(x_1, \ldots, x_k)$ over M, there is a unique D-polynomial $q(x) = q(x_1, \ldots, x_k)$ over N, such that for all $A \in D$ and all $a_j \in A$,*

$$\varphi_A(p(a)) = q(a).$$

B. *For every linear form $l(x)$ over M, there is a unique D-polynomial $q(x)$ over N, such that for all $A \in D$ and all $a_j \in A$,*

$$\varphi_A\,(l(a)) = q(a).$$

C. *The map*

$$\varphi\colon M \otimes - \to N \otimes -$$

is a natural transformation of functors $D \to \mathfrak{Set}$.

Proof. It is of course trivial that A implies B. Suppose statement B holds, and consider a homomorphism $\chi\colon A \to B$, along with finitely many elements $u_j \in M$. Define

$$l(x) = \sum u_j \otimes x_j,$$

and find the unique D-polynomial q satisfying B. Then, for any $a_j \in A$, there is a commutative diagram of the following form, proving that φ is natural:

$$\begin{array}{ccc} M \otimes A & \xrightarrow{\varphi_A} & N \otimes A \\ {\scriptstyle 1\otimes\chi}\downarrow & & \downarrow{\scriptstyle 1\otimes\chi} \\ M \otimes B & \xrightarrow[\varphi_B]{} & N \otimes B \end{array} \qquad \begin{array}{ccc} \sum u_j \otimes a_j & \longrightarrow & q(a) \\ \downarrow & & \downarrow \\ \sum u_j \otimes \chi(a_j) & \longrightarrow & q(\chi(a)) \end{array}$$

Thus, condition C holds.

Finally, suppose φ natural. We shall prove condition A. Given a D-polynomial

$$p(x) \in M \otimes \langle x_1, \ldots, x_k \rangle_D\,,$$

define

$$q(x) = \varphi_{\langle x_1,\ldots,x_k \rangle_D}\,(p(x))\,.$$

For any $A \in D$ and $a_j \in A$, define the homomorphism

$$\chi\colon\ \langle x_1, \ldots, x_k \rangle_D \to A, \quad x_j \mapsto a_j.$$

Then since φ is natural, the following diagram commutes:

$$\begin{array}{ccccccc} M\otimes\langle x_1,\dots,x_k\rangle & \xrightarrow{\varphi_{\langle x_1,\dots,x_k\rangle}} & N\otimes\langle x_1,\dots,x_k\rangle & \qquad & p(x) & \longrightarrow & q(x) \\ \downarrow{\scriptstyle 1\otimes\chi} & & \downarrow{\scriptstyle 1\otimes\chi} & & \downarrow & & \downarrow \\ M\otimes A & \xrightarrow[\varphi_A]{} & N\otimes A & & p(a) & \dashrightarrow & q(a) \end{array}$$

The uniqueness of q is evident, which proves A. □

Definition 2. — When the conditions of the theorem are fulfilled, we call φ a D-**polynomial map** from M to N. ◇

According to part B of the theorem, φ_A maps

$$\sum u_j\otimes a_j \mapsto q(a),$$

for some (unique) D-polynomial q. This observation means that the Polynomiality Principle, in naïve language, amounts to the following. *If we want the coefficients a_j (in some algebra) of the module elements u_j to transform according to certain operations, the correct setting is the category of algebras using these same operations.*

Example 1. — A $\mathfrak{Mod}$-*polynomial map* $\varphi\colon M\to N$ is just a linear transformation $M\to N$. This is because, by B above, $\varphi_{\mathbf{B}}$ will map $\sum u_j\otimes r_j$ to $\sum v_j\otimes r_j$ for all $r_j\in\mathbf{B}$, and such a map is easily seen to be linear. Conversely, any module homomorphism induces a natural transformation $M\otimes - \to N\otimes -$. △

Example 2. — Let S be a $\mathbf{B}$-algebra; then ${}_S\mathfrak{Mod}\subseteq\mathfrak{Mod}$. An ${}_S\mathfrak{Mod}$-*polynomial map* $M\to N$ is a transformation

$$M\otimes A\to N\otimes A$$

which is natural in the S-module A. This is the same as a natural transformation

$$(M\otimes S)\otimes_S - \to (N\otimes S)\otimes_S -,$$

which is an ${}_S\mathfrak{Mod}$-polynomial map $M\otimes S\to N\otimes S$; or, as we noted in the previous example, an S-linear map from $M\otimes S$ to $N\otimes S$. △

The two examples below will be the important ones.

Example 3. — A $\mathfrak{CAlg}$-*polynomial map* $M\to N$ is a *strict polynomial map*, or *polynomial law*, in the sense of [19]. Here condition B reads as follows. *For every linear form $\sum u_j\otimes x_j$ over M, there is a unique (ordinary) polynomial $\sum v_X\otimes x^X$ over N (X ranging over multi-sets), such that for all algebras A and all $a_j\in A$,*

$$\varphi_A\left(\sum u_j\otimes a_j\right)=\sum v_X\otimes a^X.$$

Intuitively, the coefficients of the elements u_j "transform as ordinary polynomials". △

Example 4. — Suppose now that **B** is numerical, and consider the category $\mathfrak{NAlg}$ of numerical algebras over **B**. An $\mathfrak{NAlg}$-*polynomial map* $M \to N$ is what will be called a *numerical map*. Condition B now reads as follows. *For every linear form* $\sum u_j \otimes x_j$ *over* M, *there is a unique numerical polynomial* $\sum v_X \otimes \binom{x}{X}$ *over* N *(*X *ranging over multi-sets), such that for all numerical algebras* A *and all* $a_j \in A$,

$$\varphi_A\left(\sum u_j \otimes a_j\right) = \sum v_X \otimes \binom{a}{X}.$$

Intuitively, the coefficients of the elements u_j "transform as numerical (binomial) polynomials". △

§2. Polynomial Maps

For historical reasons, we shall begin the exposition by defining the notion of *polynomial maps*. Over general modules, this is much too weak a notion to be useful. For example, it cannot be incorporated into Roby's framework, which is one indication it is faulty.

Presumably, it was Eilenberg and Mac Lane, who first studied non-additive maps of *abelian groups*, introducing in [6] the so-called *deviations* of a map.

Definition 3. — Let $\varphi\colon M \to N$ be a map of modules. The nth **deviation** of φ is the map

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \varphi\left(\sum_{i \in I} x_i\right)$$

of $n+1$ variables. ◇

Let us, for clarity, point out that the diamond sign itself does not work as an operator; the entity $x \diamond y$ does not possess a life of its own, and cannot exist outside the scope of an argument of a map.

It is an immediate consequence of the definition that

$$\varphi(x_1 + \cdots + x_{n+1}) = \sum_{I \subseteq [n+1]} \varphi\left(\Diamond_{i \in I} x_i\right).$$

Loosely speaking, the nth deviation measures how much φ deviates from being polynomial of degree n. We have for example

$$\varphi(x \diamond y) = \varphi(x+y) - \varphi(x) - \varphi(y) + \varphi(0),$$
$$\varphi(\diamond x) = \varphi(x) - \varphi(0),$$

and, of course,

$$\varphi(\diamond) = \varphi(0).$$

We abbreviate

$$\varphi\left(\Diamond_n x\right) = \varphi(\underbrace{x \diamond \cdots \diamond x}_{n}).$$

Definition 4. — The map $\varphi\colon M \to N$ is **polynomial** of degree n if its nth deviation vanishes:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

for any $x_1, \ldots, x_{n+1} \in M$. ◇

This definition of polynomiality is the classical one for abelian groups.[3] While this notion remains valid for arbitrary modules, it is clearly a poor one, as it does not take the scalar multiplication into account. Recall that an extra condition

$$\varphi(rx) = r\varphi(x)$$

need be imposed on a group homomorphism to make it a module homomorphism (but that this is automatic when the base ring is **Z**!). In the next section, we shall see what this equation generalises to.

§3. Numerical Maps

The base ring **B** of scalars will now be assumed numerical.

Definition 5. — The map $\varphi\colon M \to N$ is **numerical** of degree (at most) n if it satisfies the following two equations:

$$\begin{aligned} \varphi(x_1 \diamond \cdots \diamond x_{n+1}) &= 0, \qquad x_1, \ldots, x_{n+1} \in M \\ \varphi(rx) &= \sum_{k=0}^{n} \binom{r}{k} \varphi\left(\mathop{\diamond}_{k} x\right), \qquad r \in \mathbf{B},\ x \in M. \end{aligned}$$

◇

Observe that, when we speak of maps of degree n, we always mean degree *n or less*.

It is easy to prove that, over the integers, the second equation above is automatic, so that the concepts of polynomial and numerical map coincide.

Example 5. — φ is of degree 0 iff it is constant, for when $n = 0$ the above equations read:

$$\begin{aligned} \varphi(x_1) - \varphi(0) = \varphi(\diamond x_1) &= 0, \\ \varphi(rx) = \binom{r}{0}\varphi(\diamond) &= \varphi(0). \end{aligned}$$

△

Example 6. — When $n = 1$, the equations read as follows:

$$\varphi(x_1 + x_2) - \varphi(x_1) - \varphi(x_2) + \varphi(0) = \varphi(x_1 \diamond x_2) = 0,$$

[3] Of course, Eilenberg and Mac Lane themselves do not deign to make this definition, but instead move on to more important topics, like the computation of homology.

$$\varphi(rx) = \binom{r}{0}\varphi(\diamond) + \binom{r}{1}\varphi(\diamond x) = \varphi(0) + r(\varphi(x) - \varphi(0)).$$

The map

$$\psi(x) = \varphi(x) - \varphi(0)$$

is then a homomorphism. Conversely, any translate of a homomorphism is of degree 1. △

Example 7. — Let $\mathbf{B} = \mathbf{Z}$. It is a well-known fact, and not difficult to prove, that the numerical (polynomial) maps $\varphi\colon \mathbf{Z} \to \mathbf{Z}$ of degree n are precisely the ones given by numerical polynomials of degree n:

$$\varphi(x) = \sum_{k=0}^{n} c_k \binom{x}{k}.$$

△

Lemma 1. — *For r in a numerical ring and natural numbers $m \leqslant n$, the following formula holds:*

$$\sum_{k=m}^{n} (-1)^k \binom{r}{k}\binom{k}{m} = (-1)^n \binom{r}{m}\binom{r-m-1}{n-m}.$$

Proof. Induction on n. (The Numerical Transfer Principle is optional; the induction will go through in any ring.) □

Theorem 2. — *Let the map $\varphi\colon M \to N$ be polynomial of degree n. It is numerical (of degree n) iff it satisfies the equation*

$$\varphi(rx) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m}\binom{r-m-1}{n-m}\varphi(mx), \quad r \in \mathbf{B},\ x \in M.$$

Proof. This follows from the lemma:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{r}{k}\varphi\left(\underset{k}{\Diamond} x\right) &= \sum_{k=0}^{n} \binom{r}{k} \sum_{m=0}^{k} (-1)^{k-m}\binom{k}{m}\varphi(mx) \\
&= \sum_{m=0}^{n} (-1)^{-m} \left(\sum_{k=m}^{n} (-1)^k \binom{r}{k}\binom{k}{m}\right)\varphi(mx) \\
&= \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m}\binom{r-m-1}{n-m}\varphi(mx).
\end{aligned}$$

□

§4. The Augmentation Algebras

There is an algebraic way of describing numerical maps, which turns out to be very fruitful. Recall that the free module on a set M is the set

$$\mathbf{B}[M] = \left\{ \sum a_j[x_j] \,\middle|\, a_j \in \mathbf{B},\ x_j \in M \right\}$$

of formal (finite) linear combinations of elements of M. It obviously has a module structure, and if M itself is a module (or even abelian group), it also carries a multiplication, namely the **sum multiplication**

$$[x][y] = [x+y],$$

extended by linearity. It makes $\mathbf{B}[M]$ into a commutative, associative algebra with unity $[0]$.

When M is not only a module, but an algebra, there is another natural multiplication on $\mathbf{B}[M]$, namely the **product multiplication**:

$$[x] \star [y] = [xy].$$

This multiplication has the identity element $[1]$, and is of course commutative only if M is. This latter operation will make an apparition later on, in the context of Morita equivalence. In the present discussion, we will assume M to be a module only, and concentrate on the sum multiplication.

Let thus M be a module, and consider the map

$$M \to \mathbf{B}[M], \quad x \mapsto [x].$$

We may form its nth deviation

$$(x_1, \ldots, x_{n+1}) \mapsto [x_1 \diamond \ldots \diamond x_{n+1}].$$

Theorem 3. — *In the free algebra* $\mathbf{B}[M]$*, the following formula holds:*

$$[x_1 \diamond \ldots \diamond x_{n+1}] = ([x_1] - [0]) \cdots ([x_{n+1}] - [0]) .$$

Proof. Simply calculate:

$$\begin{aligned}[x_1 \diamond \ldots \diamond x_{n+1}] &= \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \left[\sum_{i \in I} x_i \right] \\ &= ([x_1] - [0]) \cdots ([x_{n+1}] - [0]) .\end{aligned}$$

□

There is a filtration of $\mathbf{B}[M]$, given by the decreasing sequence of ideals

$$\begin{aligned} I_n = {} & (\, [x_1 \diamond \ldots \diamond x_{n+1}] \mid x_i \in M \,) \\ & + \left([rx] - \sum_{k=0}^{n} \binom{r}{k} \left[\underset{k}{\diamond}\, x \right] \,\middle|\, r \in \mathbf{B},\ x \in M \right), \quad n \geqslant -1. \end{aligned}$$

Definition 6. — The n**th augmentation algebra** is the quotient algebra

$$\mathbf{B}[M]_n = \mathbf{B}[M]/I_n.$$

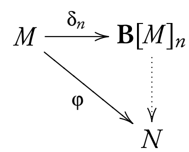
$\diamond$

If M is an algebra, then I_n is a two-sided ideal with respect to the product multiplication, so the augmentation algebras will inherit this operation.

Theorem 4. — *The map*

$$\delta_n : M \to \mathbf{B}[M]_n, \quad x \mapsto [x],$$

is the universal *numerical map of degree n, in that every numerical map* $\varphi : M \to N$ *of degree n has a unique factorisation through it.*



Proof. Given a map $\varphi : M \to N$, extend it linearly to a homomorphism

$$\varphi : \mathbf{B}[M] \to N.$$

The theorem amounts to the trivial observation that φ is numerical of degree n iff it kills I_n. $\square$

The augmentation quotients of a free module M are given by the next theorem.

Theorem 5. — *In the polynomial algebra* $\mathbf{B}[t_1, \ldots, t_k]$*, let* J_n *be the ideal generated by monomials of degree greater than n. Denote by* $(e_i)_{i=1}^k$ *the canonical basis of* $\mathbf{B}^k$*. Then*

$$\psi : \mathbf{B}[t_1, \ldots, t_k]/J_n \to \mathbf{B}[\mathbf{B}^k]_n$$
$$t^X \mapsto \left[\Diamond_{i\in X} e_i\right] = ([e] - [0])^X$$

is an isomorphism of algebras. In particular, $\mathbf{B}[\mathbf{B}^k]_n$ *is a free module.*

Proof. The map

$$\mathbf{B}[t_1, \ldots, t_k] \to \mathbf{B}[\mathbf{B}^k]_n$$
$$t^X \mapsto \left[\Diamond_{i\in X} e_i\right] = ([e] - [0])^X$$

is clearly a homomorphism of algebras, and since it annihilates J_n, it factors via $\mathbf{B}[t_1, \ldots, t_k]/J_n$. This establishes the existence of ψ.

We now define the inverse of ψ. Each t_i is nilpotent in $\mathbf{B}[t_1, \dots, t_k]/J_n$, and so the powers

$$(1 + t_i)^a$$

are defined for any $a \in \mathbf{B}$. Accordingly, for an element

$$x = a_1 e_1 + \cdots + a_k e_k \in \mathbf{B}^k,$$

we define

$$\begin{aligned} \chi\colon \mathbf{B}[\mathbf{B}^k] &\to \mathbf{B}[t_1, \dots, t_k]/J_n \\ [a_1 e_1 + \cdots + a_k e_k] &\mapsto (1 + t_1)^{a_1} \cdots (1 + t_k)^{a_k} + J_n. \end{aligned}$$

We write this more succinctly as

$$[x] \mapsto (1 + t)^x + J_n.$$

The map χ is linear by definition, and also multiplicative, since

$$\chi([x][y]) = \chi([x + y]) = (1 + t)^{x+y} = (1 + t)^x (1 + t)^y = \chi(x)\chi(y).$$

It maps I_n into J_n, because, when $x_1, \dots, x_{n+1} \in \mathbf{B}^k$,

$$\begin{aligned} \chi([x_1 \diamond \cdots \diamond x_{n+1}]) &= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} \chi\left(\left[\sum_{j \in J} x_j\right]\right) \\ &= \sum_{J \subseteq [n+1]} (-1)^{n+1-|J|} (1 + t)^{\sum_{j \in J} x_j} \\ &= \prod_{j=1}^{n+1} \left((1 + t)^{x_j} - 1\right) = 0. \end{aligned}$$

Also, for $r \in \mathbf{B}$ and $x \in \mathbf{B}^k$,

$$\begin{aligned} \chi\left([rx] - \sum_{m=0}^{n} \binom{r}{m} \left[\diamond_m x\right]\right) &= \chi\left([rx] - \sum_{m=0}^{n} \binom{r}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} [jx]\right) \\ &= (1 + t)^{rx} - \sum_{m=0}^{n} \binom{r}{m} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (1 + t)^{jx} \\ &= (1 + t)^{rx} - \sum_{m=0}^{n} \binom{r}{m} \left((1 + t)^x - 1\right)^m \\ &= (p(t) + 1)^r - \sum_{m=0}^{n} \binom{r}{m} p(t)^m, \end{aligned}$$

where, in the last step, we have let $p(t) = (1+t)^x - 1$. By the Binomial Theorem,

$$(p(t) + 1)^r = \sum_{m=0}^{\infty} \binom{r}{m} p(t)^m,$$

but since the terms of index $n+1$ and higher yield an $(n+1)$st degree polynomial, the above difference will belong to J_n. We therefore have an induced map

$$\chi\colon \mathbf{B}[\mathbf{B}^k]_n \to \mathbf{B}[t_1,\dots,t_k]/J_n,$$

and it is easy to verify that ψ and χ are inverses to each other. □

Example 8. — The isomorphism

$$\psi\colon \mathbf{B}[\mathbf{B}^2]_2 \to \mathbf{B}[t_1,t_2]/J_2$$

is given by

$$\begin{aligned} [\diamond] &\mapsto 1 & [e_1 \diamond e_1] &\mapsto t_1^2 \\ [\diamond e_1] &\mapsto t_1 & [e_1 \diamond e_2] &\mapsto t_1 t_2 \\ [\diamond e_2] &\mapsto t_2 & [e_2 \diamond e_2] &\mapsto t_2^2. \end{aligned}$$

△

§5. Properties of Numerical Maps

Let us elaborate somewhat on the behaviour of numerical maps, and investigate their elementary properties. To begin with, we note that the binomial coefficients themselves, considered as maps $\mathbf{B} \to \mathbf{B}$, are numerical. This is of course hardly surprising, as they are given by polynomials in the enveloping $\mathbf{Q}$-algebra.

Theorem 6. — *The binomial coefficient $x \mapsto \binom{x}{n}$ is numerical of degree n.*

Proof. It is numerical of degree n in $\mathbf{Z}$, and therefore also in $\mathbf{B}$ by the Numerical Transfer Principle. □

Next, not only do the nth deviations of an nth degree map vanish, but its lower order deviations are also quite pleasant.

Theorem 7. — *The map $\varphi\colon M \to N$ is numerical of degree n iff the following equation holds:*

$$\varphi(a_1x_1 \diamond \cdots \diamond a_kx_k) = \sum_{\substack{\#X=[k] \\ |X|\leqslant n}} \binom{a}{X} \varphi\left(\Diamond_X x\right), \quad a_i \in \mathbf{B},\ x_i \in M.$$

Proof. If the equation is satisfied, it follows that

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{\substack{\#X=[n+1] \\ |X|\leqslant n}} \binom{1}{X} \varphi\left(\Diamond_X x\right) = 0,$$

and

$$\varphi(a_1x_1) = \sum_{\substack{\#X=[1] \\ |X|\leqslant n}} \binom{a}{X} \varphi\left(\Diamond_X x\right) = \sum_{k=0}^{n} \binom{a_1}{k} \varphi\left(\Diamond_k x_1\right).$$

Conversely, if φ is of degree n, a calculation in the augmentation algebra $\mathbf{B}[M]_n$ yields

$$\begin{aligned}[a_1x_1 \diamond \cdots \diamond a_kx_k] &= ([a_1x_1] - [0]) \cdots ([a_kx_k] - [0]) \\ &= \sum_{q_1=1}^{\infty} \binom{a_1}{q_1} \left[\Diamond_{q_1} x_1\right] \cdots \sum_{q_k=1}^{\infty} \binom{a_k}{q_k} \left[\Diamond_{q_k} x_k\right] \\ &= \sum_{q_1=1}^{\infty} \cdots \sum_{q_k=1}^{\infty} \binom{a_1}{q_1} \cdots \binom{a_k}{q_k} \left[\Diamond_{q_1} x_1 \diamond \cdots \diamond \Diamond_{q_k} x_k\right].\end{aligned}$$

The theorem follows after application of φ. □

This proof is pure magic! It is absolutely vital that the calculation be carried out in the augmentation algebra, as there would have been no way to perform the above trick had the map φ been applied directly.

Example 9. — If φ is of degree 3, the following formulæ hold:

$$\begin{aligned}\varphi(\diamond a_1x_1) &= \binom{a_1}{1}\varphi(\diamond x_1) + \binom{a_1}{2}\varphi(x_1 \diamond x_1) + \binom{a_1}{3}\varphi(x_1 \diamond x_1 \diamond x_1) \\ \varphi(a_1x_1 \diamond a_2x_2) &= \binom{a_1}{1}\binom{a_2}{1}\varphi(x_1 \diamond x_2) \\ &\quad + \binom{a_1}{2}\binom{a_2}{1}\varphi(x_1 \diamond x_1 \diamond x_2) + \binom{a_1}{1}\binom{a_2}{2}\varphi(x_1 \diamond x_2 \diamond x_2) \\ \varphi(a_1x_1 \diamond a_2x_2 \diamond a_3x_3) &= \binom{a_1}{1}\binom{a_2}{1}\binom{a_3}{1}\varphi(x_1 \diamond x_2 \diamond x_3).\end{aligned}$$

△

We now have the following very explicit description of numerical maps.

Theorem 8. — *The map $\varphi\colon M \to N$ is numerical of degree n iff for any $u_1, \dots, u_k \in M$ there exist unique elements $v_X \in N$, X varying over multi-sets with $\#X \subseteq [k]$ and $|X| \leqslant n$, such that*

$$\varphi(a_1u_1 + \cdots + a_ku_k) = \sum_X \binom{a}{X} v_X,$$

for any $a_1, \dots, a_k \in \mathbf{B}$.

Proof. Assume φ is numerical of degree n. By the preceding theorem, we have

$$\begin{aligned}\varphi(a_1u_1 + \cdots + a_ku_k) &= \sum_{I \subseteq [k]} \varphi\left(\Diamond_{i\in I} a_iu_i\right) \\ &= \sum_{I \subseteq [k]} \sum_{\substack{\#X = I \\ |X| \leqslant n}} \binom{a}{X} \varphi\left(\Diamond_{i \in X} u_i\right)\end{aligned}$$

$$= \sum_{\substack{\#X \subseteq [k] \\ |X| \leqslant n}} \binom{a}{X} \varphi\left(\Diamond_{i \in X} u_i\right),$$

which establishes the existence of the elements v_X.

We now prove uniqueness. Let $Q \subseteq [k]$ be a multi-set, with $q_i = \deg_Q i$, and let

$$S = \{X \mid \#X \subseteq [k] \ \wedge \ \forall i : \deg_S i \leqslant q_i\}.$$

Then

$$\varphi(q_1 u_1 + \cdots + q_k u_k) = \sum_X \binom{q}{X} v_X = \sum_{X \in S} \binom{q}{X} v_X$$
$$= v_Q + \sum_{X \in S \setminus \{Q\}} \binom{q}{X} v_X.$$

We see that v_Q is determined by all v_X, such that X precedes Q in the lexicographical ordering on the set of all multi-sets on $[k]$, which can be identified with $\mathbf{N}^k$. By induction, each v_X is uniquely determined.

Conversely, assume φ is of the form specified in the theorem. It then readily follows that

$$v_X = \varphi\left(\Diamond_{i \in X} u_i\right)$$

for all X. In particular, the nth deviations of φ will vanish, and also

$$\varphi(au) = \sum_{m=0}^{n} \binom{a}{m} v_m = \sum_{m=0}^{n} \binom{a}{m} \varphi\left(\Diamond_m u\right).$$

□

And so, finally, we shall tie things together, and show that the definition we have given of numerical map, "essentially" coincides with $\mathfrak{NAlg}$-polynomiality; this latter notion entailing a natural transformation

$$M \otimes - \to N \otimes -$$

between functors $\mathfrak{NAlg} \to \mathfrak{Set}$. The subtle point here is that of degree. A numerical map, as we have defined it, always comes with a degree, which is of course not uniquely defined; it will also be numerical of any higher degree. In particular, there is no such thing as a map that is simply numerical. This is where the concept we have introduced differs from $\mathfrak{NAlg}$-polynomiality, for a map could well be $\mathfrak{NAlg}$-polynomial of infinite degree.

Example 10. — Let $U = \langle u_0, u_1, u_2, \ldots \rangle$ be free on an infinite basis. The map

$$\varphi_A : U \otimes A \to U \otimes A, \quad \sum u_k \otimes a_k \mapsto \sum u_k \otimes \binom{a_k}{k}$$

is $\mathfrak{NAlg}$-polynomial, but not numerical of any finite degree n. △

In order to exclude such anomalies, we are obliged to introduce a boundedness condition. Let $\varphi\colon M \to N$ be an $\mathfrak{NAlg}$-polynomial map. From the Polynomiality Principle, we know that for every linear form $l(x)$ over M there is a unique $\mathfrak{NAlg}$-polynomial $q(x)$ over N, such that

$$\varphi_A\,(l(a)) = q(a),$$

for all $A \in \mathfrak{NAlg}$ and all $a_j \in A$. We say that φ is of **bounded degree** n if the degree of the polynomial q is uniformly bounded above by n (independent of l).

THEOREM 9. — *The map $\varphi\colon M \to N$ is numerical of degree n iff it may be extended to a (unique) $\mathfrak{NAlg}$-polynomial map of bounded degree n.*

Proof. If

$$\varphi_A : M \otimes - \to N \otimes -$$

is an $\mathfrak{NAlg}$-polynomial map of bounded degree n, it is clear from the Polynomiality Principle that

$$\varphi_{\mathbf{B}} : M \to N$$

has the property of Theorem 8.

Conversely, let a numerical map $\varphi\colon M \to N$ be given. Given elements $u_1, \ldots, u_k \in M$, fix the elements v_X from Theorem 8. We may then extend φ in the obvious way to a natural transformation:

$$\varphi_A : M \otimes A \to N \otimes A, \quad \sum u_j \otimes a_j \mapsto \sum_X v_X \otimes \binom{a}{X}.$$

Some care is needed to ensure this map is well-defined on the tensor product, but everything works out in the end, essentially because it is postulated to work in the case $A = \mathbf{B}$. □

§6. STRICT POLYNOMIAL MAPS

As we direct our attention toward strict polynomial maps, we no longer assume a numerical base ring.

DEFINITION 7. — A $\mathfrak{CAlg}$-polynomial map $\varphi\colon M \to N$; that is, a natural transformation

$$M \otimes - \to N \otimes -$$

between functors $\mathfrak{CAlg} \to \mathfrak{Set}$; which is of bounded degree n, will be called a **strict polynomial map** of degree n. ◇

It is clear that a strict polynomial map is also numerical of the same degree (provided of course the base ring is numerical). If the base ring $\mathbf{B}$ is a $\mathbf{Q}$-algebra, the two concepts coincide, for then every algebra is numerical.

EXAMPLE 11. — A map is strict polynomial of degree 0 iff it is constant. △

Example 12. — Let $\psi\colon M \to N$ be a homomorphism. Then

$$\psi \otimes 1\colon M \otimes - \to N \otimes -$$

defines a strict polynomial map of degree 1.

From this we see that in degrees 0 and 1, numerical and strict polynomial maps always coincide. △

Example 13. — Let $\mathbf{B} = \mathbf{Z}$. The map

$$\xi\colon \mathbf{Z} \otimes - \to \mathbf{Z} \otimes -, \quad 1 \otimes x \mapsto 1 \otimes \binom{x}{2},$$

is numerical of degree 2, but not strict polynomial of any degree. The following tentative diagram, where $\beta\colon t \mapsto a$, indicates the impossibility of defining $\xi_{\mathbf{Z}[t]}$.

$$\begin{array}{ccc} \mathbf{Z} \otimes \mathbf{Z}[t] & \xrightarrow{\xi_{\mathbf{Z}[t]}} & \mathbf{Z} \otimes \mathbf{Z}[t] \\ \downarrow{\scriptstyle 1\otimes\beta} & & \downarrow{\scriptstyle 1\otimes\beta} \\ \mathbf{Z} \otimes \mathbf{Z} & \xrightarrow[\xi_{\mathbf{Z}}]{} & \mathbf{Z} \otimes \mathbf{Z} \end{array} \qquad \begin{array}{ccc} 1 \otimes t & \longrightarrow & ? \\ \downarrow & & \downarrow \\ 1 \otimes a & \longrightarrow & 1 \otimes \binom{a}{2} \end{array}$$

Note that $\mathbf{Z}[t]$ is not a numerical ring; there is no such thing as $\binom{t}{2}$! △

Example 14. — Contrary to the situation for numerical maps, strict polynomial maps are not determined by the underlying maps. The most simple example is probably the following. Let $\mathbf{B} = \mathbf{Z}$, and define

$$\varphi_A : \mathbf{Z}/2 \otimes A \to \mathbf{Z}/2 \otimes A, \quad 1 \otimes x \mapsto 1 \otimes x(x-1).$$

This is a non-trivial strict polynomial map of degree 2, and its underlying map is zero!

We see here at play the well-known distinction between polynomials and polynomial maps, the former class being richer than the latter. The point is that the strict polynomial structure provides extra data, which makes the zero map strict polynomial of degree 2 in a non-trivial way. △

As mentioned earlier, strict polynomial maps were invented by Norbert Roby, who designated them *lois polynomes*. The elementary facts enumerated below, where we consider a strict polynomial map $\varphi\colon M \to N$, are all taken from his article [19].

1. From the Polynomiality Principle, the following proposition is immediately deduced. *For any $u_1, \dots, u_k \in M$ there exist unique elements $v_X \in N$ (only finitely many of which are non-zero), X varying over multi-sets with $X \subseteq [k]$, such that*

$$\varphi(u_1 \otimes a_1 + \cdots + u_k \otimes a_k) = \sum_X v_X \otimes a^X$$

for all a_j in all algebras.

We call v_X the **multi-deviation** of φ of type X, with respect to the elements u_i, and we shall denote this by the symbol

$$v_X = \varphi_{u^{[X]}}.$$

(This notation makes the subscript look like a divided power, which, in fact, it is. See below.)

2. φ is of degree n iff $\varphi_{u^{[X]}} = 0$ whenever $|X| > n$.

3. φ is said to be **homogeneous** of degree n if

$$\varphi(az) = a^n \varphi(z)$$

for all a in all algebras A and all $z \in M \otimes A$. This amounts to saying that $\varphi_{u^{[X]}} \neq 0$ only when $|X| = n$.

4. When φ is homogeneous of degree n, note that

$$\varphi_{u^{[n]}} = \varphi(u).$$

5. Any φ will have a unique decomposition into homogeneous components, namely:

$$\varphi(u_1 \otimes a_1 + \cdots + u_k \otimes a_k) = \sum_{n=0}^{\infty} \sum_{|X|=n} \varphi_{u^{[X]}} \otimes a^X.$$

There is some subtlety here. It is important to note that the above sum is only *locally finite*. For a given

$$z = u_1 \otimes a_1 + \cdots + u_k \otimes a_k,$$

the sum contains only a finite number of terms, but this number depends on z, and, worse, need not be bounded above. Restricting attention to strict polynomial maps of some given degree n, as we shall do, these difficulties are avoided.

6. There is a fundamental relationship between homogeneous maps and divided power algebras. For any module M there is a *universal* homogeneous map

$$\gamma_n \colon M \to \Gamma^n(M), \quad \sum u_i \otimes a_i \mapsto \sum_{|X|=n} u^{[X]} \otimes a^X$$

of degree n, through which every map $\varphi \colon M \to N$ of degree n factors uniquely:

$$\begin{array}{ccc} M \otimes A & \xrightarrow{\gamma_n} & \Gamma^n(M) \otimes A \\ & \searrow^{\varphi} & \vdots \\ & & N \otimes A \end{array} \qquad \begin{array}{ccc} \sum u_i \otimes a_i & \longrightarrow & \sum_{|X|=n} u^{[X]} \otimes a^X \\ & \searrow & \downarrow \\ & & \sum_{|X|=n} \varphi_{u^{[X]}} \otimes a^X \end{array}$$

In other words, there is a canonical isomorphism between the module of homogeneous polynomial maps of degree n from M to N and the module of homomorphisms from $\Gamma^n(M)$ to N.

7. Given φ, the map
$$\Gamma^n(M) \to N, \quad u^{[X]} \mapsto \varphi_{u^{[X]}}$$
is a module homomorphism.

§7. The Divided Power Algebras

Divided power algebras are intimately connected to strict polynomial maps. Roby, in all his prolificacy, has a great deal to say about them, and we once again refer the reader to [19].

Like the free algebra, the divided power algebra $\Gamma(M)$ comes equipped with two natural multiplications.

Let first M be a module. We recall that the **divided power multiplication** on $\Gamma(M)$ is given simply by juxtaposition:
$$x^{[m]} \cdot y^{[n]} = x^{[m]} y^{[n]}.$$
The homogeneous components $\Gamma^n(M)$ are not closed under this operation.

Let now M be an algebra. Then there is another multiplication on the nth divided power module $\Gamma^n(M)$, which we now describe. Note that there is a canonical map
$$\delta\colon M \times M \to \Gamma^n(M) \otimes \Gamma^n(M), \quad (x, y) \mapsto x^{[n]} \otimes y^{[n]},$$
which is universal for bihomogeneous maps of bidegree (n, n) out of $M \times M$. Because the map
$$\zeta\colon M \times M \to \Gamma^n(M \otimes M), \quad (x, y) \mapsto (x \otimes y)^{[n]},$$
is bihomogeneous of degree (n, n), it will have a unique factorisation through δ:
$$\begin{array}{ccccc} M \times M & \xrightarrow{\delta} & \Gamma^n(M) \otimes \Gamma^n(M) & & \\ & \searrow^{\zeta} & \downarrow & & \\ & & \Gamma^n(M \otimes M) & \longrightarrow & \Gamma^n(M) \end{array}$$
Composition with the canonical (linear) map
$$\Gamma^n(M \otimes M) \to \Gamma^n(M), \qquad (x \otimes y)^{[n]} \mapsto (xy)^{[n]},$$
results in the following multiplication on $\Gamma^n(M)$:
$$\Gamma^n(M) \otimes \Gamma^n(M) \to \Gamma^n(M), \quad x^{[n]} \otimes y^{[n]} \mapsto x \star y = (xy)^{[n]}.$$

It will be called the **product multiplication** on $\Gamma^n(M)$.

Let now A and B be multi-sets of cardinality n, and let the variables x_a and y_b be indexed over $\#A$ and $\#B$, respectively. It is then easy to verify that the following formula holds:

$$x^{[A]} \star y^{[B]} = \sum_{\mu\colon A \to B} (xy)^{[\mu]}$$

(where the sum is taken over multations, rather than sums of such). We do not prove this formula here, as we shall establish a more general result in Chapter 7.

It deserves to be pointed out, and emphasised, that the nth divided power module $\Gamma^n(M)$ is *not* generated by the pure divided powers $z^{[n]}$, for $z \in M$, as the following example shows.

Example 15. — Consider in $\Gamma^3(\mathbf{Z}^2)$ a pure power

$$(a_1e_1 + a_2e_2)^{[3]} = a_1^3 e_1^{[3]} + a_1^2 a_2 e_1^{[2]} e_2 + a_1 a_2^2 e_1 e_2^{[2]} + a_2^3 e_2^{[3]}.$$

The coefficients of $e_1^{[2]}e_2$ and $e_1e_2^{[2]}$ have the same parity (even or odd). Therefore, it will be impossible to isolate $e_1^{[2]}e_2$ as a linear combination of pure powers. △

$\Gamma^n(M)$ is, however, "universally" generated by pure powers, in the following sense.

Theorem 10: The Divided Power Lemma.

- *A natural transformation*

$$\zeta\colon \Gamma^n(M) \otimes - \to N \otimes -,$$

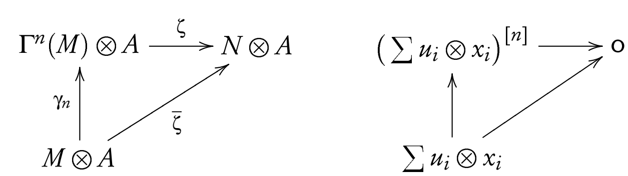
between functors $\mathfrak{CAlg} \to \mathfrak{Mod}$*, is uniquely determined by its effect on* pure *divided powers* $z^{[n]}$ *(when* $z \in M \otimes A$ *for some algebra* A*).*

- *More generally, a natural transformation*

$$\zeta\colon \Gamma^m(M) \otimes \Gamma^n(M) \otimes - \to N \otimes -$$

is uniquely determined by its effect on tensor products $z^{[m]} \otimes w^{[n]}$ *of pure powers.*

Proof. It suffices to show that if ζ vanishes on pure powers, it is identically zero. Indeed, linear maps $\Gamma^n(M) \to N$ correspond to homogeneous maps $M \to N$:



Since $\overline{\zeta} = \zeta\gamma_n = 0$, also $\zeta = 0$.

For the second part, proceed similarly, while noting that linear maps

$$\Gamma^m(M) \otimes \Gamma^n(M) \to N$$

correspond to bihomogeneous maps

$$M \oplus M \to N.$$

□

Chapter 6

POLYNOMIAL FUNCTORS

Och när jag stod där gripen, kall av skräck
och fylld av ängslan inför hennes tillstånd
begynte plötsligt mimans fonoglob
att tala till mig på den dialekt
ur högre avancerad tensorlära
som hon och jag till vardags brukar mest.

— Harry Martinson, *Aniara*

With all preliminary work disposed of, we may finally begin our study of polynomial functors proper. The first layer of bricks has been laid out with minute detail, and will provide the rock solid foundations upon which to erect our glorious temple. The chief aim of this monograph, it will be recalled, is to study module functors. But, in order to describe these, it was necessary first to gain a proper understanding of module *maps*. Only then could we hope for some insight into the machinery of functors.

We now consider ourselves amply prepared for the task. Our incorporation of numerical and strict polynomial maps into the same framework (Professor Roby's) will be seen to lead to a corresponding unification of the notions of numerical and strict polynomial functor. Here, again, can be seen the deficiency of Professors Eilenberg and Mac Lane's original notion, as functors that are just polynomial do not allow for such a unified treatment.

The beginning of the chapter is comprised by a quick introduction to module functors in general. This is less of a luxurious indulgence than it may seem, but rather an important reminder that our attention is restricted to a special class of functors, albeit rather large and general. Namely, we shall only consider those module functors that are *determined by their values on free and finitely generated modules*; this latter category admitting a smooth description, and being pleasant in a multitude of ways. The question then arises as to what functors possess this amiable property. We are by no means the first to study these; from [3] we draw the characterisation of them as the *right-exact functors that commute with inductive limits*.

Corresponding to the classes of numerical and strict polynomial *maps* are the classes of numerical and strict polynomial *functors*, which we then set out to explore. The chapter closes with a swift investigation of *analytic* functors, which are characterised as the inductive limits of polynomial functors.

§1. Module Functors

Let

$$\mathfrak{FMod}$$

be the category of free modules, finitely or infinitely generated; and let

$$\mathfrak{XMod}$$

be the category of those modules that are finitely generated and free (the letter X intended to suggest "eXtra nice modules"!). A **module functor** is a functor

$$\mathfrak{XMod} \to \mathfrak{Mod}$$

— and there should really be no need to point out that linearity will *not* be assumed.

This may seem a meagre substitute for a functor defined on all modules, but the restriction is not as heavy as one might think. As it turns out, a functor defined on the subcategory $\mathfrak{XMod}$ has a canonical well-behaved extension to the whole module category $\mathfrak{Mod}$. We now describe this extension process, and thus convince ourselves (and hopefully the reader) that there is no serious imposition in considering only functors $\mathfrak{XMod} \to \mathfrak{Mod}$, as will be done henceforth.

First, let us recall what it means for a functor, not necessarily additive, to be right-exact.

Definition 1. — A functor F between abelian categories is **right-exact** if for any exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

the associated sequence

$$F(A \oplus B) \xrightarrow[-F(1_B)]{F(\alpha+1_B)} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0$$

is also exact. ◇

This definition agrees with the usual one in the case of an additive functor. In fact, the usual definition actually *implies* additivity of the functor, which renders it useless for our purposes.

Theorem 1.

1. *Any functor $\mathfrak{XMod} \to \mathfrak{Mod}$ has a unique extension to a functor $\mathfrak{FMod} \to \mathfrak{Mod}$ which commutes with inductive limits.*

2. *Any functor* $\mathfrak{FMod} \to \mathfrak{Mod}$ *has a unique extension to a right-exact functor* $\mathfrak{Mod} \to \mathfrak{Mod}$.

Proof. Since the result is well-known, we shall be content to sketch an outline of its proof.

The first part follows from Lazard's Theorem, which states that every flat module is an inductive limit of finitely generated, free modules. A functor $G\colon \mathfrak{XMod} \to \mathfrak{Mod}$ may be extended to $\overline{G}\colon \mathfrak{FMod} \to \mathfrak{Mod}$ by putting

$$\overline{G}(\varinjlim M_\alpha) = \varinjlim G(M_\alpha),$$

for any inductive system (M_α) of finitely generated, free modules. This definition is probably independent of the inductive system chosen.

The second part of the theorem is an immediate consequence of Theorem 2.14 in [3]. (The crucial point is that $\mathfrak{FMod}$ is a subcategory of projective generators that is closed under direct sums.) The extension procedure (which essentially uses parts of the Dold–Puppe construction originally presented in [5]) may be summarised as follows.

Let a functor $F\colon \mathfrak{FMod} \to \mathfrak{Mod}$ be given. Given a module M, choose a free resolution:

$$Q \xrightarrow{\psi} P \longrightarrow M \longrightarrow 0$$

Let π and ξ denote the canonical projections:

$$P \xleftarrow{\pi} P \oplus Q \xrightarrow{\xi} Q$$

Define the extension $\overline{F}\colon \mathfrak{Mod} \to \mathfrak{Mod}$ by the equation

$$\overline{F}(M) = F(P) \Big/ \Big[F(\pi)\big(\operatorname{Ker} F(\pi + \psi\xi)\big)\Big].$$

Again, this definition is independent of the particular free resolution chosen.

That $\overline{F}$ extends F is clear. When M is free, we may take the obvious free resolution:

$$0 \xrightarrow{0} M \longrightarrow M \longrightarrow 0$$

Here $\pi = 1_M$ and $\xi = 0$, and so

$$\begin{aligned}\overline{F}(M) &= F(M) \Big/ \Big[F(\pi)\big(\operatorname{Ker} F(\pi + \psi\xi)\big)\Big] = F(M) \Big/ \Big[F(1_M)(\operatorname{Ker} F(1_M))\Big] \\ &= F(M) \Big/ \Big[1_{F(M)}(\operatorname{Ker} 1_{F(M)})\Big] = F(M)/0 = F(M).\end{aligned}$$

□

§2. Polynomial Functors

We now turn to interpreting our three notions of polynomiality in terms of functors, in order from the weakest to the strongest.

Definition 2. — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **polynomial** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. ◇

Example 1. — If F is polynomial of degree 0, then $F(\alpha) = F(0)$ for every homomorphism α. We have

$$1_{F(M)} = F(1_M) = F(0\colon M \to M) = F(0\colon N \to M)F(0\colon M \to N),$$

and similarly

$$1_{F(N)} = F(0\colon M \to N)F(0\colon N \to M);$$

hence $F(M) \cong F(N)$. The polynomial functors of degree 0 are thus the constant ones. △

Example 2. — If F is of degree 1, then

$$F(\alpha+\beta) - F(\alpha) - F(\beta) + F(0) = 0,$$

or, equivalently,

$$F(\alpha+\beta) - F(0) = (F(\alpha) - F(0)) + (F(\beta) - F(0)).$$

The functor

$$E(\gamma) = F(\gamma) - F(0)$$

is then additive.

Conversely, a functor F of the form

$$F(M) = K \oplus E(M),$$

with E additive, is polynomial of degree 1. △

§3. Numerical Functors

In order to discuss numerical functors, we assume of course a numerical base ring.

Definition 3. — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **numerical** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. ◇

Note that, over the integers, the notions of polynomial and numerical functor coincide.

Note also the inconspicuous assumption on uniformly bounded degree. We shall presently see what happens when this assumption is dropped.

Example 3. — The functor F is numerical of degree 0 iff it is constant:

$$F(M) = K.$$

△

Example 4. — The functor F is numerical of degree 1 iff it is of the form

$$F(M) = K \oplus E(M),$$

with E linear. Taking it on faith that there exist functors which are additive, but not linear, we see that numericality is a stronger notion than polynomiality already in degree 1. △

Example 5. — The most notorious examples of polynomial functors are no doubt the classical algebraic functors: the *tensor power* $T^n(M)$, the *symmetric power* $S^n(M)$, the *exterior power* $\Lambda^n(M)$, and the *divided power* $\Gamma^n(M)$. They are all numerical of degree n, because they commute with extension of scalars, and are of bounded degree. For example, the map

$$T^n_A\colon A \otimes T^n(M) = T^n_A(M) \to T^n_A(N) = A \otimes T^n(N)$$

is clearly natural in all (numerical) algebras A. △

A **natural transformation** $\eta\colon F \to G$ of numerical functors is a family of homomorphisms

$$\eta = (\eta_M\colon F(M) \to G(M) \mid M \in \mathfrak{XMod}),$$

such that for any modules M and N, any numerical algebra A, and any

$$\omega \in A \otimes \operatorname{Hom}(M, N),$$

the following diagram commutes:

$$\begin{array}{ccc} A \otimes F(M) & \xrightarrow{1 \otimes \eta_M} & A \otimes G(M) \\ \downarrow{\scriptstyle F(\omega)} & & \downarrow{\scriptstyle G(\omega)} \\ A \otimes F(N) & \xrightarrow[1 \otimes \eta_N]{} & A \otimes G(N) \end{array} \tag{1}$$

We shall denote the category of numerical functors of degree n by

$$\mathfrak{Num}_n.$$

It is easy to see that it is abelian (the case $\mathbf{B} = \mathbf{Z}$ is well known). It is closed under direct sums, and we will see in Chapter 9 that it possesses a small projective generator, so by the Morita Equivalence, it is in fact a module category.

Definition 4. — The numerical functor F is **quasi-homogeneous** of degree n if the extension functor

$$F\colon \mathbf{Q}\otimes_{\mathbf{Z}} \mathfrak{XMod} \to \mathbf{Q}\otimes_{\mathbf{Z}} \mathfrak{Mod}$$

satisfies the equation

$$F(r\alpha) = r^n F(\alpha),$$

for any $r \in \mathbf{Q}\otimes_{\mathbf{Z}} \mathbf{B}$ and homomorphism α. ◇

As remarked in connection with the category $\mathfrak{Lab}\mathfrak{y}^n$, demanding

$$F(r\alpha) = r^n F(\alpha)$$

just for $r \in \mathbf{B}$ would be insufficient to yield interesting results. Being quasi-homogeneous is, in a sense, the closest a numerical functor can come to being homogeneous.

The category of quasi-homogeneous functors of degree n will be denoted by

$$\mathfrak{QHom}_n.$$

§4. Properties of Numerical Functors

Let us hasten to point out that our definition of natural transformation is unnecessarily complicated. A consequence of Theorem 8 of Chapter 5 is that a polynomial functor is uniquely determined by its underlying functor. In view of this, the following theorem is hardly surprising. The reason for adopting the more complicated condition as definition, is to conform to the situation for strict polynomial functors.

Theorem 2. — *The diagram (1) commutes for any (ordinary) natural transformation* $\eta\colon F \to G$.

Proof. Consider homomorphisms

$$\alpha_1, \ldots, \alpha_k\colon M \to N.$$

Assume that

$$F(a_1\otimes\alpha_1 + \cdots + a_k\otimes\alpha_k) = \sum_{\mu} \binom{a_1}{m_1}\cdots\binom{a_k}{m_k}\otimes\beta_\mu$$
$$G(a_1\otimes\alpha_1 + \cdots + a_k\otimes\alpha_k) = \sum_{\nu} \binom{a_1}{n_1}\cdots\binom{a_k}{n_k}\otimes\gamma_\nu,$$

for any $a_1, \ldots, a_k$ in any numerical algebra A, where we have abbreviated

$$\mu = (m_1, \ldots, m_k) \quad \text{and} \quad \nu = (n_1, \ldots, n_k).$$

The naturality of η ensures that

$$\sum_{\mu} \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} \eta_N \beta_\mu = \sum_{\nu} \binom{a_1}{n_1} \cdots \binom{a_k}{n_k} \gamma_\nu \eta_M.$$

Specialise first to the case $a_2 = a_3 = \cdots = 0$, to obtain

$$\sum_{m_1} \binom{a_1}{m_1} \eta_N \beta_{(m_1, 0, \dots)} = \sum_{n_1} \binom{a_1}{n_1} \gamma_{(n_1, 0, \dots)} \eta_M.$$

Successively putting $a_1 = 0, 1, 2, \dots$ leads to

$$\eta_N \beta_{(m_1, 0, \dots)} = \gamma_{(m_1, 0, \dots)} \eta_M$$

for all m_1. Proceeding inductively, one shows that

$$\eta_N \beta_\mu = \gamma_\mu \eta_M$$

for all μ. The commutativity of the diagram (1), for

$$\omega = a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k,$$

is then demonstrated by the following instantiation:

$$\begin{array}{ccc}
b \otimes x & \longrightarrow & b \otimes \eta_M(x) \\
\downarrow & & \downarrow \\
\sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \beta_\mu(x) & \longrightarrow & \left[\begin{array}{l} \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \eta_N \beta_\mu(x) \\ = \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \gamma_\mu \eta_M(x) \end{array} \right]
\end{array}$$

□

The subsequent theorem is the first to illustrate a recurring theme: that a numerical functor of degree n is essentially determined by its action on $\mathbf{B}^n$.

Theorem 3. — *The following conditions are equivalent on a polynomial functor F of degree n.*

A.

$$F(r\alpha) = \sum_{k=0}^{n} \binom{r}{k} F\left(\underset{k}{\Diamond} \alpha \right),$$

for any scalar r and homomorphism α (numerical functor).

B.

$$F(r\alpha) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m\alpha),$$

for any scalar r and homomorphism α.

A'.

$$F(r\cdot \mathrm{1}_{\mathbf{B}^n}) = \sum_{k=0}^{n} \binom{r}{k} F\left(\Diamond_k \mathrm{1}_{\mathbf{B}^n}\right),$$

for any scalar r.

B'.

$$F(r\cdot \mathrm{1}_{\mathbf{B}^n}) = \sum_{m=0}^{n} (-1)^{n-m}\binom{r}{m}\binom{r-m-1}{n-m} F(m\cdot \mathrm{1}_{\mathbf{B}^n}),$$

for any scalar r.

Proof. That A and B are equivalent follows from Theorem 2 of Chapter 5, as does the equivalence of A′ and B′. Clearly B implies B′, so there remains to establish that B′ implies B.

Hence assume B′, and put

$$Z_m = (-1)^{n-m}\binom{r}{m}\binom{r-m-1}{n-m}.$$

In the case $q \leqslant n$ the equation

$$F(r\cdot \mathrm{1}_{\mathbf{B}^q}) = \sum_{m=0}^{n} Z_m F(m\cdot \mathrm{1}_{\mathbf{B}^q})$$

holds, because $\mathrm{1}_{\mathbf{B}^q}$ factors through $\mathrm{1}_{\mathbf{B}^n}$.

Consider now the case $q > n$. By induction, assume the formula holds for $q-1$. Letting π_i as usual denote canonical projections, we calculate:

$$\begin{aligned}
F(r\cdot \mathrm{1}_{\mathbf{B}^q}) &= F(r\pi_1 + \cdots + r\pi_q) \\
&= -\sum_{I\subset[q]} (-1)^{q-|I|} F\left(\sum_{i\in I} r\pi_i\right) \\
&= -\sum_{I\subset[q]} (-1)^{q-|I|} \sum_{m=0}^{n} Z_m F\left(\sum_{i\in I} m\pi_i\right) \\
&= -\sum_{m=0}^{n} Z_m \sum_{I\subset[q]} (-1)^{q-|I|} F\left(\sum_{i\in I} m\pi_i\right) \\
&= \sum_{m=0}^{n} Z_m F(m\pi_1 + \cdots + m\pi_q) = \sum_{m=0}^{n} Z_m F(m\cdot \mathrm{1}_{\mathbf{B}^q}).
\end{aligned}$$

The third and sixth steps are because the qth deviation vanishes. This shows that the equation holds for $\mathrm{1}_{\mathbf{B}^q}$, for any q.

Finally, in the case of an arbitrary homomorphism $\alpha\colon \mathbf{B}^p \to \mathbf{B}^q$, we have

$$F(r\alpha) = F(r\cdot \mathrm{1}_{\mathbf{B}^q})F(\alpha)$$

$$= \sum_{m=0}^{n} Z_m F(m \cdot 1_{\mathbf{B}^q}) F(\alpha) = \sum_{m=0}^{n} Z_m F(m\alpha),$$

and the proof is finished. □

The following very pleasant formula is an immediate consequence of the corresponding formula for maps.

THEOREM 4. — *The module functor F is numerical of degree n iff for any scalars a_i and homomorphisms α_i, the following equation holds:*

$$F(a_1\alpha_1 \diamond \cdots \diamond a_k\alpha_k) = \sum_{\substack{\#X=[k] \\ |X| \leqslant n}} \binom{a}{X} F\left(\Diamond_X \alpha\right).$$

Proof. Theorem 7 of Chapter 5. □

EXAMPLE 6. — If F is of degree 3, the following formulæ hold:

$$\begin{aligned}
F(\diamond a_1\alpha_1) &= \binom{a_1}{1} F(\diamond\alpha_1) + \binom{a_1}{2} F(\alpha_1 \diamond \alpha_1) \\
&\quad + \binom{a_1}{3} F(\alpha_1 \diamond \alpha_1 \diamond \alpha_1) \\
F(a_1\alpha_1 \diamond a_2\alpha_2) &= \binom{a_1}{1}\binom{a_2}{1} \varphi(\alpha_1 \diamond \alpha_2) + \binom{a_1}{2}\binom{a_2}{1} F(\alpha_1 \diamond \alpha_1 \diamond \alpha_2) \\
&\quad + \binom{a_1}{1}\binom{a_2}{2} F(\alpha_1 \diamond \alpha_2 \diamond \alpha_2) \\
F(a_1\alpha_1 \diamond a_2\alpha_2 \diamond a_3\alpha_3) &= \binom{a_1}{1}\binom{a_2}{1}\binom{a_3}{1} F(\alpha_1 \diamond \alpha_2 \diamond \alpha_3).
\end{aligned}$$

△

§5. The Hierarchy of Numerical Functors

We say that a map φ, or a family of such, is **multiplicative** if

$$\varphi(z)\varphi(w) = \varphi(zw),$$

whenever z and w are entities («quelques choses») such that the equation makes sense, and also

$$\varphi(1) = 1,$$

where the symbol 1 is to be interpreted in a natural way (usually differently on each side). An ordinary functor is the prime example of such a multiplicative family.

Also, we say that a family of maps is of **(uniformly) bounded degree**, if every map in the family is numerical of some fixed degree n.

Theorem 5. — *Consider the following constructs, where A ranges over all numerical algebras:*

A. *A family of ordinary functors $E_A : {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$, commuting with extension of scalars.*

B. *A functor $J: \mathfrak{XMod} \to \mathfrak{Mod}$, with arrow maps*

$$J_A: \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N))$$

that are multiplicative and natural in A.

C. *A functor $F: \mathfrak{XMod} \to \mathfrak{Mod}$, with arrow maps*

$$F_A : A \otimes \operatorname{Hom}_{\mathbf{B}}(M, N) \to A \otimes \operatorname{Hom}_{\mathbf{B}}(F(M), F(N))$$

that are multiplicative and natural in A (numerical maps).

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow maps are assumed to have uniformly bounded degree, all three are equivalent.

Proof. Given E, define J by

$$J(M) = E_{\mathbf{B}}(M)$$

and the diagram:

$$\begin{array}{ccc} \operatorname{Hom}_A(A \otimes M, A \otimes N) & \xrightarrow{E_A} & \operatorname{Hom}_A(E_A(A \otimes M), E_A(A \otimes N)) \\ & \overset{J}{\searrow} & \updownarrow \\ & & \operatorname{Hom}_A(A \otimes E_{\mathbf{B}}(M), A \otimes E_{\mathbf{B}}(N)) \end{array}$$

Conversely, given J, define the functors E by the equations

$$E_A(M) = A \otimes J(M)$$

and

$$\operatorname{Hom}_A(A \otimes M, A \otimes N) \xrightarrow{E_A = J} \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N)).$$

Also, it is easy to define J from F; simply let

$$J(M) = F(M),$$

and use the following diagram:

$$\begin{array}{ccc} A \otimes \operatorname{Hom}_{\mathbf{B}}(M, N) & \xrightarrow{F} & A \otimes \operatorname{Hom}_{\mathbf{B}}(F(M), F(N)) \\ \updownarrow & & \downarrow \\ \operatorname{Hom}_A(A \otimes M, A \otimes N) & \xrightarrow[J]{} & \operatorname{Hom}_A(A \otimes F(M), A \otimes F(N)) \end{array}$$

The left column in the diagram is an isomorphism as long as M and N are free.

The difficult part is defining F from J, provided that J is indeed of bounded degree n. The following proof is modelled on the corresponding proof for strict polynomial functors in [20]. Let M and N be two modules, and let A be any numerical algebra. Find a free resolution

$$\mathbf{B}^{(\lambda)} \longrightarrow \mathbf{B}^{(\kappa)} \longrightarrow J(M) \longrightarrow 0\ ,$$

and apply the contra-variant, left-exact functor

$$\mathrm{Hom}_A(A \otimes -, A \otimes J(N)),$$

to obtain a commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N)) & \xrightarrow{\ \iota\ } & (A \otimes J(N))^{\kappa} & \xrightarrow{\ \sigma\ } & (A \otimes J(N))^{\lambda} \\ & {\scriptstyle J}\big\uparrow & & \big\uparrow{\scriptstyle \zeta} & & \\ & A \otimes \mathrm{Hom}(M,N) & \xrightarrow[\delta_n]{} & A \otimes \mathbf{B}[\mathrm{Hom}(M,N)]_n & & \end{array}$$

The homomorphism

$$\iota J \colon A \otimes \mathrm{Hom}(M,N) \to (A \otimes J(N))^{\kappa}$$

may be split up into components

$$(\iota J)_k \colon A \otimes \mathrm{Hom}(M,N) \to A \otimes J(N),$$

for each $k \in \kappa$. Those are numerical of degree n, and will factor over δ_n via some linear ζ_k. Together they yield a linear map

$$\zeta \colon A \otimes \mathbf{B}[\mathrm{Hom}(M,N)]_n \to (A \otimes J(N))^{\kappa},$$

making the above square commute.

Now, $\sigma\zeta\delta_n = \sigma\iota J = 0$, which gives $\sigma\zeta = 0$. By the exactness of the upper row, ζ factors via some homomorphism

$$\xi \colon \mathbf{B}[\mathrm{Hom}(M,N)]_n \to \mathrm{Hom}(J(M), J(N)).$$

Because

$$\iota J = \zeta\delta_n = \iota\xi\delta_n$$

and ι is one-to-one, we also have $J = \xi\delta_n$. The following diagram will therefore commute:

$$\begin{array}{ccc} \mathrm{Hom}(J(M), J(N)) & \xrightarrow{\ \iota\ } & J(N)^{(\kappa)} \\ {\scriptstyle J}\big\uparrow & \overset{\xi}{\nwarrow} & \big\uparrow{\scriptstyle \zeta} \\ \mathrm{Hom}(M,N) & \xrightarrow[\delta_n]{} & \mathbf{B}[\mathrm{Hom}(M,N)]_n \end{array}$$

Since J factors over $\mathbf{B}[\mathrm{Hom}(M,N)]_n$, it is numerical of degree n, and so may be used to construct F. □

We thus obtain the following hierarchy of functors.

- *Numerical functors*, as defined previously, have bounded degree, and satisfy all three conditions A, B and C.
- A functor satisfying condition C, but with no assumption on the degree, will be called **locally numerical**.
- A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **analytic**.

Example 7. — The classical algebraic functors T, S, Λ and Γ are all analytic, for they evidently satisfy condition A of the theorem.

Of these, only Λ is locally numerical. This is because, when $n > p$, the module

$$\Lambda^n(\mathbf{B}^p) = 0,$$

and hence, for given p and q, the map

$$\Lambda\colon \operatorname{Hom}(\mathbf{B}^p, \mathbf{B}^q) \to \operatorname{Hom}(\Lambda(\mathbf{B}^p), \Lambda(\mathbf{B}^q))$$

is numerical of degree $\max(p, q)$. △

§6. Strict Polynomial Functors

The base ring $\mathbf{B}$ is now no longer assumed numerical.

Definition 5. — The functor $F\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$ is said to be **strict polynomial** of degree n if the arrow maps

$$F\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$$

have been given a (multiplicative) strict polynomial structure. ⋄

A strict polynomial functor is also numerical of the same degree, provided the base ring is numerical. If the base ring $\mathbf{B}$ is a $\mathbf{Q}$-algebra, the two concepts coincide, for then every algebra is numerical.

We remark that, comparable to the situation for maps, strict polynomial functors *are not determined by their underlying functors*.

Example 8. — Because numerical and strict polynomial maps coincide in degrees 0 and 1, the same holds true for functors. △

Example 9. — The classical algebraic functors T^n, S^n, Λ^n and Γ^n are in fact strict polynomial of degree n, because they are natural in *all* algebras, not just numerical ones. △

EXAMPLE 10. — The following singular example may serve as a warning. Let **B** be a numerical ring, let p be a prime, and consider the functor

$$F(M) = S^p(M)/pS^p(M).$$

F inherits from S^p the property of being homogeneous of degree p. On the other hand, because $p \mid r^p - r$ by Fermat's Little Theorem, the underlying functor of F takes

$$F(r\alpha) = \overline{S^p(r\alpha)} = \overline{r^p S^p(\alpha)} = \overline{rS^p(\alpha)} = rF(\alpha).$$

Moreover, by the Binomial Theorem,

$$(a+b)^p \equiv a^p + b^p \mod p,$$

so

$$F(\alpha+\beta) = \overline{S^p(\alpha+\beta)} = \overline{S^p(\alpha) + S^p(\beta)} = F(\alpha) + F(\beta).$$

The underlying functor of F is linear!

Quite obviously, F may also be given the structure of homogeneous functor of degree 1, furnishing us with two different strict polynomial structures on the same functor, and even of different degrees. △

By a **natural transformation** $\eta\colon F \to G$ of strict polynomial functors, we mean a family of homomorphisms

$$\eta = (\eta_M\colon F(M) \to G(M) \mid M \in \mathfrak{XMod}),$$

such that for any modules M and N, any algebra A, and any

$$\omega \in A \otimes \mathrm{Hom}(M, N),$$

the following diagram commutes:

$$\begin{array}{ccc} A \otimes F(M) & \xrightarrow{1\otimes\eta_M} & A \otimes G(M) \\ \downarrow{\scriptstyle F(\omega)} & & \downarrow{\scriptstyle G(\omega)} \\ A \otimes F(N) & \xrightarrow[1\otimes\eta_N]{} & A \otimes G(N) \end{array}$$

We shall denote by

$$\mathfrak{SPol}_n$$

the category of strict polynomial functors of degree n. It is well known to be abelian, and, like the category of numerical functors, it is in fact a module category. However, rather than consider arbitrary strict polynomial functors, we shall usually limit our attention to homogeneous ones, as any strict polynomial functor decomposes as a direct sum of such.

§7. The Hierarchy of Strict Polynomial Functors

As in the numerical case, we have the following three equivalent characterisations of strict polynomial functors. The words *(uniformly) bounded degree* will here be taken to mean "strict polynomial of bounded degree".

Theorem 6. — *Consider the following constructs, where A ranges over all algebras:*

A. A family of ordinary functors $E_A : {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$, commuting with extension of scalars.

B. A functor $J\colon \mathfrak{XMod} \to \mathfrak{Mod}$ with arrow maps

$$J_A\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N))$$

that are multiplicative and natural in A.

C. A functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ with arrow maps

$$F_A : A \otimes \mathrm{Hom}_{\mathbf{B}}(M, N) \to A \otimes \mathrm{Hom}_{\mathbf{B}}(F(M), F(N))$$

that are multiplicative and natural in A (strict polynomial maps).

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow maps are assumed to have uniformly bounded degree, all three are equivalent.

Proof. The proof is exactly analogous to the one given for polynomial functors, except that, in the proof that B implies C, the module

$$\bigoplus_{k=0}^{n} \Gamma^k \mathrm{Hom}(M, N)$$

is used in place of $\mathbf{B}[\mathrm{Hom}(M, N)]_n$. The details are found in [20]. □

As in the numerical case, we obtain the following hierarchy.

- *Strict polynomial functors*, as defined previously, have bounded degree, and satisfy all three conditions A, B and C.
- A functor satisfying condition C, but with no assumption on the degree, will be called **locally strict polynomial**.
- A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **strict analytic**.

Example 11. — The functors T, S, and Γ are in fact strict analytic, and Λ is locally strict polynomial. △

§8. Homogeneous Functors

There are some superficial similarities between numerical and strict polynomial functors — the first and most glaringly obvious one being their respective definitions. Parts of their theories indeed run exactly in parallel. Therefore, it may come as a surprise that, in fact, some very fundamental differences exist between the two genera.

There is especially one property of strict polynomial functors that is lacking for numerical ones, and that is the ability to split into a sum of homogeneous components. We saw this happen already in the case of maps.

Definition 6. — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **homogeneous** of degree n if the arrow maps

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

have been given a (multiplicative) homogeneous structure. ⋄

The category of homogeneous functors will be denoted by

$$\mathfrak{Hom}_n.$$

We shall always prefer this category over $\mathfrak{SPol}_n$, since, according to the following theorem, nothing essential will be lost by considering homogeneous functors only.

Theorem 7. — *A strict polynomial functor decomposes as a unique direct sum of homogeneous functors. The only possible natural transformation between homogeneous functors of different degrees is the zero transformation. Consequently,*

$$\mathfrak{SPol}_n = \bigoplus_{k=0}^{n} \mathfrak{Hom}_k.$$

Proof. See [20]. □

Example 12. — If F_k is homogeneous of degree k, the direct sum

$$\bigoplus_{k=0}^{n} F_k$$

will provide a strict polynomial functor of degree n. By the theorem, this is the generic situation. On the other hand, a numerical functor is in general "more than the sum of its parts". An example of this phenomenon cannot be given at this stage, since we do not yet know of a functor that is numerical, but not strict polynomial. △

§9. Analytic Functors

We now examine the analytic and strict analytic functors. They will come into play later, when we consider operads.

Theorem 8. — *The strict analytic functors are precisely the direct sums (or, equivalently, inductive limits) of strict polynomial functors.*

Proof. See [20]. □

Example 13. — The strict analytic functors T, S, Λ, and Γ all decompose as infinite direct sums of homogeneous functors. This is the generic situation. △

Lemma 1. — *Let F be an analytic functor, let $u \in F(P)$, and define the subfunctor G by*

$$G(M) = \langle F(\alpha)(u) \,|\, \alpha\colon P \to M \rangle .$$

Consider the natural transformation

$$\xi\colon \operatorname{Hom}(P, -) \to F,$$

given by

$$\begin{aligned} \xi_N\colon \operatorname{Hom}(P, N) &\to F(N) \\ \alpha &\mapsto F(\alpha)(u). \end{aligned}$$

If ξ_N is numerical of degree n, then so is

$$G\colon (M, N) \to \operatorname{Hom}(G(M), G(N))$$

for all M. In particular:

- *If all ξ_N are numerical, then G is locally numerical.*
- *If all ξ_N are numerical of bounded degree, then G is numerical.*

Proof. Observe that the modules $G(M)$ are invariant under the action of F. Thus, G is indeed a subfunctor of F.

Suppose ξ_N is numerical of degree n. Then, for all homomorphisms

$$\alpha, \alpha_i\colon P \to N$$

and scalars r, the following equations hold:

$$\begin{cases} F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1})(u) = 0 \\ F(r\alpha)(u) = \sum_{m=0}^{n} \binom{r}{m} F\left(\Diamond_m \alpha\right)(u). \end{cases}$$

This implies that, for all homomorphisms

$$\beta, \beta_i\colon M \to N, \qquad \gamma\colon P \to M,$$

and scalars r, the following equations hold:

$$\begin{cases} F(\beta_1 \diamond \cdots \diamond \beta_{n+1})F(\gamma)(u) = 0 \\ F(r\beta)F(\gamma)(u) = \sum_{m=0}^{n} \binom{r}{m} F\left(\diamond_m \beta\right) F(\gamma)(u). \end{cases}$$

Hence

$$F(\beta_1 \diamond \cdots \diamond \beta_{n+1}) = 0$$

and

$$F(r\beta) = \sum_{m=0}^{n} \binom{r}{m} F\left(\underset{m}{\diamond} \beta\right)$$

on $G(M)$, which means that every

$$G\colon (M, N) \to \mathrm{Hom}(G(M), G(N))$$

is indeed numerical of degree n. □

Theorem 9. — *The analytic functors are precisely the inductive limits of numerical functors.*

Proof. Step 1: Inductive limits of numerical, or even analytic, functors are analytic. Let the functors F_i, for $i \in I$, be analytic. For any

$$\alpha \in \mathrm{Hom}_A(A \otimes M, A \otimes N),$$

we have

$$F_i(\alpha)\colon A \otimes F_i(M) \to A \otimes F_i(N).$$

Therefore

$$\varinjlim F_i(\alpha)\colon A \otimes \varinjlim F_i(M) \to A \otimes \varinjlim F_i(N),$$

since tensor products commute with inductive limits, which yields a map

$$\varinjlim F_i\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes \varinjlim F_i(M), A \otimes \varinjlim F_i(N)),$$

establishing that $\varinjlim F_i$ is analytic.

Step 2: Analytic functors are inductive limits of locally numerical functors. Let F be analytic. The maps

$$F\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes F(M), A \otimes F(N))$$

are then multiplicative and natural in A. To show F is the inductive limit of locally numerical functors, it is sufficient to construct, for any given module P and element $u \in F(P)$, a locally numerical subfunctor G of F, such that $u \in G(P)$.

To this end, define G as in the lemma:

$$G(M) = \langle F(\alpha)(u) \,|\, \alpha\colon P \to M \rangle.$$

Clearly $u \in G(P)$. By the lemma, G is locally numerical, if only we can show that

$$\xi_M\colon \operatorname{Hom}(P, M) \to F(M)$$

is always numerical (of possibly unbounded degree).

We make use of the following fact. The module $\operatorname{Hom}(P, M)$ is finitely generated and free, because P and M are. Let $\varepsilon_1, \ldots, \varepsilon_k$ be a basis.

Let $s_1, \ldots, s_k$ be free variables, and let

$$A = \mathbf{B}\binom{s_1, \ldots, s_k}{-}.$$

Since

$$F\left(\sum s_i \otimes \varepsilon_i\right) \in \operatorname{Hom}_A(A \otimes F(P), A \otimes F(M)),$$

we may write

$$F\left(\sum s_i \otimes \varepsilon_i\right)(1_A \otimes u) = \sum_{|X| \leqslant n} \binom{s}{X} \otimes v_X \in A \otimes F(M)$$

for some n, and hence

$$\xi_M\left(\sum s_i \varepsilon_i\right) = F\left(\sum s_i \varepsilon_i\right)(u) = \sum_{|X| \leqslant n} \binom{s}{X} v_X.$$

Since the ε_i generate $\operatorname{Hom}(P, M)$, it follows that ξ_M is numerical of degree n.

Step 3: Locally numerical functors are inductive limits of numerical functors. Let F be locally numerical, and, given P and $u \in F(P)$, define G and ξ as before. We shall show that G is numerical by showing that ξ is numerical of some fixed degree.

Let $\alpha_i\colon P \to M$ be homomorphisms, let

$$B = \mathbf{B}\binom{s_1, \ldots, s_k}{-}, \qquad C = \mathbf{B}\binom{s_1, \ldots, s_k, t}{-}$$

be free numerical rings, and consider the algebra homomorphism

$$\tau\colon B \to C, \quad s_i \mapsto t s_i.$$

There is a commutative diagram:

$$\begin{array}{ccccc}
B & \qquad & B \otimes \operatorname{Hom}(P, M) & \xrightarrow{F} & B \otimes \operatorname{Hom}(F(P), F(M)) \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau\otimes 1} & & \downarrow{\scriptstyle \tau\otimes 1} \\
C & & C \otimes \operatorname{Hom}(P, M) & \xrightarrow[F]{} & C \otimes \operatorname{Hom}(F(P), F(M))
\end{array}$$

As a consequence, we obtain, for any homomorphisms $\alpha_i\colon P \to M$:

$$\begin{array}{ccc}
\sum s_i \otimes \alpha_i & \longrightarrow & F\left(\sum s_i \otimes \alpha_i\right) \\
\big\downarrow & & \big\downarrow \\
\sum ts_i \otimes \alpha_i & \longrightarrow & \left[\begin{array}{c}(\tau \otimes 1)F\left(\sum s_i \otimes \alpha_i\right) \\ = F\left(\sum ts_i \otimes \alpha_i\right)\end{array}\right]
\end{array}$$

Consider now

$$F\colon \mathbf{B}\binom{s_1, \ldots, s_k}{-} \otimes \mathrm{Hom}(P, M) \to \mathbf{B}\binom{s_1, \ldots, s_k}{-} \otimes \mathrm{Hom}(F(P), F(M)),$$

and write

$$F\left(\sum s_i \otimes \alpha_i\right) = \sum_X \binom{s}{X} \otimes \beta_X,$$

for some homomorphisms $\beta_X\colon F(P) \to F(M)$.

Similarly, from contemplating

$$F\colon \mathbf{B}\binom{t}{-} \otimes \mathrm{Hom}(P, P) \to \mathbf{B}\binom{t}{-} \otimes \mathrm{Hom}(F(P), F(P)),$$

we may write

$$F(t \otimes 1_P) = \sum_{m \leqslant n} \binom{t}{m} \otimes \gamma_m,$$

for some number n and homomorphisms $\gamma_m\colon F(P) \to F(P)$. Observe that n is fixed, and only depends on F.

We now have

$$\begin{aligned}
\sum_X \binom{ts}{X} \otimes \beta_X &= (\tau \otimes 1)\left(\sum_X \binom{s}{X} \otimes \beta_X\right) \\
&= (\tau \otimes 1)F\left(\sum s_i \otimes \alpha_i\right) \\
&= F\left(\sum ts_i \otimes \alpha_i\right) \\
&= F\left(\sum s_i \otimes \alpha_i\right) F(t \otimes 1_P) \\
&= \left(\sum_X \binom{s}{X} \otimes \beta_X\right)\left(\sum_{m \leqslant n} \binom{t}{m} \otimes \gamma_m\right) \\
&= \sum_X \sum_{m \leqslant n} \binom{s}{X}\binom{t}{m} \otimes \beta_X \gamma_m.
\end{aligned}$$

The right-hand side, and therefore also the left-hand side, is of degree n in t, whence $\beta_X = 0$ when $|X| > n$.

Consequently,

$$\xi_M\left(\sum s_i\alpha_i\right) = F\left(\sum s_i\alpha_i\right)(u) = \sum_{|X|\leqslant n}\binom{s}{X}\beta_X(u),$$

and ξ is numerical of degree n. □

Chapter 7

DEVIATIONS AND CROSS-EFFECTS

> [...] je donnerais bien cent sous au mathématicien qui me démontrerait par une équation algébrique l'existence de l'enfer.
>
> — Honoré de Balzac, *La Peau de chagrin*

We have now arrived at the arguably most technical chapter of the monograph, being virtually nothing more than a collection of formulæ. These will be established with the pronounced goal of examining the internal structure of functors, essentially "taking them apart to see what makes them tick". Mechanics employ a technical term for this process: *reverse engineering*.

The central concept in the theory of polynomial functors has hitherto been that of *cross-effects*. Ever since they were first designed by Professors Eilenberg and Mac Lane in 1954 ([6]), they have been the algebraist's vivisection tool *par préférence*, being to module *functors* what the deviations are to module *maps*.

Recall that the first deviation measures how much a module map φ deviates from being affine, as testified by the equation

$$\varphi(x+y) = \varphi(\diamond) + \varphi(\diamond x) + \varphi(\diamond y) + \varphi(x \diamond y).$$

A corresponding equation for functors then ensues, namely the direct sum decomposition

$$F(X \oplus Y) = F^{\dagger}() \oplus F^{\dagger}(X) \oplus F^{\dagger}(Y) \oplus F^{\dagger}(X \mid Y),$$

valid for any module functor F, and any modules X and Y. The quantities $\varphi(\diamond)$ and $F^{\dagger}()$ are the "constant terms". Disregarding these, the first cross-effect $F^{\dagger}(X \mid Y)$, like the first deviation $\varphi(x \diamond y)$, measures the deviation from additivity. Indeed, *polynomial* functors of degree n may be defined, as did originally Eilenberg and Mac Lane ([6]), as those with vanishing nth cross-effects.

But the cross-effects carry an inherent deficiency, which greatly reduces their utility, in that they do not take scalar multiplication into account. Of course, neither did the deviations, but there we were able to save the day by inventing a second equation involving binomial coefficients, marking the transition from *polynomial* to *numerical* maps. This flexibility is lost when passing to cross-effects, which would seem to render them somewhat obsolete.

Nevertheless, we shall later see how to put them to good use.

§1. The Deviations

We first make a more detailed study of deviations in the context of functors. We shall obtain formulæ for the composition of deviations, the deviations of a tensor product, and the deviations of a composition. The results are valid for any module functors F and G over any base ring $\mathbf{B}$.

The symbol

$$M \sqsubseteq X \times Y$$

shall denote that M is a subset of $X \times Y$, and that both the canonical projections are *onto*.

Lemma 1. — *Let m and n be natural numbers, and let $L \subseteq [m] \times [n]$. Then*

$$\sum_{L \subseteq K \sqsubseteq [m] \times [n]} (-1)^{|K|} = 0,$$

unless L is of the form $P \times Q$, for some $P \subseteq [m]$, $Q \subseteq [n]$.

Proof. If L is not of the given form, there exists an (a, b), which is not in L, but such that some (a, j) and some (i, b) are in L. Then, for any set $K \subseteq [m] \times [n]$ containing (a, b), K will satisfy the given set inclusions iff $K \backslash \{(a, b)\}$ does. Because the cardinalities of these sets differ by 1, the corresponding terms in the above sum will have opposite signs, and hence cancel. □

Lemma 2. — *Let m, n, p, and q be natural numbers. Then*

$$\sum_{[p] \times [q] \subseteq K \sqsubseteq [m] \times [n]} (-1)^{|K|} = (-1)^{m+n+p+q+pq}.$$

Proof. Let $Y(m, n, k)$ denote the number of sets K of cardinality k satisfying

$$[p] \times [q] \subseteq K \sqsubseteq [m] \times [n].$$

The formula is evidently true for $m = p$ and $n = q$, for then $Y(p, q, pq) = 1$, and all other $Y(p, q, k) = 0$.

We now do recursion. Consider the pair $(m, n) \in [m] \times [n]$. The sets K containing (m, n) will fall into two classes: those where (m, n) is *mandatory* in order to satisfy $K \sqsubseteq [m] \times [n]$, and those where it is not. For the latter class we may proceed as in the preceding proof. Taking such a K and removing (m, n) will yield another set counted in the sum above, but of cardinality decreased by 1. Since these two types of sets exactly pair off, with opposing signs, their contribution to the given sum is 0.

Consider then those K of which (m, n) *is* a mandatory element. They fall into three categories:

- Some $(m,j) \in K$, for $1 \leqslant j \leqslant n-1$, but no $(i,n) \in K$, for $1 \leqslant i \leqslant m-1$. The number of such sets is $Y(m, n-1, k-1)$.
- No $(m,j) \in K$, for $1 \leqslant j \leqslant n-1$, but some $(i,n) \in K$, for $1 \leqslant i \leqslant m-1$. The number of such sets is $Y(m-1, n, k-1)$.
- No $(m,j) \in K$, for $1 \leqslant j \leqslant n-1$, and no $(i,n) \in K$, for $1 \leqslant i \leqslant m-1$. The number of such sets is $Y(m-1, n-1, k-1)$.

Induction yields

$$\begin{aligned}
&\sum_k (-1)^k Y(m,n,k) \\
&= \sum_k (-1)^k \big(Y(m, n-1, k-1) + Y(m-1, n, k-1) + Y(m-1, n-1, k-1)\big) \\
&= -\big((-1)^{m+n-1+p+q+pq} + (-1)^{m-1+n+p+q+pq} + (-1)^{m-1+n-1+p+q+pq}\big) \\
&= (-1)^{m+n+p+q+pq},
\end{aligned}$$

as desired. □

Theorem 1: The Deviation Formula.

$$F(\alpha_1 \diamond \cdots \diamond \alpha_m) \circ F(\beta_1 \diamond \cdots \diamond \beta_n) = \sum_{K \subseteq [m] \times [n]} F\left(\Diamond_{(i,j) \in K} \alpha_i \beta_j \right).$$

Proof. We have

$$\begin{aligned}
\sum_{K \subseteq [m] \times [n]} F\left(\Diamond_{(i,j) \in K} \alpha_i \beta_j \right) &= \sum_{K \subseteq [m] \times [n]} \sum_{L \subseteq K} (-1)^{|K|-|L|} F\left(\sum_{(i,j) \in L} \alpha_i \beta_j \right) \\
&= \sum_{L \subseteq [m] \times [n]} \sum_{L \subseteq K \subseteq [m] \times [n]} (-1)^{|K|-|L|} F\left(\sum_{(i,j) \in L} \alpha_i \beta_j \right) \\
&= \sum_{L \subseteq [m] \times [n]} (-1)^{|L|} F\left(\sum_{(i,j) \in L} \alpha_i \beta_j \right) \sum_{L \subseteq K \subseteq [m] \times [n]} (-1)^{|K|} \\
&= \sum_{P \times Q \subseteq [m] \times [n]} (-1)^{|P||Q|} F\left(\sum_{(i,j) \in P \times Q} \alpha_i \beta_j \right) (-1)^{m+n+|P|+|Q|+|P||Q|} \\
&= \sum_{P \subseteq [m]} (-1)^{m-|P|} F\left(\sum_{i \in P} \alpha_i \right) \sum_{Q \subseteq [n]} (-1)^{n-|Q|} F\left(\sum_{j \in Q} \beta_j \right) \\
&= F(\alpha_1 \diamond \cdots \diamond \alpha_m) F(\beta_1 \diamond \cdots \diamond \beta_n).
\end{aligned}$$

In the fifth step the lemmata were used to evaluate the inner sum. □

LEMMA 3. — *Let n be a natural number, and let $P, Q \subseteq [n]$. Then*

$$\sum_{\substack{P\subseteq I\\ Q\subseteq J\\ I\cup J=[n]}} (-1)^{|I|+|J|} = 0,$$

unless $P = Q$.

Proof. Compute

$$\begin{aligned}\sum_{\substack{P\subseteq I\\ Q\subseteq J\\ I\cup J=[n]}} (-1)^{|I|+|J|} &= \sum_{P\subseteq I\subseteq[n]} \sum_{\substack{Q\subseteq J\subseteq[n]\\ [n]\setminus I\subseteq J}} (-1)^{|I|+|J|}\\ &= \sum_{P\subseteq I\subseteq[n]} (-1)^{|I|} \sum_{Q\cup([n]\setminus I)\subseteq J\subseteq[n]} (-1)^{|J|},\end{aligned}$$

and note that the inner sum vanishes unless $Q \cup ([n]\setminus I) = [n]$, or, equivalently, $I \subseteq Q$. Hence

$$\sum_{\substack{P\subseteq I\\ Q\subseteq J\\ I\cup J=[n]}} (-1)^{|I|+|J|} = \sum_{P\subseteq I\subseteq Q} (-1)^{|I|+n} = 0,$$

when $P \neq Q$. □

LEMMA 4. — *Let n be a natural number, and let $P \subseteq [n]$. Then*

$$\sum_{\substack{P\subseteq I,J\\ I\cup J=[n]}} (-1)^{|I|+|J|} = (-1)^{n-|P|}.$$

Proof. By the preceding proof,

$$\sum_{\substack{P\subseteq I,J\\ I\cup J=[n]}} (-1)^{|I|+|J|} = \sum_{I=P} (-1)^{|I|+n} = (-1)^{n-|P|}.$$

□

THEOREM 2.

$$(F\otimes G)(\alpha_1 \diamond \cdots \diamond \alpha_n) = \sum_{I\cup J=[n]} F\left(\Diamond_{i\in I} \alpha_i\right) \otimes G\left(\Diamond_{j\in J} \alpha_j\right).$$

Proof. Compute:

$$\sum_{I\cup J=[n]} F\left(\Diamond_{i\in I} \alpha_i\right) \otimes G\left(\Diamond_{j\in J} \alpha_j\right)$$

$$= \sum_{I\cup J=[n]} \left(\sum_{P\subseteq I}(-1)^{|I|-|P|}F\left(\sum_{i\in P}\alpha_i\right)\right) \otimes \left(\sum_{Q\subseteq J}(-1)^{|J|-|Q|}G\left(\sum_{j\in Q}\alpha_j\right)\right)$$

$$= \sum_{P,Q\subseteq[n]} (-1)^{|P|+|Q|} \left(F\left(\sum_{i\in P}\alpha_i\right)\otimes G\left(\sum_{j\in Q}\alpha_j\right)\right) \sum_{\substack{P\subseteq I\\ Q\subseteq J\\ I\cup J=[n]}} (-1)^{|I|+|J|}$$

$$= \sum_{P\subseteq[n]} (-1)^{|P|+|P|}(-1)^{n-|P|} \left(F\left(\sum_{i\in P}\alpha_i\right)\otimes G\left(\sum_{j\in P}\alpha_j\right)\right)$$

$$= \sum_{P\subseteq[n]} (-1)^{n-|P|}(F\otimes G)\left(\sum_{i\in P}\alpha_i\right) = (F\otimes G)(\alpha_1\diamond\cdots\diamond\alpha_n).$$

In the fourth step, the lemmata were used. □

Let X be a set. We shall write

$$M \lhd X$$

to indicate that $M \subseteq 2^X$ (that is, M is a family of subsets of X), such that $M \nsubseteq 2^Y$ for any $Y \subseteq X$. Equivalently,

$$\bigcup M = \bigcup_{Z\in M} Z = X.$$

Lemma 5. — *Let n be a natural number, and let $P \subseteq [n]$. Then*

$$\sum_{P\subseteq J\lhd[n]} (-1)^{|J|} = 0,$$

unless P is of the form 2^Q, for some $Q \subseteq [n]$.

Proof. If $P \subset 2^{\bigcup P}$, choose $A \in 2^{\bigcup P}\setminus P$. Because the set J satisfies

$$P \subseteq J \lhd [n]$$

iff it satisfies

$$P \subseteq J \cup \{A\} \lhd [n],$$

sets J with and without A will cancel each other out in the sum. □

Lemma 6. — *Let n be a natural number, and let $Q \subseteq [n]$. Then*

$$\sum_{2^Q\subseteq J\lhd[n]} (-1)^{|J|} = (-1)^{n-|Q|-2^{|Q|}}.$$

Proof. The formula is clearly valid when $n = |Q|$. Now assume $n > |Q|$, and consider for the moment the sets J satisfying

$$J \backslash (2^{[n-1]} \cup \{\{n\}\}) \neq \emptyset.$$

(There is thus a set $Y \in J$ with $n \in Y$, but $Y \neq \{n\}$.) Because such a set J satisfies

$$2^Q \subseteq J \lhd [n]$$

iff it satisfies

$$2^Q \subseteq J \cup \{n\} \lhd [n],$$

sets J with and without $\{n\}$ will cancel each other out.

There remain the sets J with

$$J \subseteq 2^{[n-1]} \cup \{\{n\}\}.$$

Since $J \lhd [n]$, necessarily $\{n\} \in J$, and we may write $J = K \cup \{\{n\}\}$ for $K \subseteq 2^{[n-1]}$. We then have

$$\sum_{2^Q \subseteq J \lhd [n]} (-1)^{|J|} = \sum_{2^Q \subseteq K \lhd [n-1]} (-1)^{|K|+1},$$

and the formula follows by induction. □

THEOREM 3.

$$(F \circ G)(\alpha_1 \diamond \cdots \diamond \alpha_n) = \sum_{J \lhd [n]} F\left(\Diamond_{I \in J} G\left(\Diamond_{i \in I} \alpha_i \right) \right).$$

Proof. Compute:

$$\begin{aligned}
&\sum_{J \lhd [n]} F\left(\Diamond_{I \in J} G\left(\Diamond_{i \in I} \alpha_i \right) \right) \\
&\quad = \sum_{J \lhd [n]} \sum_{P \subseteq J} (-1)^{|J|-|P|} F\left(\sum_{I \in P} G\left(\Diamond_{i \in I} \alpha_i \right) \right) \\
&\quad = \sum_{P \subseteq 2^{[n]}} (-1)^{|P|} F\left(\sum_{I \in P} G\left(\Diamond_{i \in I} \alpha_i \right) \right) \sum_{P \subseteq J \lhd [n]} (-1)^{|J|} \\
&\quad = \sum_{Q \subseteq [n]} (-1)^{2^{|Q|}} F\left(\sum_{I \in 2^Q} G\left(\Diamond_{i \in I} \alpha_i \right) \right) (-1)^{n-|Q|-2^{|Q|}} \\
&\quad = \sum_{Q \subseteq [n]} (-1)^{n-|Q|} F\left(G\left(\sum_{i \in Q} \alpha_i \right) \right) \\
&\quad = (F \circ G)(\alpha_1 \diamond \cdots \diamond \alpha_n).
\end{aligned}$$

In the fourth step, the lemmata were used. □

§2. The Cross-Effects

An arbitrary module functor may be analysed in terms of its *cross-effects*. We here indicate how this is done. Again, no assumptions are placed on either the functors or the base ring.

Example 1. — Consider the second symmetric functor S^2, and let X and Y be modules. We see that

$$S^2(X) = \langle x_1 x_2 \mid x_i \in X \rangle$$

contains the monomials obtained from X, and

$$S^2(Y) = \langle y_1 y_2 \mid y_i \in Y \rangle$$

the monomials from Y only. If S^2 were additive, we would have

$$S^2(X \oplus Y) = S^2(X) \oplus S^2(Y).$$

But not so; mixed terms of the form xy will appear in $S^2(X \oplus Y)$, terms that were not present in either $S^2(X)$ or $S^2(Y)$. This is the *cross-effect* $(S^2)^\dagger(X \mid Y)$. △

The cross-effects may be defined as either of four equally canonical modules. Given a direct sum $M = M_1 \oplus \cdots \oplus M_n$, let

$$\pi_j \colon M \to M$$

be projection on the jth summand,

$$\rho_j \colon M \to M/M_j$$

retraction from the jth summand, and

$$\tau_j \colon M/M_j \to M$$

insertion of 0 into the jth summand.

Theorem 4. — *For a module functor F, the following four modules are naturally isomorphic:*

$$\begin{aligned}
A &= \operatorname{Im}\big[F(\pi_1 \diamond \cdots \diamond \pi_n) \colon F(M) \to F(M)\big] \\
B &= \operatorname{Ker}\big[(F(\rho_1), \ldots, F(\rho_n)) \colon F(M) \to \textstyle\bigoplus F(M/M_j)\big] \\
C &= \operatorname{Coker}\big[F(\tau_1) + \cdots + F(\tau_n) \colon \textstyle\bigoplus F(M/M_j) \to F(M)\big] \\
D &= \operatorname{Coim}\big[F(\pi_1 \diamond \cdots \diamond \pi_n) \colon F(M) \to F(M)\big].
\end{aligned}$$

Proof. We only show that modules $A = B$, and leave the remaining cases to the reader.

Suppose

$$z \in \operatorname{Ker}(F(\rho_1), \ldots, F(\rho_n)).$$

Note that if $j \neq i$, then $\pi_i \tau_j \rho_j = \pi_i$. Consequently, if $j \notin I$, then

$$F\left(\sum_{i\in I} \pi_i\right)(z) = F\left(\sum_{i\in I} \pi_i\right) F(\tau_j) F(\rho_j)(z) = 0.$$

It follows that

$$\begin{aligned} F(\pi_1 \diamond \cdots \diamond \pi_n)(z) &= \sum_{I\subseteq[n]} (-1)^{n-|I|} F\left(\sum_{i\in I} \pi_i\right)(z) \\ &= F\left(\pi_1 + \cdots + \pi_n\right)(z) = F(1)(z) = z. \end{aligned}$$

Conversely, assume

$$z = F(\pi_1 \diamond \cdots \diamond \pi_n)(y) \in \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n).$$

Then, since $\rho_j \pi_j = 0$, we get

$$\begin{aligned} F(\rho_j)(z) = F(\rho_j) F(\pi_1 \diamond \cdots \diamond \pi_n)(y) &= \sum_{I\subseteq[n]} (-1)^{n-|I|} F\left(\rho_j \sum_{i\in I} \pi_i\right)(y) \\ = \sum_{I\subseteq[n]} (-1)^{n-|I|} F\left(\sum_{i\in I\setminus\{j\}} \rho_j \pi_i\right)(y) &= 0, \end{aligned}$$

because sets I with and without j will give cancelling terms. □

Definition 1. — The $(n-1)$st **cross-effect** of F is the multi-functor

$$F^\dagger\left(M_1 \mid \cdots \mid M_n\right)$$

of n arguments, defined as any of the four modules above. ◇

For our purposes, it shall be most convenient to view the cross-effect as

$$F^\dagger\left(M_1 \mid \cdots \mid M_n\right) = \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n).$$

It is also the definition which generalises most readily to yield the *multi-cross-effects* of a strict analytic functor.

Let us now describe, for given homomorphisms $\alpha_i \colon M_i \to M_i'$, how to form the map

$$F^\dagger\left(\alpha_1 \mid \cdots \mid \alpha_n\right) : F^\dagger\left(M_1 \mid \cdots \mid M_n\right) \to F^\dagger\left(M_1' \mid \cdots \mid M_n'\right).$$

It is easy to verify the commutativity of the following diagram:

$$\begin{array}{ccc} F(\bigoplus M_i) & \xrightarrow{F(\bigoplus \alpha_i)} & F(\bigoplus M_i') \\ {\scriptstyle F(\diamond \pi_i)}\downarrow & & \downarrow{\scriptstyle F(\diamond \pi_i')} \\ F(\bigoplus M_i) & \xrightarrow[F(\bigoplus \alpha_i)]{} & F(\bigoplus M_i') \end{array}$$

Therefore, there will be an induced homomorphism

$$\operatorname{Im} F\left(\Diamond\, \pi_i\right) \to \operatorname{Im} F\left(\Diamond\, \pi_i'\right),$$

which we define to be $F^\dagger(\alpha_1 \mid \cdots \mid \alpha_n)$. It will now be readily checked that, in fact,

$$F^\dagger(\alpha_1 \mid \cdots \mid \alpha_n) = F(\alpha_1 \diamond \cdots \diamond \alpha_n).$$

When presented with a (finite) family $(M_i)_{i\in I}$ of modules, we shall write

$$F^\dagger(M_i|_I)$$

for the cross-effect of the modules M_i.

Theorem 5: The Cross-Effect Decomposition.

$$F(M_1 \oplus \cdots \oplus M_k) = \bigoplus_{I\subseteq[k]} F^\dagger(M_i|_{i\in I}).$$

Proof. See [6]. □

This decomposition is evidently functorial, and there is a corresponding decomposition of natural transformations $\zeta\colon F \to G$:

$$\zeta_{M_1\oplus\cdots\oplus M_k} = \bigoplus_{I\subseteq[k]} \left[\zeta^\dagger_{M_i|_{i\in I}}\colon F^\dagger(M_i|_{i\in I}) \to G^\dagger(M_i|_{i\in I})\right].$$

Theorem 6. — *F is polynomial of degree n iff its nth cross-effect vanishes.*

Proof. Suppose the nth cross-effect vanishes, and consider $n+1$ homomorphisms

$$\alpha_1, \ldots, \alpha_{n+1}\colon M \to N.$$

Let $M_j = M$ and $N_j = N$, for $j = 1, \ldots, n+1$, let

$$\pi_j\colon \bigoplus N_i \to \bigoplus N_i$$

denote the jth projection, and define

$$\sigma\colon \bigoplus N_i \to N, \quad (y_1, \ldots, y_{n+1}) \mapsto \sum y_i.$$

The equality

$$F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1}) = F(\sigma) \circ F(\pi_1 \diamond \cdots \diamond \pi_{n+1}) \circ F((\alpha_1, \ldots, \alpha_{n+1}))$$

is easily checked, for maps

$$F(N) \leftarrow F\left(\bigoplus N_i\right) \leftarrow F\left(\bigoplus N_i\right) \leftarrow F(M).$$

Since the middle component is zero by assumption, we conclude that

$$F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1}) = 0,$$

and so F is polynomial of degree n.

The other direction is trivial. □

Example 2. — For any functor F,

$$F^\dagger() = F(0)$$

is the "constant term" of F.

If F is *reduced*, that is, $F(0) = 0$; then the zeroth cross-effect coincides with the functor itself:

$$F^\dagger(X) = F(X).$$

In the general case,

$$F(X) = F^\dagger() \oplus F^\dagger(X).$$

△

Example 3. — If F is polynomial of degree 1, then all cross-effects of order higher than 0 vanish, and

$$F(X \oplus Y) = F^\dagger() \oplus F^\dagger(X) \oplus F^\dagger(Y).$$

△

Example 4. — For a functor of degree 2, all cross-effects of order higher than 1 vanish, and

$$F(X \oplus Y) = F^\dagger() \oplus F^\dagger(X) \oplus F^\dagger(Y) \oplus F^\dagger(X \mid Y).$$

To take a concrete example, we have for S^2:

$$\begin{aligned}(S^2)^\dagger() &= 0\\ (S^2)^\dagger(X) &= \langle x_1 x_2 \mid x_i \in X\rangle\\ (S^2)^\dagger(Y) &= \langle y_1 y_2 \mid y_i \in Y\rangle\\ (S^2)^\dagger(X \mid Y) &= \langle x_1 y_1 \mid x_i \in X,\ y_i \in Y\rangle .\end{aligned}$$

The equation

$$(S^2)^\dagger(X \mid Y \mid Z) = 0$$

allows for the following interpretation in words: there are no monomials of degree 2 involving elements from all three modules X, Y, and Z. △

§3. The Multi-Deviations

The deviations, which exist for arbitrary maps and functors, find for strict polynomial maps and functors their equivalent in the *multi-deviations*, introduced earlier. We shall derive formulæ for the composition of multi-deviations, the multi-deviations of a tensor product, and the multi-deviations of a composition. Perhaps surprisingly, the proofs will here be substantially shorter, because no combinatorial tinkering will be necessary.

Throughout this section and the next, F and G will denote strict analytic functors over a commutative base ring. Let $\alpha_i\colon M \to N$ be homomorphisms. Recall that the maps

$$F_{\alpha^{[A]}}\colon F(M) \to F(N)$$

are defined by the universal validity of the equation

$$F\left(\sum_i a_i \otimes \alpha_i\right) = \sum_A a^A \otimes F_{\alpha^{[A]}}.$$

We have mentioned the formula

$$x^{[A]} \star y^{[B]} = \sum_{\mu\colon A\to B} (xy)^{[\mu]}$$

for the product multiplication on the divided power algebras. Its functorial generalisation is as follows.

Theorem 7.

$$F_{\alpha^{[A]}} \circ F_{\beta^{[B]}} = \sum_{\mu\colon A\to B} F_{(\alpha\circ\beta)^{[\mu]}}.$$

Proof. Identify the coefficient of $a^A b^B$ in

$$\begin{aligned}\left(\sum_A a^A \otimes F_{\alpha^{[A]}}\right) \circ \left(\sum_B b^B \otimes F_{\beta^{[B]}}\right) &= F\left(\sum_i a_i \otimes \alpha_i\right) \circ F\left(\sum_j b_j \otimes \beta_j\right)\\ &= F\left(\sum_{i,j} a_i b_j \otimes \alpha_i\beta_j\right)\\ &= \sum_{\mu\colon A\to B} (ab)^{\mu} \otimes F_{(\alpha\circ\beta)^{[\mu]}}.\end{aligned}$$

□

Theorem 8.

$$(F\otimes G)_{\alpha^{[X]}} = \sum_{A\sqcup B=X} F_{\alpha^{[A]}} \otimes G_{\alpha^{[B]}}.$$

Proof. Identify the coefficient of a^X in

$$\begin{aligned}(F\otimes G)\left(\sum_i a_i \otimes \alpha_i\right) &= F\left(\sum_i a_i \otimes \alpha_i\right) \otimes G\left(\sum_i a_i \otimes \alpha_i\right)\\ &= \left(\sum_A a^A \otimes F_{\alpha^{[A]}}\right) \otimes \left(\sum_B a^B \otimes G_{\alpha^{[B]}}\right).\end{aligned}$$

□

Let G be a functor, and let α_k, for $k \in X$, be homomorphisms. If E is a partition of X, we define the symbol

$$G_\alpha^{[E]} = \bigodot_{Y\in E} G_{\alpha[Y]} = \prod_{Y\in\#E} G_{\alpha[Y]}^{[\deg_E Y]}.$$

Thus, for example, if

$$E = \{\{1,1\},\{2,3\},\{2,3\}\}$$

is a partition of $X = \{1,1,2,2,3,3\}$, then

$$G_\alpha^{[E]} = G_{\alpha[\{1,1\}]}^{[1]} G_{\alpha[\{2,3\}]}^{[2]} = G_{\alpha_1^{[2]}} G_{\alpha_2\alpha_3}^{[2]}.$$

Theorem 9.

$$(F\circ G)_{\alpha[X]} = \sum_{E\in\operatorname{Par} X} F_{G_\alpha^{[E]}}.$$

Proof. Identify the coefficient of a^X in

$$\begin{aligned}(F\circ G)\left(\sum_i a_i\otimes\alpha_i\right) &= F\left(G\left(\sum_i a_i\otimes\alpha_i\right)\right)\\ &= F\left(\sum_Y a^Y\otimes G_{\alpha[Y]}\right)\\ &= \sum_X\sum_{E\in\operatorname{Par} X}\left(\prod_{Y\in E} a^Y\right)\otimes F_{\bigodot_{Y\in E} G_{\alpha[Y]}}.\end{aligned}$$

□

§4. The Multi-Cross-Effects

The cross-effects of a strict analytic functor may be further dissected into so called *multi-cross-effects*. Let us begin with an example to illustrate the concept.

Example 5. — Let X and Y be modules, and consider the third symmetric functor S^3. We know from before that the cross-effect $(S^3)^\dagger(X \mid Y)$ contains the cubic monomials built by elements of both X and Y. This module splits canonically into two components:

$$(S^3)^\dagger(X\mid Y) = \left\langle x_1x_2y_1 \mid x_i\in X,\ y_j\in Y\right\rangle \oplus \left\langle x_1y_1y_2 \mid x_i\in X,\ y_j\in Y\right\rangle.$$

These are the multi-cross-effects

$$(S^3)^\dagger_{\{X,X,Y\}}(X\mid Y) \qquad\text{and}\qquad (S^3)^\dagger_{\{X,Y,Y\}}(X\mid Y),$$

respectively. △

When given a direct sum

$$M = M_1 \oplus \cdots \oplus M_n,$$

$\pi_i \colon M \to M$ will as usual denote the ith projection. We recall that the cross-effects of an arbitrary module functor F are given by

$$F^\dagger(M_1 \mid \cdots \mid M_n) = \operatorname{Im} F(\pi_1 \diamond \cdots \diamond \pi_n).$$

Definition 2. — Let A be a multi-set with $\#A \subseteq [n]$. The **multi-cross-effect** of F of type A is the multi-functor

$$F_A^\dagger(M_1 \mid \cdots \mid M_n) = \operatorname{Im} F_{\pi[A]}$$

of n arguments. ◇

When $(M_i)_{i \in I}$ is a family of modules, we may write

$$F_A^\dagger(M_i|_I)$$

for the multi-cross-effect of type A of the modules M_i.

Theorem 10: The Multi-Cross-Effect Decomposition.

$$F^\dagger(M_1 \mid \cdots \mid M_n) = \bigoplus_{\#A = [n]} F_A^\dagger(M_1 \mid \cdots \mid M_n),$$

and consequently,

$$F(M_1 \oplus \cdots \oplus M_n) = \bigoplus_{\#A \subseteq [n]} F_A^\dagger(M_1 \mid \cdots \mid M_n).$$

Proof. It immediately follows from the equation

$$F\left(\sum_i a_i \otimes \pi_i\right) = \sum_A a^A \otimes F_{\pi[A]}$$

that the identity map on

$$F(M_1 \oplus \cdots \oplus M_n)$$

decomposes as

$$1 = F(1) = F\left(\sum_i \pi_i\right) = \sum_A F_{\pi[A]}.$$

Furthermore, the equation

$$\sum_{A,B} a^A b^B \otimes F_{\pi[A]} F_{\pi[B]} = F\left(\sum_i a_i \otimes \pi_i\right) F\left(\sum_j b_j \otimes \pi_j\right)$$

$$= F\left(\sum_k a_k b_k \otimes \pi_k\right) = \sum_C (ab)^C \otimes F_{\pi[C]}$$

shows that

$$F_{\pi[A]} F_{\pi[B]} = \begin{cases} 0 & \text{when } A \neq B, \\ F_{\pi[A]} & \text{when } A = B. \end{cases}$$

Consequently, the maps $F_{\pi[A]}$ provide a direct sum decomposition. □

This decomposition is evidently functorial, and there is a corresponding decomposition of natural transformations $\zeta\colon F \to G$:

$$\zeta_{M_1 \oplus \cdots \oplus M_n} = \bigoplus_{\# A \subseteq [n]} (\zeta_A^\dagger)_{M_1 | \cdots | M_n}.$$

Note in particular the following special case of the theorem:

$$F(\mathbf{B}^n) = \bigoplus_{\# A \subseteq [n]} F_A^\dagger(\mathbf{B}|_n),$$

which is the one that will be most frequently used.

Theorem 11. — *F is homogeneous of degree n iff its multi-cross-effects of type A vanish whenever $|A| \neq n$.*

Proof. If F is homogeneous of degree n, then plainly

$$F_A^\dagger\,(M_1 \mid \cdots \mid M_n) = \operatorname{Im} F_{\pi[A]} \neq 0$$

only if $|A| = n$.

Conversely, suppose that the multi-cross-effects of F of type A vanish when $|A| \neq n$. Because there is a direct sum decomposition

$$1_{F(M)} = F(1_M) = \sum_m F_{1_M^{[m]}},$$

and we made the assumption that

$$\operatorname{Im} F_{1_M^{[m]}} = 0$$

when $m \neq n$, we deduce that, in fact,

$$F_{1_M^{[m]}} = 0$$

when $m \neq n$, and hence

$$1_{F(M)} = F_{1_M^{[n]}}.$$

Let $\alpha\colon M \to N$ be a homomorphism. We get

$$F(a \otimes \alpha) = F(1 \otimes \alpha) F(a \otimes 1_M)$$

$$
\begin{aligned}
&= (1 \otimes F(\alpha)) \sum_m a^m \otimes F_{1_M^{[m]}} \\
&= (1 \otimes F(\alpha))(a^n \otimes 1_{F(M)}) = a^n \otimes F(\alpha).
\end{aligned}
$$

□

Of course, F is strict polynomial of degree n iff the multi-cross-effects of type A vanish whenever $|A| > n$.

Example 6. — When F is homogeneous of degree 0, then

$$F^\dagger_{\{\}}() = F^\dagger() = F(0),$$

with all other multi-cross-effects vanishing. △

Example 7. — When F is homogeneous of degree 1, then

$$F^\dagger_{\{X\}}(X) = F^\dagger(X) = F(X)$$

are the only non-vanishing multi-cross-effects. △

Example 8. — Even in the case of homogeneity degree 2, the multi-cross-effects coincide with the cross-effects, and thus provide no further decomposition. So, for example:

$$
\begin{aligned}
F^\dagger_{\{X,X\}}(X \mid Y) &= F^\dagger(X) \\
F^\dagger_{\{Y,Y\}}(X \mid Y) &= F^\dagger(Y) \\
F^\dagger_{\{X,Y\}}(X \mid Y) &= F^\dagger(X \mid Y).
\end{aligned}
$$

△

Example 9. — The utility of the multi-cross-effects becomes apparent once we reach homogeneity degree 3. In this case, we have

$$
\begin{aligned}
F^\dagger(X) &= F^\dagger_{\{X,X,X\}}(X \mid Y \mid Z) \\
F^\dagger(X \mid Y) &= F^\dagger_{\{X,X,Y\}}(X \mid Y \mid Z) \oplus F^\dagger_{\{X,Y,Y\}}(X \mid Y \mid Z) \\
F^\dagger(X \mid Y \mid Z) &= F^\dagger_{\{X,Y,Z\}}(X \mid Y \mid Z).
\end{aligned}
$$

We see that $F^\dagger(X \mid Y)$ all of a sudden decomposes, as in the introductory example. △

Example 10. — As our final, very general, example, we choose the complete symmetric functor S. Abbreviating

$$X^p Y^q = \left\langle x_1 \dots x_p y_1 \dots y_q \mid x_i \in X,\ y_j \in Y \right\rangle,$$

we obtain the following decomposition of $S(X \oplus Y)$:

$$
\begin{array}{ll}
S^\dagger() \; \{ & S^\dagger_{\{\}}(X \mid Y) = 1 \\
S^\dagger(X) \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. & \begin{array}{rcl} S^\dagger_{\{X\}}(X \mid Y) & = & X \\ S^\dagger_{\{X,X\}}(X \mid Y) & = & X^2 \\ S^\dagger_{\{X,X,X\}}(X \mid Y) & = & X^3 \\ & \vdots & \end{array} \\
S^\dagger(Y) \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. & \begin{array}{rcl} S^\dagger_{\{Y\}}(X \mid Y) & = & Y \\ S^\dagger_{\{Y,Y\}}(X \mid Y) & = & Y^2 \\ S^\dagger_{\{Y,Y,Y\}}(X \mid Y) & = & Y^3 \\ & \vdots & \end{array} \\
S^\dagger(X \mid Y) \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right. & \begin{array}{rcl} S^\dagger_{\{X,Y\}}(X \mid Y) & = & XY \\ S^\dagger_{\{X,X,Y\}}(X \mid Y) & = & X^2Y \\ S^\dagger_{\{X,Y,Y\}}(X \mid Y) & = & XY^2 \\ S^\dagger_{\{X,X,X,Y\}}(X \mid Y) & = & X^3Y \\ S^\dagger_{\{X,X,Y,Y\}}(X \mid Y) & = & X^2Y^2 \\ S^\dagger_{\{X,Y,Y,Y\}}(X \mid Y) & = & XY^3 \\ & \vdots & \end{array}
\end{array}
$$

△

Chapter 8

PROJECTIVE GENERATORS

The first person he met was Rabbit.

"Hallo, Rabbit," he said, "is that you?"

"Let's pretend it isn't," said Rabbit, "and see what happens."

"I've got a message for you."

"I'll give it to him."

"We're all going on an Expotition with Christopher Robin!"

"What is it when we're on it?"

"A sort of boat, I think," said Pooh.

"Oh! that sort."

"And we're going to discover a Pole or something. Or was it a Mole? Anyhow we're going to discover it."

"We are, are we?" said Rabbit.

"Yes. And we've got to bring Pro-things to eat with us. In case we want to eat them. [. . .]"

— Alan Alexander Milne, *Winnie-the-Pooh*

All functors are equal, but some are more equal than others. Indeed, some functors turn out to possess the pleasant property of being a projective generator of the category they live in. Expounding upon these amiable functors is the purpose of the present chapter.

If the theory of polynomial functors be likened to a palace, then this chapter, along with the next, could not unjustly be called the oriental garden encircling it. Japanese influences are here highly perceptible. The course of the current chapter will have us indulge in a veritable orgy of lemmata, which all bear the unmistakable mark of Yoneda, while the theorems of the next are elaborate instances of the classical Morita Equivalence. Of course, the counterexamples, if there were any, would probably be due to Nagata.

§1. The Fundamental Module Functor

Let us begin by exhibiting a projective generator for the category of all module functors. As usual, $\mathbf{B}$ is then permitted to be an arbitrary, unital ring.

Theorem 1. — *Let K be a fixed module. The functor* $\mathbf{B}[\mathrm{Hom}(K,-)]$*, given by*

$$M \mapsto \mathbf{B}[\mathrm{Hom}(K,M)]$$

$$[\chi\colon M \to N] \mapsto \begin{bmatrix} [\chi_*]\colon \mathbf{B}[\mathrm{Hom}(K, M)] \to \mathbf{B}[\mathrm{Hom}(K, N)] \\ [\alpha] \mapsto [\chi \circ \alpha] \end{bmatrix},$$

is a module functor.

Define

$$\mathbf{B}^\infty = \bigoplus_{n=1}^{\infty} \mathbf{B} = \bigcup_{n=1}^{\infty} \mathbf{B}^n = \varinjlim_{n\to\infty} \mathbf{B}^n,$$

the module of all finite vectors over **B**. The functor

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)]$$

will be called the **fundamental module functor**. It is a projective generator for the category of module functors, as will be demonstrated in the next chapter.

§2. The Classical Yoneda Correspondence

Recalling the classical Yoneda correspondence, we strongly emphasise that *natural transformations are always assumed linear*, whereas functors are not.

Theorem 2: The Classical Yoneda Lemma. — *Let K be a fixed module, and F a functor. The map*

$$\begin{aligned} \mathrm{Y}_{K,F}\colon\ \mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(K, -)], F\right) &\to F(K) \\ \eta &\mapsto \eta_K([1_K]) \end{aligned}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccccc} K & \qquad & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K, -)], F) & \xrightarrow{\mathrm{Y}_{K,F}} & F(K) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle[\beta^*]^*} & & \downarrow{\scriptstyle F(\beta)} \\ L & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(L, -)], F) & \xrightarrow[\mathrm{Y}_{L,F}]{} & F(L) \end{array}$$

$$\begin{array}{ccccc} F & \qquad & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K, -)], F) & \xrightarrow{\mathrm{Y}_{K,F}} & F(K) \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi_*} & & \downarrow{\scriptstyle\xi_K} \\ G & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K, -)], G) & \xrightarrow[\mathrm{Y}_{K,G}]{} & G(K) \end{array}$$

Proof. This is, in fact, the original Yoneda Lemma in disguise. Note that *arbitrary* natural transformations

$$\mathrm{Hom}(K, -) \to F$$

correspond to *linear* transformations

$$\mathbf{B}[\mathrm{Hom}(K, -)] \to F.$$

□

§3. The Fundamental Numerical Functor

We now turn to numerical functors, and, as is customary, assume a numerical base ring.

Theorem 3. — *Let K be a fixed module. The functor $\mathbf{B}[\mathrm{Hom}(K,-)]_n$, given by*

$$\begin{gathered} M \mapsto \mathbf{B}[\mathrm{Hom}(K,M)]_n \\ [\chi\colon M \to N] \mapsto \begin{bmatrix} [\chi_*]\colon \mathbf{B}[\mathrm{Hom}(K,M)]_n \to \mathbf{B}[\mathrm{Hom}(K,N)]_n \\ [\alpha] \mapsto [\chi \circ \alpha] \end{bmatrix}, \end{gathered}$$

is numerical of degree n.

Proof. Since $\mathbf{B}[\mathrm{Hom}(K,-)]_n$ is the composition of $\mathbf{B}[-]_n$ with the Hom-functor, it suffices to prove $\mathbf{B}[-]_n$ is of degree n. Let $\chi_j\colon M \to N$ be homomorphisms, and let $x \in M$; then

$$[\chi_1 \diamond \cdots \diamond \chi_{n+1}]([x]) = [\chi_1(x) \diamond \cdots \diamond \chi_{n+1}(x)] = 0.$$

Moreover, if $a \in \mathbf{B}$ and $\chi\colon M \to N$, then

$$[a\chi]([x]) = [a\chi(x)] = \sum_{k=0}^{n} \binom{a}{k} \left[\mathop{\diamond}_{k} \chi(x) \right] = \sum_{k=0}^{n} \binom{a}{k} \left[\mathop{\diamond}_{k} \chi \right]([x]).$$

We infer that $\mathbf{B}[-]_n$ is numerical of degree n. □

The case $K = \mathbf{B}^n$ is especially important. The functor

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n$$

will be called the **fundamental numerical functor** of degree n. It will presently be seen to be a projective generator.

Example 1. — We give an example of a functor which is polynomial, but not numerical, of degree 1. Let the base ring be $\mathbf{R}$, and define for real vector spaces

$$\begin{gathered} F\colon {}_{\mathbf{R}}\mathfrak{Mod} \to {}_{\mathbf{R}}\mathfrak{Mod} \\ V \mapsto \mathbf{R}[V]/\langle [x+y]-[x]-[y] \rangle . \end{gathered}$$

We thus impose additivity, but not linearity. Since F is additive, its first cross-effect will vanish. But F is not numerical of degree 1, for

$$\begin{gathered} F(\sqrt{2}\colon \mathbf{R} \to \mathbf{R})\colon [1] \mapsto [\sqrt{2}] \\ \sqrt{2}F(1\colon \mathbf{R} \to \mathbf{R})\colon [1] \mapsto \sqrt{2}[1], \end{gathered}$$

and $[\sqrt{2}] \neq \sqrt{2}[1]$ in

$$F(\mathbf{R}) = \mathbf{R}[\mathbf{R}]/\langle [x+y]-[x]-[y] \rangle .$$

In fact, F is not numerical of *any* degree. △

§4. The Numerical Yoneda Correspondence

Theorem 4: The Numerical Yoneda Lemma. — *Let K be a fixed module, and F a numerical functor of degree n. The map*

$$\begin{aligned} \mathrm{Y}_{K,F}\colon \ \mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F\right) &\to F(K) \\ \eta &\mapsto \eta_K([\iota_K]) \end{aligned}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccccc} K & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F) & \xrightarrow{\mathrm{Y}_{K,F}} & F(K) \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle[\beta^*]^*}\downarrow & & \downarrow{\scriptstyle F(\beta)} \\ L & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(L,-)]_n, F) & \xrightarrow[\mathrm{Y}_{L,F}]{} & F(L) \end{array}$$

$$\begin{array}{ccccc} F & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F) & \xrightarrow{\mathrm{Y}_{K,F}} & F(K) \\ {\scriptstyle\xi}\downarrow & & {\scriptstyle\xi_*}\downarrow & & \downarrow{\scriptstyle\xi_K} \\ G & & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, G) & \xrightarrow[\mathrm{Y}_{K,G}]{} & G(K) \end{array}$$

Proof. The proof is the usual one. Consider the following commutative diagram:

$$\begin{array}{cccccccc} K & & \mathbf{B}[\mathrm{Hom}(K,K)]_n & \xrightarrow{\eta_K} & F(K) & & [\iota_K] & \longrightarrow & \eta_K([\iota_K]) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle[\alpha_*]}\downarrow & & \downarrow{\scriptstyle F(\alpha)} & & \downarrow & & \downarrow \\ M & & \mathbf{B}[\mathrm{Hom}(K,M)]_n & \xrightarrow[\eta_M]{} & F(M) & & [\alpha] & \longrightarrow & \eta_M([\alpha]) = F(\alpha)(\eta_K([\iota_K])) \end{array}$$

Upon inspection, we find that $\mathrm{Y}_{K,F}$ has the inverse

$$y \mapsto \begin{bmatrix} \eta_M\colon \mathbf{B}[\mathrm{Hom}(K,M)]_n \to F(M) \\ [\alpha] \mapsto F(\alpha)(y) \end{bmatrix}.$$

When defining this inverse, the numericality of F is used in an essential way to ensure that the map

$$\mathrm{Hom}(K,M) \to \mathrm{Hom}(F(K), F(M))$$

factor through $\mathbf{B}[\mathrm{Hom}(K,M)]_n$.

The naturality of Y is obvious. □

In particular, putting $F = \mathbf{B}[\mathrm{Hom}(K,-)]_n$, we obtain a module isomorphism

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n) \cong \mathbf{B}[\mathrm{Hom}(K,K)]_n = \mathbf{B}[\mathrm{End}\,K]_n,$$

given by the map

$$\mathrm{Y}\colon \eta \mapsto \eta_K([1_K])$$

with inverse

$$\mathrm{Y}^{-1}\colon [\sigma] \mapsto \begin{bmatrix} [\sigma^*]\colon \mathbf{B}[\mathrm{Hom}(K,-)]_n \to \mathbf{B}[\mathrm{Hom}(K,-)]_n \\ [\alpha] \mapsto [\alpha \circ \sigma]. \end{bmatrix}.$$

We note that

$$\begin{aligned} \mathrm{Y}^{-1}([\sigma][\tau]) = \mathrm{Y}^{-1}([\sigma+\tau]) &= [(\sigma+\tau)^*] \\ &= [\sigma^* + \tau^*] = [\sigma^*][\tau^*] = \mathrm{Y}^{-1}([\sigma])\mathrm{Y}^{-1}([\tau]), \end{aligned}$$

so that the Yoneda correspondence is actually a ring isomorphism under the sum multiplication.

Now, this is probably very interesting and all, but for our purposes it will be of no major consequence, and we mentioned the above fact just in passing. The really interesting question is what happens to the product multiplication on $\mathbf{B}[\mathrm{End}\,K]_n$:

$$\begin{aligned} \mathrm{Y}^{-1}([\sigma] \star [\tau]) = \mathrm{Y}^{-1}([\sigma\tau]) &= [(\sigma\tau)^*] \\ &= [\tau^*] \circ [\sigma^*] = \mathrm{Y}^{-1}([\tau]) \circ \mathrm{Y}^{-1}([\sigma]). \end{aligned}$$

The product multiplication is reversed by Y.

Theorem 5. — *The Yoneda correspondence provides an isomorphism of rings*

$$\big(\mathrm{Nat}\,\mathbf{B}[\mathrm{Hom}(K,-)]_n\big)^\circ \cong \mathbf{B}[\mathrm{End}\,K]_n,$$

where the former is equipped with composition, and the latter with the product multiplication.

§5. The Deviated Power Functors

The fundamental numerical functor

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n$$

is not atomic, but built from simpler components. A similar decomposition is available for the functor

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty,-)],$$

and, in order to subsume this case, we shall, in what follows, allow for the possibility $n = \infty$. The symbol

$$\mathfrak{Num}_\infty$$

will then denote arbitrary module functors, and we agree that

$$\mathfrak{Laby}_\infty = \mathfrak{Laby}.$$

In the case $n = \infty$, numericality of the base ring shall of course not be needed.

We draw the reader's attention to a little peculiarity. When including the case $n = \infty$, we allow for a non-commutative base ring $\mathbf{B}$. However, since we limit ourselves to the study of finitely generated and free modules, *all our modules are automatically bimodules*. Homomorphisms are still one-sided, and we settle on the consideration of *right module homomorphisms* only, as this turns out to give the most natural theory.

When $(M_i)_{i \in I}$ is a finite family of modules, we define

$$\left[\mathop{\Diamond}_{i\in I} M_i\right]_n = \left\langle \left[\mathop{\Diamond}_{i\in I} u_i\right] \middle| u_i \in M_i \right\rangle \subseteq \mathbf{B}\left[\bigoplus_{i\in I} M_i\right]_n.$$

As a degenerate case, we have

$$\left[\mathop{\Diamond}_{i\in\emptyset} M_i\right]_n = \langle[\diamond]\rangle = \langle[0]\rangle \cong \mathbf{B}.$$

Let X be a finite set. Define the corresponding **deviated power functor** by

$$\Delta^X(M) = \left[\mathop{\Diamond}_{x\in X} M\right]_n \subseteq \mathbf{B}\left[\bigoplus_{x\in X} M\right]_n.$$

Definition 1. — The **deviated power category** $\mathfrak{Dev}$ is the category of functors Δ^X, where $X \in \mathfrak{Laby}_n$. ◇

The definition we have given of deviated powers, in terms of deviations, is algebraic, but awkward. It turns out that the most natural, and fruitful, way of viewing them is as mazes.

Let M be a module, and X and Y be finite sets. We will denote by

$${}_M\mathfrak{Laby}_n(X, Y)$$

the module of mazes $X \to Y$, where the passages have been labelled with elements of M (rather than $\mathbf{B}$). We can, and will, still impose the labyrinth axioms — two if $n = \infty$, and four if $n < \infty$. It should be noted that there is no way of composing such mazes (as there is no multiplication in M), so we do not obtain a category.

Evidently, the assignment

$$M \mapsto {}_M\mathfrak{Laby}_n(X, Y)$$

is functorial. When $\alpha\colon M \to N$, we shall denote by

$${}_\alpha\mathfrak{Laby}_n(X, Y)\colon {}_M\mathfrak{Laby}_n(X, Y) \to {}_N\mathfrak{Laby}_n(X, Y)$$

the induced homomorphism (acting on labels of passages).

Putting $Y = *$, the canonical one-element set, will provide us with the sought description of the deviated power functors. We are forced to accept the degenerate case $X = \emptyset$. An element of ${}_M\mathfrak{Laby}_n(\emptyset, *)$ will be an (unlabelled) "dead end", which we otherwise took great care to forbid, but it is necessary to allow this pathology in order to handle the functor $\Delta^{\emptyset}$.

Let two direct sums M^X and M^Y be given. Recall that, for $p \in X$ and $q \in Y$, there is a canonical transportation map

$$\sigma_{qp} \colon M^X \to M^Y, \quad \sigma_{qp}\left((u_x)_{x\in X}\right)_y = \begin{cases} 0 & \text{if } y \neq q \\ u_p & \text{if } y = q. \end{cases}$$

Theorem 6. — *Let X be a finite set, and M a module. The map*

$$\Xi_{X,M} \colon \Delta^X(M) \to {}_M\mathfrak{Laby}_n(X, *)$$
$$\left[\mathop{\Diamond}_{x\in X} u_x\right] \mapsto \bigcup_{x\in X} \left\{ x \xrightarrow{u_x} * \right\}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccccc} X & & \Delta^X(M) & \xrightarrow{\Xi_{X,M}} & {}_M\mathfrak{Laby}_n(X,*) \\ {\scriptstyle P}\big\downarrow & & {\scriptstyle [\Diamond_{[p\colon x\to y]\in P}\, \bar{p}\sigma_{yx}]}\big\downarrow & & \big\downarrow{\scriptstyle (P^\circ)^*} \\ Y & & \Delta^Y(M) & \xrightarrow[\Xi_{Y,M}]{} & {}_M\mathfrak{Laby}_n(Y,*) \end{array}$$

$$\begin{array}{ccccc} M & & \Delta^X(M) & \xrightarrow{\Xi_{X,M}} & {}_M\mathfrak{Laby}_n(X,*) \\ {\scriptstyle \alpha}\big\downarrow & & {\scriptstyle [\alpha]}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{Laby}_n(\alpha)} \\ N & & \Delta^X(N) & \xrightarrow[\Xi_{X,N}]{} & {}_N\mathfrak{Laby}_n(X,*) \end{array}$$

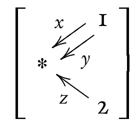
Proof. Define a map

$$\mathbf{B}\left[\bigoplus_{x\in X} M\right]_n \to \bigoplus_{Z\subseteq X} {}_M\mathfrak{Laby}_n(X,*)$$
$$[(u_x)_{x\in X}] \mapsto \left(\bigcup_{z\in Z}\left\{ z \xrightarrow{u_z} * \right\}\right)_{Z\subseteq X},$$

and note that $\Xi_{X,M}$ is the restriction of this to $\Delta^X(M)$. Hence Ξ is well-defined, and it has an obvious inverse.

The second diagram above evidently commutes. That the first one does is a consequence of the Deviation Formula and the definition of the maze product. □

Example 2. — The maze



$$\in {}_M\mathfrak{Laby}_n(\{1,2\},*)$$

corresponds to the deviation

$$[(x,0) \diamond (y,0) \diamond (0,z)] \in \Delta^2(M).$$

△

We see that deviations of module elements correspond to mazes. That also deviations of homomorphisms can be naturally interpreted as mazes should come as no surprise.

Theorem 7. — *Given homomorphisms* $\alpha_k\colon M \to N$, *let*

$$S = \bigcup \left\{ * \xrightarrow{\alpha_k} * \right\} \in {}_{\operatorname{Hom}(M,N)}\mathfrak{Laby}_n(*,*).$$

The following diagram commutes:

$$\begin{array}{ccc} \Delta^X(M) & \xrightarrow{\Xi_{X,M}} & {}_M\mathfrak{Laby}_n(X,*) \\ {\scriptstyle [\diamond\,\alpha_k]}\downarrow & & \downarrow{\scriptstyle S_*} \\ \Delta^X(N) & \xrightarrow[\Xi_{X,N}]{} & {}_N\mathfrak{Laby}_n(X,*) \end{array}$$

Proof. The formula

$$[\alpha_1 \diamond \cdots \diamond \alpha_m]([u_1 \diamond \cdots \diamond u_n]) = \sum_{K \sqsubseteq [m]\times[n]} \left[\mathop{\Diamond}_{(i,j)\in K} \alpha_i(u_j) \right]$$

can be obtained as a special case of the Deviation Formula (it is, in any case, proved in exactly the same way). The commutativity of the above diagram is then an immediate consequence. □

Let $P\colon X \to Y$ be a maze. We denote by

$$\Delta^P\colon \Delta^Y \to \Delta^X$$

the corresponding natural transformation

$$P^*\colon {}_{-}\mathfrak{Laby}_n(Y,*) \to {}_{-}\mathfrak{Laby}_n(X,*).$$

With the current description of deviated powers, the verification of the following theorem becomes a mere triviality.

THEOREM 8. — *The functor*

$$\Xi\colon \mathfrak{Laby}_n \to \mathfrak{Dev}_n,$$

mapping

$$X \mapsto \Delta^X$$
$$[P\colon X \to Y] \mapsto \left[\Delta^P\colon \Delta^Y \to \Delta^X\right],$$

is an anti-isomorphism of categories.

Proof. That every transformation

$$_M\mathfrak{Laby}_n(Y,*) \to {}_M\mathfrak{Laby}_n(X,*),$$

which is natural in M, is of the form P^*, and uniquely so, is not difficult to see. Recall that we may choose M to be a free algebra. □

EXAMPLE 3. — We have thus made three important identifications. The set $\{1,2\} \in \mathfrak{Laby}_n$ corresponds under the category anti-isomorphism to the functor Δ^2, which, as we know, is naturally isomorphic to the functor

$$_{-}\mathfrak{Laby}_n(\{1,2\},*).$$

In what follows, such identifications will be made without comment. △

The importance of the deviated power functors stems from the fact that they provide a splitting of the fundamental (numerical) functor into atomic components (and here, for once, the original definition is actually useful). To subsume the case $n = \infty$, we put $[\infty] = \mathbf{Z}^+$.

THEOREM 9.

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n = \bigoplus_{X\subseteq[n]} \Delta^X.$$

Proof.

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,M)]_n = \mathbf{B}\left[\bigoplus_{k\in[n]} M\right]_n = \bigoplus_{X\subseteq[n]}\left[\Diamond_{x\in X} M\right]_n = \bigoplus_{X\subseteq[n]} \Delta^X(M).$$

□

§6. THE LABYRINTHINE YONEDA CORRESPONDENCE

As usual, σ denotes transportation maps.

Theorem 10: The Labyrinthine Yoneda Lemma. — *Let X be a set, and F a functor. The map*

$$\begin{aligned} \mathrm{Y}_{X,F}\colon\ \mathrm{Nat}(\Delta^X, F) &\to F^\dagger(\mathbf{B}|_X) \\ \zeta &\mapsto \zeta_{\mathbf{B}^X}\left(\bigcup_{x\in X}\left\{\, x \xrightarrow{\ 1_x\ } * \,\right\}\right) \end{aligned}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccc} X & \qquad & \mathrm{Nat}(\Delta^X, F) & \xrightarrow{\ \mathrm{Y}_{X,F}\ } & F^\dagger(\mathbf{B}|_X) \\ {\scriptstyle P}\big\downarrow & & {\scriptstyle (\Delta^P)^*}\big\downarrow & & \big\downarrow{\scriptstyle F\left(\Diamond_{[p\colon x\to y]\in P}\,\bar{p}\sigma_{yx}\right)} \\ Y & & \mathrm{Nat}(\Delta^Y, F) & \xrightarrow[\ \mathrm{Y}_{Y,F}\ ]{} & F^\dagger(\mathbf{B}|_Y) \end{array}$$

$$\begin{array}{ccc} F & \qquad & \mathrm{Nat}(\Delta^X, F) & \xrightarrow{\ \mathrm{Y}_{X,F}\ } & F^\dagger(\mathbf{B}|_X) \\ {\scriptstyle \xi}\big\downarrow & & {\scriptstyle \xi_*}\big\downarrow & & \big\downarrow{\scriptstyle \xi^\dagger_{\mathbf{B}|_X}} \\ G & & \mathrm{Nat}(\Delta^X, G) & \xrightarrow[\ \mathrm{Y}_{X,G}\ ]{} & G^\dagger(\mathbf{B}|_X) \end{array}$$

Proof. A natural transformation

$$\zeta\colon \Delta^X \to F$$

corresponds to a natural transformation

$$\eta_M\colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^X, M)] \to F(M),$$

taking

$$\left[\Diamond_{y\in Y}[\alpha_y\colon \mathbf{B}_y \to M]\right] \mapsto \begin{cases} 0 & \text{if } Y \subset X, \\ \zeta_M\left(\bigcup_{y\in Y}\left[\, * \xleftarrow{\ \alpha_y(1_y)\ } y \,\right]\right) & \text{if } Y = X. \end{cases}$$

The original Yoneda map takes η to

$$\begin{aligned} \eta_{\mathbf{B}^X}([1_{\mathbf{B}^X}]) &= \sum_{Y\subseteq X} \eta_{\mathbf{B}^X}\left(\left[\Diamond_{y\in Y} 1_{\mathbf{B}_y}\right]\right) = \eta_{\mathbf{B}^X}\left(\left[\Diamond_{x\in X} 1_{\mathbf{B}_x}\right]\right) \\ &= \zeta_{\mathbf{B}^X}\left(\bigcup_{x\in X}\left[\, * \xleftarrow{\ 1_{\mathbf{B}_x}(1_x)\ } x \,\right]\right) = \zeta_{\mathbf{B}^X}\left(\bigcup_{x\in X}\left[\, * \xleftarrow{\ 1_x\ } x \,\right]\right). \end{aligned}$$

Thus, $\mathrm{Y}_{X,F}$ is the X-component of the original Yoneda map, and it is clear that

$$\mathrm{Y}_{X,F}(\zeta) = \eta_{\mathbf{B}^X}\left(\left[\mathop{\Diamond}_{x\in X} \mathrm{I}_{\mathbf{B}_x}\right]\right) \in F^{\dagger}(\mathbf{B}|_X).$$

That the second diagram above commutes is clear, so there remains to investigate the first. From the original Yoneda Lemma, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^X,)], F) & \xrightarrow{\mathrm{Y}_{\mathbf{B}^X,F}} & F(\mathbf{B}^X) \\ {\scriptstyle \left[\left(\Diamond_{[p\colon x\to y]\in P}\,\bar{p}\sigma_{yx}\right)^*\right]^*}\Big\downarrow & & \Big\downarrow{\scriptstyle F\left(\Diamond_{[p\colon x\to y]\in P}\,\bar{p}\sigma_{yx}\right)} \\ \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^Y, -)], F) & \xrightarrow[\mathrm{Y}_{\mathbf{B}^Y,F}]{} & F(\mathbf{B}^Y) \end{array}$$

We shall show that the following diagram commutes, from which the claim follows:

$$\begin{array}{ccc} \mathrm{Nat}(\Delta^X, F) & \longrightarrow & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^X,)], F) \\ {\scriptstyle (\Delta^P)^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \left[\left(\Diamond_{[p\colon x\to y]\in P}\,\bar{p}\sigma_{yx}\right)^*\right]^*} \\ \mathrm{Nat}(\Delta^Y, F) & \longrightarrow & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^Y, -)], F) \end{array}$$

Consider a natural transformation

$$\zeta\colon \Delta^X \to F,$$

corresponding to an

$$\eta\colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^X, -)] \to F$$

as above. The homomorphism

$$\left[\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right)^*\right]^*$$

maps η to the natural transformation

$$\eta_M\left[\left(\mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right)^*\right] : \mathbf{B}[\mathrm{Hom}(\mathbf{B}^Y, M)] \to F(M),$$

which takes

$$\begin{aligned} \left[\mathop{\Diamond}_{y\in Y} [\alpha_y\colon \mathbf{B}_y \to M]\right] \mapsto \eta_M &\left(\left[\mathop{\Diamond}_{y\in Y} \alpha_y \circ \mathop{\Diamond}_{[p\colon x\to y]\in P} \bar{p}\sigma_{yx}\right]\right) \\ = \eta_M &\left(\left[\mathop{\Diamond}_{[p\colon x\to y]\in P} \alpha_y \circ \bar{p}\sigma_{yx}\right]\right) \end{aligned}$$

$$= \zeta_M \left(\bigcup_{[p\colon x\to y]\in P} \left[* \xleftarrow{\alpha_y(\bar{p}\sigma_{yx}(1_x))} x \right] \right)$$
$$= \zeta_M \left(\bigcup_{[p\colon x\to y]\in P} \left[* \xleftarrow{\alpha_y(1_y)\cdot\bar{p}} x \right] \right),$$

where the last step is because

$$\alpha_y(\bar{p}\sigma_{yx}(1_x)) = \alpha_y(\bar{p}\cdot 1_y) = \alpha_y(1_y\cdot\bar{p}) = \alpha_y(1_y)\cdot\bar{p}.$$

On the other hand,

$$\zeta\Delta^P\colon \Delta^Y \to F$$

corresponds to a natural transformation

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^Y, M)] \to F(M),$$

that takes

$$\left[\mathop{\Diamond}_{y\in Y} [\alpha_y\colon \mathbf{B}_y \to M] \right] \mapsto \zeta_M \left(\bigcup_{y\in Y} \left[* \xleftarrow{\alpha_y(1_y)} y \right] \cdot \bigcup_{[p\colon x\to y]\in P} \left[y \xleftarrow{\bar{p}} x \right] \right).$$

The commutativity of the diagram is demonstrated. □

§7. The Fundamental Homogeneous Functor

We recall that, given $a_i \in A$ (where A is some algebra) and homomorphisms $\alpha_i \in \mathrm{Hom}(M, N)$, the equation

$$\Gamma^n\left(\sum_i a_i \otimes \alpha_i\right) = \sum_{|X|=n} a^X \otimes (\Gamma^n)_{\alpha[X]}$$

defines the multi-deviations

$$(\Gamma^n)_{\alpha[X]}\colon \Gamma^n(M) \to \Gamma^n(N).$$

As a matter of notational convenience, let us agree to write

$$\alpha^{[X]} = (\Gamma^n)_{\alpha[X]}.$$

The symbol $\alpha^{[X]}$ may thus be interpreted either as an element of

$$\Gamma^n \mathrm{Hom}(M, N)$$

or as a map

$$\Gamma^n(M) \to \Gamma^n(N)$$

(and usually as both).

Theorem 11. — *Let K be a fixed module. The functor $\Gamma^n \operatorname{Hom}(K,-)$, given by*

$$M \mapsto \Gamma^n \operatorname{Hom}(K, M)$$
$$[\chi\colon M \to N] \mapsto \left[\begin{matrix}(\chi_*)^{[n]}\colon \Gamma^n \operatorname{Hom}(K,M) \to \Gamma^n \operatorname{Hom}(K,N) \\ \alpha^{[n]} \mapsto (\chi \circ \alpha)^{[n]}\end{matrix}\right],$$

is homogeneous of degree n.

It will prove convenient to have available a formula for the strict polynomial structure. Thus, for any algebra A:

$$\left[\begin{matrix}\sum_i a_i \otimes \chi_i \\ \in A \otimes \operatorname{Hom}(M,N)\end{matrix}\right] \mapsto \left[\begin{matrix}\left(\sum_i a_i \otimes (\chi_i)_*\right)^{[n]} = \sum_{|X|=n} a^X \otimes (\chi_*)^{[X]} \\ \in A \otimes \operatorname{Hom}(\Gamma^n \operatorname{Hom}(K,M), \Gamma^n \operatorname{Hom}(K,N))\end{matrix}\right].$$

Again, the case $K = \mathbf{B}^n$ is especially important. The functor

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, -)$$

will be called the **fundamental homogeneous functor** of degree n. It will presently be seen to be a projective generator.

§8. The Homogeneous Yoneda Correspondence

Theorem 12: The Homogeneous Yoneda Lemma. — *Let K be a fixed module, and F a homogeneous functor of degree n. The map*

$$\Upsilon_{K,F}\colon \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-), F) \to F(K)$$
$$\eta \mapsto \eta_K(1_K^{[n]})$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccccc} K & \quad & \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-),F) & \xrightarrow{\Upsilon_{K,F}} & F(K) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle ((\beta^*)^{[n]})^*} & & \downarrow{\scriptstyle F(\beta)} \\ L & & \operatorname{Nat}(\Gamma^n \operatorname{Hom}(L,-),F) & \xrightarrow[\Upsilon_{L,F}]{} & F(L) \end{array}$$

$$\begin{array}{ccccc} F & \quad & \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-),F) & \xrightarrow{\Upsilon_{K,F}} & F(K) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \xi_K} \\ G & & \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-),G) & \xrightarrow[\Upsilon_{K,G}]{} & G(K) \end{array}$$

Proof. Let $y \in F(K)$, and consider the strict polynomial map

$$\Xi\colon \operatorname{Hom}(K,M) \xrightarrow{F} \operatorname{Hom}(F(K),F(M)) \longrightarrow F(M),$$

where the second (linear) map is evaluation at y. Since this map is homogeneous of degree n, it gives rise to a linear map

$$\Gamma^n \operatorname{Hom}(K,M) \to F(M), \quad \alpha^{[n]} \mapsto F(\alpha)(y).$$

We may therefore define

$$\begin{gathered} \Xi\colon F(K) \to \operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-),F) \\ y \mapsto \begin{bmatrix} \zeta_M\colon \Gamma^n \operatorname{Hom}(K,M) \to F(M) \\ \alpha^{[n]} \mapsto F(\alpha)(y) \end{bmatrix}. \end{gathered}$$

It should be evident that ζ is indeed natural.

Let us now show that the above formula gives the inverse of Y. On the one hand, it is clear that

$$\mathrm{Y}\Xi(y) = \Xi(y)_K(1_K^{[n]}) = F(1_K)(y) = y.$$

On the other hand, let

$$\eta\colon \Gamma^n \operatorname{Hom}(K,-) \to F$$

be given. There is a commutative diagram:

$$\begin{array}{ccccccc} K & & \Gamma^n \operatorname{Hom}(K,K) & \xrightarrow{\eta_K} & F(K) & & 1_K^{[n]} \longrightarrow \eta_K(1_K^{[n]}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle (\alpha_*)^{[n]}} & & \downarrow{\scriptstyle F(\alpha)} & & \downarrow \qquad\qquad \downarrow \\ M & & \Gamma^n \operatorname{Hom}(K,M) & \xrightarrow[\eta_M]{} & F(M) & & \alpha^{[n]} \longrightarrow \eta_M(\alpha^{[n]}) = F(\alpha)(\eta_K(1_K^{[n]})) \end{array}$$

We deduce that the natural transformation $\Xi\mathrm{Y}(\eta)$ maps

$$\alpha^{[n]} \mapsto F(\alpha)(\mathrm{Y}(\eta)) = F(\alpha)(\eta_K(1_K^{[n]})) = \eta_M(\alpha^{[n]}),$$

and hence $\Xi\mathrm{Y}(\eta) = \eta$.

The naturality of Y is obvious. □

In particular, putting $F = \Gamma^n \operatorname{Hom}(K,-)$, we obtain a module isomorphism

$$\operatorname{Nat}(\Gamma^n \operatorname{Hom}(K,-)) \cong \Gamma^n \operatorname{Hom}(K,K) = \Gamma^n(\operatorname{End} K),$$

given by the map

$$\mathrm{Y}\colon \eta \mapsto \eta_K(1_K^{[n]}),$$

with inverse

$$\mathrm{Y}^{-1}\colon \sigma^{[n]} \mapsto \begin{bmatrix} (\sigma^*)^{[n]}\colon \Gamma^n \operatorname{Hom}(K,M) \to \Gamma^n \operatorname{Hom}(K,M) \\ \alpha^{[n]} \mapsto (\alpha \circ \sigma)^{[n]} \end{bmatrix}.$$

As in the numerical case, we observe that

$$\begin{aligned} \Upsilon^{-1}(\sigma^{[n]} \star \tau^{[n]}) &= \Upsilon^{-1}((\sigma\tau)^{[n]}) = ((\sigma\tau)^*)^{[n]} \\ &= (\tau^*\sigma^*)^{[n]} = (\tau^*)^{[n]} \circ (\sigma^*)^{[n]} = \Upsilon^{-1}(\tau^{[n]}) \circ \Upsilon^{-1}(\sigma^{[n]}), \end{aligned}$$

and we have deduced the following theorem.

Theorem 13. — *The Yoneda correspondence provides an isomorphism of rings*

$$\big(\operatorname{Nat}(\Gamma^n \operatorname{Hom}(K, -))\big)^\circ \cong \Gamma^n(\operatorname{End} K),$$

where the former is equipped with composition, and the latter with the product multiplication.

§9. The Divided Power Functors

When $(M_a)_{a\in A}$ is a (finite) multi-set of modules, we define

$$\bigodot_{a\in A} M_a = \bigotimes_{a\in \# A} \Gamma^{\deg a}(M_a),$$

which might be called a confluent product of modules. As a degenerate case, we have

$$\bigodot_{a\in\emptyset} M_a = \mathbf{B}.$$

Let A be a multi-set. Define the corresponding **divided power functor** by

$$\Gamma^A(M) = \bigodot_{a\in A} M = \bigotimes_{a\in\# A} \Gamma^{\deg a}(M).$$

Definition 2. — The n**th divided power category** $\mathfrak{Div}_n$ is the full subcategory of $\mathfrak{Hom}_n$ consisting of the functors Γ^A, where $A \in \mathfrak{MSet}_n$. ◇

In order to study the deviated powers, we took to establishing an isomorphism between $\Delta^X(M)$ and a module of formal mazes, labelled with elements from M. Analogously, it will prove convenient to view elements of $\Gamma^X(M)$ as formal multations. We recall that, at the very beginning, we saw an example of the reverse procedure, that of viewing multations as divided powers. There is indeed a very intimate connection between multations and divided powers.

Let M be a module, let $\xi\colon A \to K$ be a multation, and consider module elements $x_k \in M$, for $k \in K$. Recall that we have defined the symbol

$$x^{\otimes[\xi]} = \bigotimes_{a\in\# A} \bigodot_{(a,k)\in\xi} x_k \in \Gamma^A(M).$$

If ξ is a sum of multations, we choose to interpret this quantity as the corresponding sum of divided powers. (This is why we prefer the symbol to the left, as this could conceivably be linear in ξ, whereas it looks more doubtful that the right one is.)

Consider now a multation $\mu\colon A \to B$, and free variables x_{ba}. Let M be a module. As usual, σ_{ba} will denote transportation maps $M^{\#B} \to M^{\#A}$, but this time *acting on the right*. Recall that the equation

$$\left(\sum_{\substack{a\in\#A\\ b\in\#B}} x_{ba}\otimes\sigma_{ba}\right)^{[n]} = \sum_{\mu} x^{\mu}\otimes\sigma^{[\mu]}$$

defines homomorphisms

$$\sigma^{[\mu]}\colon \Gamma^n(M^{\#B}) \to \Gamma^n(M^{\#A}).$$

(Here μ will range over multations $X \to Y$, where $\#X \subseteq \#A$, $\#Y \subseteq \#B$, and $|X| = |Y| = n$.) We note that

$$\sigma^{[\mu]}\colon ((x_b)_{b\in\#B})^{[n]} \mapsto \bigodot_{(a,b)\in\mu} (x_b)\sigma_{ba},$$

and hence $\sigma^{[\mu]}$ may be viewed as a map

$$\Gamma^{\mu}(M)\colon \Gamma^{B}(M) \to \Gamma^{A}(M),$$

taking

$$x^{\otimes[B]} \mapsto x^{\otimes[\mu]}.$$

It is clear that Γ^{μ} is a natural transformation.

Example 4. — The multation

$$\mu = \begin{bmatrix} 1 & 1 & 2 & 2\\ 1 & 2 & 2 & 2 \end{bmatrix}$$

provides a natural transformation

$$\Gamma^{\mu}\colon \Gamma^{\{1,2,2,2\}} \to \Gamma^{\{1,1,2,2\}}$$
$$x_1 \otimes x_2^{[3]} \mapsto x_1x_2 \otimes x_2^{[2]}.$$

△

Theorem 14. — *The functor*

$$\Xi\colon \mathfrak{MSet}_n \to \mathfrak{Div}_n,$$

mapping

$$A \mapsto \Gamma^{A}$$
$$[\mu\colon A \to B] \mapsto \begin{bmatrix} \Gamma^{\mu}\colon \Gamma^{B} \to \Gamma^{A}\\ x^{\otimes[\xi]} \mapsto x^{\otimes[\xi\mu]} \end{bmatrix}$$

is an anti-isomorphism of categories.

Proof. Since Γ^μ is found as the coefficient of x^μ in

$$\left(\sum_{\substack{a\in\#A\\ b\in\#B}} x_{ba}\otimes\sigma_{ba}\right)^{[n]},$$

while μ itself is the coefficient of x^μ in

$$\left(\sum_{\substack{a\in\#A\\ b\in\#B}} x_{ba}\otimes\begin{bmatrix}a\\ b\end{bmatrix}\right)^{[n]},$$

it is evident that the correspondence is functorial. Also, it should be clear that any natural transformation $\Gamma^B\to\Gamma^A$ can be expressed as a sum of maps Γ^μ, for, as usual, M may be chosen to be a free algebra.

Finally, let us verify the given formula for Γ^μ. Let $\xi\colon B\to C$ be a multation; then we already know that

$$\Gamma^\xi\colon\Gamma^C\to\Gamma^B$$

maps

$$x^{\otimes[C]}\mapsto x^{\otimes[\xi]}.$$

Hence, by functoriality,

$$\Gamma^\mu(x^{\otimes[\xi]})=\Gamma^\mu(\Gamma^\xi(x^{\otimes[C]}))=\Gamma^{\xi\mu}(x^{\otimes[C]})=x^{\otimes[\xi\mu]}.$$

□

The theorem should be compared with the category anti-isomorphism

$$\mathfrak{Set}_n\overset{\circ}{\cong}\{T^A\mid A\in\mathfrak{Set}_n\},$$

a result that appears to hark back to Weyl. Here, to complete the analogy, $\mathfrak{Set}_n$ denotes the category of n-element sets and their *bijections*.

Dr. Salomonsson presents an alternative proof in [20]; more conceptual, but with the disadvantage of obscuring the underlying combinatorics.

Example 5. — In the previous example, we studied the multation

$$\mu=\begin{bmatrix}1&1&2&2\\1&2&2&2\end{bmatrix},$$

and the corresponding natural transformation

$$\Gamma^\mu\colon\Gamma^{\{1,2,2,2\}}\to\Gamma^{\{1,1,2,2\}}$$
$$x_1\otimes x_2^{[3]}\mapsto x_1x_2\otimes x_2^{[2]}.$$

We now use the theorem to find out what happens to an element

$$y \otimes xy^{[2]} \in \Gamma^{\{1,2,2,2\}}(M).$$

First, write the divided power as a multation

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ y & x & y & y \end{bmatrix};$$

then, simply compose the multations:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ y & x & y & y \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ x & y & y & y \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 & 2 & 2 \\ y & y & x & y \end{bmatrix}.$$

Therefore,

$$\Gamma^{\mu}(y \otimes xy^{[2]}) = xy \otimes y^{[2]} + 2(y^{[2]} \otimes xy).$$

△

Like their cousins, the deviated powers, the divided power functors provide a splitting of the fundamental functor into its atomic components.

THEOREM 15.

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, -) = \bigoplus_{\substack{\#A \subseteq [n] \\ |A| = n}} \Gamma^A.$$

Proof.

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, M) = \Gamma^n(M^n) = \bigoplus_{\substack{\#A \subseteq [n] \\ |A| = n}} \Gamma^A(M).$$

□

§10. THE MULTI-SET YONEDA CORRESPONDENCE

And finally, to close the chapter, yet another Yoneda Lemma. As usual, σ denotes transportation maps.

THEOREM 16: THE MULTI-SET YONEDA LEMMA. — *Let A be a multi-set, and F a strict analytic functor. The map*

$$\begin{aligned} \mathrm{Y}_{A,F} \colon \operatorname{Nat}(\Gamma^A, F) &\to F_A^{\dagger}(\mathbf{B}|_{\#A}) \\ \zeta &\mapsto \zeta_{\mathbf{B}^{\#A}}(1^{\otimes [A]}) \end{aligned}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{array}{ccccc} A & \qquad & \mathrm{Nat}(\Gamma^A, F) & \xrightarrow{\Upsilon_{A,F}} & F_A^\dagger(\mathbf{B}|_{\#A}) \\ \mu\downarrow & & (\Gamma^\mu)^*\downarrow & & \downarrow F_{\sigma[\mu]} \\ B & & \mathrm{Nat}(\Gamma^B, F) & \xrightarrow[\Upsilon_{B,F}]{} & F_B^\dagger(\mathbf{B}|_{\#B}) \end{array}$$

$$\begin{array}{ccccc} F & \qquad & \mathrm{Nat}(\Gamma^A, F) & \xrightarrow{\Upsilon_{A,F}} & F_A^\dagger(\mathbf{B}|_{\#A}) \\ \xi\downarrow & & \xi_*\downarrow & & \downarrow (\xi_A^\dagger)_{\mathbf{B}|_{\#A}} \\ G & & \mathrm{Nat}(\Gamma^A, G) & \xrightarrow[\Upsilon_{A,G}]{} & G_A^\dagger(\mathbf{B}|_{\#A}) \end{array}$$

Proof. A natural transformation

$$\zeta\colon \Gamma^A \to F$$

corresponds to a natural transformation

$$\eta_M\colon \Gamma^{|A|}\operatorname{Hom}(\mathbf{B}^{\#A}, M) \to F(M),$$

taking

$$\alpha^{[X]} \mapsto \begin{cases} 0 & \text{if } X \neq A, \\ \zeta_M\left(\alpha(1)^{\otimes[X]}\right) & \text{if } X = A, \end{cases}$$

where $\alpha_a\colon \mathbf{B}_a \to M$, for $a \in \#A$. The original Yoneda map takes η to

$$\eta_{\mathbf{B}^{\#A}}(1_{\mathbf{B}^{\#A}}^{[|A|]}) = \sum_{\substack{X\subseteq\#A \\ |X|=|A|}} \eta_{\mathbf{B}^{\#A}}(1^{\otimes[X]}) = \eta_{\mathbf{B}^{\#A}}(1^{\otimes[A]}) = \zeta_{\mathbf{B}^{\#A}}(1^{\otimes[A]}).$$

Thus, $\Upsilon_{A,F}$ is the A-component of the original Yoneda map, and it is clear that

$$\Upsilon_{A,F}(\zeta) = \zeta_{\mathbf{B}^{\#A}}(1^{\otimes[A]}) \in F_A^\dagger(\mathbf{B}|_{\#A}).$$

That the second diagram above commutes is clear, so there remains to investigate the first. Evidently, since μ is a multation, A and B must be multisets of the same cardinality n. From the original Yoneda Lemma, there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Nat}(\Gamma^n \operatorname{Hom}(\mathbf{B}^{\#A}, -), F) & \xrightarrow{\Upsilon_{\mathbf{B}^{\#A},F}} & F(\mathbf{B}^{\#A}) \\ \left((\sigma^*)^{[\mu]}\right)^*\downarrow & & \downarrow F_{\sigma[\mu]} \\ \mathrm{Nat}(\Gamma^n \operatorname{Hom}(\mathbf{B}^{\#B}, -), F) & \xrightarrow[\Upsilon_{\mathbf{B}^{\#B},F}]{} & F(\mathbf{B}^{\#B}) \end{array}$$

We shall show that the following diagram commutes, from which the claim follows:

$$\begin{array}{ccc} \mathrm{Nat}(\Gamma^A, F) & \longrightarrow & \mathrm{Nat}(\Gamma^n \operatorname{Hom}(\mathbf{B}^{\#A}, -), F) \\ {\scriptstyle (\Gamma^\mu)^*}\big\downarrow & & \big\downarrow{\scriptstyle \left((\sigma^*)^{[\mu]}\right)^*} \\ \mathrm{Nat}(\Gamma^B, F) & \longrightarrow & \mathrm{Nat}(\Gamma^n \operatorname{Hom}(\mathbf{B}^{\#B}, -), F) \end{array}$$

Consider a natural transformation

$$\zeta\colon \Gamma^A \to F,$$

corresponding to an

$$\eta\colon \Gamma^n \operatorname{Hom}(\mathbf{B}^{\#A}, -) \to F$$

as above. The homomorphism

$$\left((\sigma^*)^{[\mu]}\right)^*$$

maps η to the natural transformation

$$\eta_M(\sigma^*)^{[\mu]}\colon \Gamma^n \operatorname{Hom}(\mathbf{B}^{\#B}, M) \to F(M),$$

which, for homomorphisms $\alpha_b\colon \mathbf{B}_b \to M$, takes

$$\begin{aligned} \alpha^{[B]} &\mapsto \eta_M(\alpha^{[B]} \circ (\sigma^*)^{[\mu]}) \\ &= \eta_M((\alpha\sigma)^{[\mu]}) = \zeta_M((\alpha\sigma)(1)^{[\mu]}) = \zeta_M(\alpha(1)^{\otimes[\mu]}). \end{aligned}$$

On the other hand,

$$\zeta\Gamma^\mu\colon \Gamma^B \to F$$

corresponds to a natural transformation

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^{\#B}, M) \to F(M),$$

that takes

$$\alpha^{[B]} \mapsto \zeta_M \Gamma^\mu(\alpha(1)^{\otimes[B]}) = \zeta_M(\alpha(1)^{\otimes[\mu]}).$$

The commutativity of the diagram is demonstrated. □

Chapter 9

MODULE REPRESENTATIONS

Et la glace où se fige un réel mouvement
Reste froide malgré son détestable ouvrage.
La force du miroir trompa plus d'un amant
Qui crut aimer sa belle et n'aima qu'un mirage.

— Guillaume Apollinaire, *La Force du Miroir*

Exhibited on display in the preceding chapter were some rather special functors, most notably

$$\mathbf{B}[\operatorname{Hom}(\mathbf{B}^n, -)]_n \quad \text{and} \quad \Gamma^n \operatorname{Hom}(\mathbf{B}^n, -).$$

We rather vaguely suggested they were projective generators for the categories $\mathfrak{Num}_n$ and $\mathfrak{Hom}_n$, respectively, postponing the proof to the present chapter. After proving this, and invoking the celebrated Morita Equivalence, it can be inferred that numerical and homogeneous functors of degree n may be viewed as modules over the rings $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ and $\Gamma^n(\mathbf{B}^{n\times n})$, respectively.

A first incarnation of this curious result, published in a Georgian journal of some obscurity, appeared in 1988, when Professor Pirashvili, [17], established an equivalence of the category of *polynomial* functors (over $\mathbf{Z}$) with a module category. Fifteen years later, Dr. Salomonsson and Professor Ekedahl discovered that strict polynomial functors likewise admit a module representation. This was based on previous work by Professors Friedlander and Suslin, which only subsumed the special case of the base ring being a field. We refer to Salomonsson's doctoral thesis [20] for the general result.

We shall state and prove an equivalent version of Salomonsson's theorem, using the modern conveniences brought about by the invention of multations, along with a modified version of Pirashvili's, making it apply to numerical, rather than polynomial, functors.

The formidable consequence of these two theorems is that polynomial functors — of modules! — may in fact be viewed as modules themselves, albeit over two different rings. Techniques of module theory may then enter the game in order to study these functors.

§1. The Divided Power Map

Presupposing for a moment the key rôle the rings $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ and $\Gamma^n(\mathbf{B}^{n\times n})$ are about to play, one obvious way of relating our two species of functors — numerical and strict polynomial — is to find homomorphisms between the two rings. We shall pave the way in this section, preparing for a full exploration later.

Let us begin with somewhat greater generality. Start with a module M. We propose a study of the **divided power map**

$$\begin{aligned} \gamma_n \colon M &\to \Gamma^n(M) \\ x &\mapsto x^{[n]}. \end{aligned}$$

Theorem 1. — *The divided power map is numerical of degree n and therefore induces a linear map*

$$\begin{aligned} \gamma_n \colon \mathbf{B}[M]_n &\to \Gamma^n(M) \\ [x] &\mapsto x^{[n]}. \end{aligned}$$

This is a natural transformation of (numerical) functors.

Proof. Since γ_n is homogeneous of degree n, it is also numerical of the same degree. □

Lemma 1. — *If $x_1, \dots, x_n \in M$, then*

$$(x_1 \diamond \cdots \diamond x_n)^{[n]} = x_1 \cdots x_n.$$

Proof. The Principle of Inclusion and Exclusion. □

Theorem 2. — *Let M be finitely generated and free. The cokernel of the homomorphism*

$$(\pi, \gamma_n) \colon \mathbf{B}[M]_n \to \mathbf{B}[M]_{n-1} \oplus \Gamma^n(M)$$

is

$$\operatorname{Coker}(\pi, \gamma_n) \cong \Gamma^n(M)/\langle x_1 \dots x_n \mid x_i \in M \rangle.$$

In particular, (π, γ_n) is an injection of finite index.

Proof. Let $\{e_1, \dots, e_k\}$ be a basis for M. Then the elements

$$[f_1 \diamond \cdots \diamond f_m], \quad f_i \in \{e_1, \dots, e_k\},$$

for $0 \leqslant m \leqslant n$, constitute a basis for $\mathbf{B}[M]_n$. The image of (π, γ_n) is generated by the images

$$(\pi, \gamma_n)([f_1 \diamond \cdots \diamond f_m]) = \Big([f_1 \diamond \cdots \diamond f_m], [f_1 \diamond \cdots \diamond f_m]^{[n]}\Big), \quad 0 \leqslant m < n;$$

and

$$(\pi, \gamma_n)([f_1 \diamond \cdots \diamond f_n]) = \Big(0, (f_1 \diamond \cdots \diamond f_n)^{[n]}\Big) = (0, f_1 \cdots f_n).$$

The relations

$$[f_1 \diamond \cdots \diamond f_m] \equiv -[f_1 \diamond \cdots \diamond f_m]^{[n]} \mod \operatorname{Im}(\pi, \gamma_n)$$

let us represent each element of the cokernel by a sum of divided nth powers, while the relations

$$f_1 \ldots f_n \equiv 0 \mod \operatorname{Im}(\pi, \gamma_n)$$

yield the desired factor module of $\Gamma^n(M)$. □

Theorem 3. — *Let M be finitely generated and free. The kernel of the homomorphism*

$$\gamma_n \colon \mathbf{B}[M]_n \to \Gamma^n(M)$$

is

$$\operatorname{Ker} \gamma_n = \mathbf{Q} \otimes_{\mathbf{Z}} \langle [rz] - r^n[z] \mid r \in \mathbf{B},\ z \in M \rangle \cap \mathbf{B}[M]_n.$$

Proof. Let $\{e_1, \ldots, e_k\}$ be a basis for M. Then the elements

$$[f_1 \diamond \cdots \diamond f_m], \quad f_i \in \{e_1, \ldots, e_k\},$$

for $0 \leqslant m \leqslant n$, constitute a basis for $\mathbf{B}[M]_n$.

Denote

$$L = \mathbf{Q} \otimes_{\mathbf{Z}} \langle [rz] - r^n[z] \mid r \in \mathbf{B}, z \in M \rangle;$$

then evidently

$$L \cap \mathbf{B}[M]_n \subseteq \operatorname{Ker} \gamma_n.$$

We now show the reverse inclusion.

Calculating modulo L, we have

$$\begin{aligned} \Big[\mathop{\Diamond}_{m} z\Big] &= \sum_{I \subseteq [m]} (-1)^{m-|I|} \Big[\sum_{i \in I} z\Big] = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} [rz] \\ &\equiv \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} r^n [z] = m! \left\{ {n \atop m} \right\} [z], \end{aligned}$$

where $\left\{ {n \atop m} \right\}$ denotes a Stirling number of the second kind.

We may then write

$$\begin{aligned} m! \left\{ {n \atop m} \right\} [f_1 \diamond \cdots \diamond f_m] &= m! \left\{ {n \atop m} \right\} \sum_{I \subseteq [m]} (-1)^{m-|I|} \Big[\sum_{i \in I} f_i\Big] \\ &= \sum_{I \subseteq [m]} (-1)^{m-|I|} \Big[\mathop{\Diamond}_{m} \sum_{i \in I} f_i\Big] \\ &= \xi + \xi', \end{aligned}$$

where ξ is a sum of mth deviations, and ξ' collects the higher-order deviations. We calculate ξ:

$$\begin{aligned}\xi &= \sum_{I\subseteq[m]} (-1)^{m-|I|} \sum_{\substack{\#A\subseteq I\\ |A|=m}} \binom{m}{A} \left[\Diamond_{a\in A} f_a\right]\\
&= \sum_{\substack{\#A\subseteq[m]\\ |A|=m}} \left(\sum_{\#A\subseteq I\subseteq[m]} (-1)^{m-|I|}\right) \binom{m}{A} \left[\Diamond_{a\in A} f_a\right]\\
&= \sum_{\substack{\#A=[m]\\ |A|=m}} \binom{m}{A} \left[\Diamond_{a\in A} f_a\right] = \binom{m}{1,\ldots,1}[f_1\diamond\cdots\diamond f_m] = m![f_1\diamond\cdots\diamond f_m].\end{aligned}$$

We thus have

$$m!\left\{{n \atop m}\right\}[f_1\diamond\cdots\diamond f_m] \equiv m![f_1\diamond\cdots\diamond f_m] + \xi' \mod L,$$

and, consequently, provided $1 < m < n$ (so that $\left\{{n \atop m}\right\} > 1$),

$$[f_1\diamond\cdots\diamond f_m] \equiv \frac{1}{m!\left(\left\{{n \atop m}\right\} - 1\right)} \cdot \xi' \mod L.$$

Now suppose $\omega \in \operatorname{Ker}\gamma_n$. Using the above relation, together with

$$[\diamond] = [0] \equiv 0 \mod L$$

and

$$[\diamond f] = [f] - [0] \equiv \frac{1}{n!\left\{{n \atop n}\right\}}\left[\Diamond_n f\right] = \frac{1}{n!}\left[\Diamond_n f\right] \mod L,$$

we may express ω as a (fractional) linear combination of nth deviations of the basis elements e_i:

$$\omega \equiv \sum_{\substack{\#A\subseteq[k]\\ |A|=n}} c_A \left[\Diamond_{a\in A} e_a\right] \mod L.$$

Apply γ_n:

$$0 = \gamma_n(\omega) = \sum_{\substack{\#A\subseteq[k]\\ |A|=n}} c_A \gamma_n\left(\Diamond_{a\in A} e_a\right) = \sum_{\substack{\#A\subseteq[k]\\ |A|=n}} c_A e^A.$$

Because the elements $e^{[A]}$ constitute a basis for $\Gamma^n(M)$, it must be that all coefficients $c_A = 0$, and hence $\omega \in L$. The proof is finished. □

We remark that the divided power map may be extended, without much ado, to a map

$$\gamma_n\colon \mathbf{Q}\otimes_{\mathbf{Z}} \mathbf{B}[M]_n \to \mathbf{Q}\otimes_{\mathbf{Z}} \Gamma^n(M).$$

This map (almost trivially) possesses an inverse. We shall, however, be interested in obtaining an inverse under a slightly less generous localisation.

Theorem 4. — *The homomorphism*

$$\begin{aligned} \varepsilon_n \colon \Gamma^n(M) &\to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[M]_n \\ x^{[A]} &\mapsto \frac{1}{\deg A} \left[\underset{A}{\Diamond}\, x \right] \end{aligned}$$

constitutes a section of the divided power map:

$$\gamma_n \varepsilon_n = 1_{\Gamma^n(M)}.$$

This leads to a direct sum decomposition

$$\operatorname{Im} \varepsilon_n \cong \Gamma^n(M) \oplus (\operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n).$$

Proof. The relation $\gamma_n \varepsilon_n = 1$ is immediate, and then the following exact sequence splits:

$$0 \longrightarrow \operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n \longrightarrow \operatorname{Im} \varepsilon_n \underset{\varepsilon_n}{\overset{\gamma_n}{\rightleftarrows}} \Gamma^n(M) \longrightarrow 0$$

□

It will now be appropriate to specialise the preceding discussion to the particular rings $\mathbf{B}[\mathbf{B}^{n \times n}]_n$ and $\Gamma^n(\mathbf{B}^{n \times n})$.

Theorem 5. — *The maps*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n \times n}]_n \to \Gamma^n(\mathbf{B}^{n \times n}), \qquad \varepsilon_n \colon \Gamma^n(\mathbf{B}^{n \times n}) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[\mathbf{B}^{n \times n}]_n$$

are homomorphisms of algebras, when both rings are equipped with the product multiplication.

Proof. Calculate:

$$\gamma_n([x]) \star \gamma_n([y]) = x^{[n]} \star y^{[n]} = (xy)^{[n]} = \gamma_n([xy]) = \gamma([x] \star [y]).$$

To show ε_n preserves multiplication, it will be enough to consider pure powers $x^{[n]}$ and $y^{[n]}$. Use the Deviation Formula:

$$\begin{aligned} \varepsilon_n(x^{[n]}) \star \varepsilon_n(y^{[n]}) &= \frac{1}{n!} \left[\underset{n}{\Diamond}\, x \right] \star \frac{1}{n!} \left[\underset{n}{\Diamond}\, y \right] \\ &= \frac{1}{(n!)^2} \cdot n! \left[\underset{n}{\Diamond}\, xy \right] \\ &= \frac{1}{n!} \left[\underset{n}{\Diamond}\, xy \right] = \varepsilon_n((xy)^{[n]}) = \varepsilon_n(x^{[n]} \star y^{[n]}). \end{aligned}$$

□

§2. Module Functors

Lemma 2. — *A module functor that vanishes on* $\mathbf{B}^\infty$ *is identically zero.*

Proof. Let

$$F\colon \mathfrak{XMod} \to \mathfrak{Mod}$$

be a module functor. Since F is extended to arbitrary modules through inductive limits, we have by definition

$$F(\mathbf{B}^\infty) = \varinjlim F(\mathbf{B}^n).$$

Note that $F(M)$ is always a direct summand in $F(M \oplus N)$, for modules M and N. This follows from the Cross-Effect Decomposition:

$$F(M \oplus N) = \underbrace{F^\dagger() \oplus F^\dagger(M)}_{F(M)} \oplus F^\dagger(N) \oplus F^\dagger(M \mid N).$$

Since $\mathbf{B}^n$ is always a direct summand of $\mathbf{B}^\infty$, any $F(\mathbf{B}^n)$ is a direct summand of $F(\mathbf{B}^\infty)$. □

Theorem 6. — *The fundamental module functor*

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)]$$

is a small projective generator for

$$\mathfrak{Num}_\infty = \mathrm{Fun}(\mathfrak{XMod}, \mathfrak{Mod}),$$

through which there is a Morita equivalence

$$\mathrm{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \sim {}_{\mathbf{B}[\mathbf{B}^{\infty\times\aleph_0}]}\mathfrak{Mod},$$

where $\mathbf{B}[\mathbf{B}^{\infty\times\aleph_0}]$ *carries the product multiplication.*

More precisely, the functor F *corresponds to the abelian group* $F(\mathbf{B}^\infty)$, *with module structure given by the equation*

$$[\tau] \cdot x = F(\tau)(x).$$

Proof. Step 1: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)]$ *is projective.* We must show that

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)], -)$$

is right-exact, or preserves epimorphisms. Hence let $\eta\colon F \to G$ be epic, so that each η_M is onto. The following diagram, constructed by aid of the Yoneda Lemma, shows that η_* is epic:

$$\begin{array}{ccc}
\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)], F) & \overset{Y}{\longleftrightarrow} & F(\mathbf{B}^\infty) \\
\Big\downarrow{\scriptstyle \eta_*} & & \Big\downarrow{\scriptstyle \eta_{\mathbf{B}^\infty}} \\
\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^\infty, -)], G) & \underset{Y}{\longleftrightarrow} & G(\mathbf{B}^\infty)
\end{array}$$

Step 2: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)]$ *is a generator.* By the lemma,

$$0 = \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)], F) \cong F(\mathbf{B}^{\infty})$$

implies $F = 0$.

Step 3: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)]$ *is small.* Compute, using the Yoneda Lemma:

$$\begin{aligned}\mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)], \bigoplus F_k\right) &\cong \left(\bigoplus F_k\right)(\mathbf{B}^{\infty}) = \bigoplus F_k(\mathbf{B}^{\infty}) \\ &\cong \bigoplus \mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)], F_k\right).\end{aligned}$$

Step 4: The Morita equivalence. As $\mathrm{Fun}(\mathfrak{X}\mathfrak{Mod}, \mathfrak{Mod})$ is an abelian category with arbitrary direct sums, we have a Morita equivalence:

$$\mathrm{Fun}(\mathfrak{X}\mathfrak{Mod}, \mathfrak{Mod}) \underset{\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)]\otimes_S -}{\overset{\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)], -)}{\rightleftarrows}} {}_S\mathfrak{Mod}$$

The new base ring is

$$S = \left(\mathrm{Nat}\, \mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)]\right)^{\circ} \cong \mathbf{B}[\mathrm{End}\, \mathbf{B}^{\infty}] = \mathbf{B}[\mathbf{B}^{\infty \times \aleph_0}].$$

Plainly, the functor F corresponds to the abelian group

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, -)], F) \cong F(\mathbf{B}^{\infty}).$$

Step 5: The module structure. Under the Yoneda map, an element $x \in F(\mathbf{B}^{\infty})$ will correspond to the natural transformation

$$\begin{aligned}\eta_M \colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, M)] &\to F(M) \\ [\alpha] &\mapsto F(\alpha)(x).\end{aligned}$$

Likewise, a scalar $[\tau] \in \mathbf{B}[\mathbf{B}^{\infty \times \aleph_0}]$ will correspond to

$$\begin{aligned}\sigma_M \colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, M)] &\to \mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, M)] \\ [\alpha] &\mapsto [\alpha \circ \tau].\end{aligned}$$

The product of the scalar σ and the module element η is the transformation

$$\begin{aligned}(\eta \circ \sigma)_M \colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^{\infty}, M)] &\to F(M) \\ [\alpha] &\mapsto F(\alpha \circ \tau)(x),\end{aligned}$$

which under the Yoneda map corresponds to

$$(\eta \circ \sigma)_{\mathbf{B}^{\infty}}([1_{\mathbf{B}^{\infty}}]) = F(1_{\mathbf{B}^{\infty}} \circ \tau)(x) = F(\tau)(x) \in F(\mathbf{B}^{\infty}).$$

The scalar multiplication on $F(\mathbf{B}^{\infty})$ is thus given by the formula

$$[\tau] \cdot x = F(\tau)(x),$$

and the proof is finished. □

Example 1. — The functor T^2 corresponds to the abelian group

$$T^2(\mathbf{B}^\infty) = \mathbf{B}^\infty \otimes \mathbf{B}^\infty,$$

with module structure

$$[\tau] \cdot (x \otimes y) = T^2(\tau)(x \otimes y) = \tau(x) \otimes \tau(y),$$

for any $[\tau] \in \mathbf{B}[\mathbf{B}^{\infty \times \aleph_0}]$. △

§3. Numerical Functors

We now repeat the feat, but for numerical functors.

Lemma 3. — *A polynomial functor of degree n that vanishes on $\mathbf{B}^n$ is identically zero.*

Proof. Suppose that F is polynomial of degree n, and that $F(\mathbf{B}^n) = 0$. We shall show that $F(\mathbf{B}^q) = 0$ for all natural numbers q.

Consider first the case $q \leqslant n$. Then $\mathbf{B}^q$ is a direct summand of $\mathbf{B}^n$, so $F(\mathbf{B}^q)$ is a direct summand of $F(\mathbf{B}^n) = 0$.

Proceed by induction, and suppose $F(\mathbf{B}^{q-1}) = 0$ for some $q - 1 \geqslant n$. Decompose

$$1_{\mathbf{B}^q} = \pi_1 + \cdots + \pi_q,$$

where $\pi_j \colon \mathbf{B}^q \to \mathbf{B}^q$ as usual denotes the jth projection. Since F is polynomial of degree n, and therefore of degree $q - 1$,

$$0 = F(\pi_1 \diamond \cdots \diamond \pi_q) = \sum_{J \subseteq [q]} (-1)^{q-|J|} F\left(\sum_J \pi_j\right).$$

Consider a J with $|J| \leqslant q - 1$. Since $\sum_J \pi_j$ factors through $\mathbf{B}^{q-1}$, the homomorphism $F\left(\sum_J \pi_j\right)$ factors through $F(\mathbf{B}^{q-1}) = 0$. Only $J = [q]$ will give a non-trivial contribution to the sum above, yielding

$$0 = F(\pi_1 + \cdots + \pi_q) = F(1_{\mathbf{B}^q}) = 1_{F(\mathbf{B}^q)};$$

hence $F(\mathbf{B}^q) = 0$. □

Theorem 7. — *The fundamental numerical functor*

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n$$

is a small projective generator for $\mathfrak{Num}_n$, through which there is a Morita equivalence

$$\mathfrak{Num}_n \sim {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod},$$

where $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ carries the product multiplication.

More precisely, the functor F corresponds to the abelian group $F(\mathbf{B}^n)$, with module structure given by the equation

$$[\tau] \cdot x = F(\tau)(x).$$

Proof. The preceding proof goes through exactly as before, but with Lemma 3 in place of Lemma 2. □

Example 2. — Let $n = 2$. The functor T^2 corresponds to the abelian group

$$T^2(\mathbf{B}^2) = \mathbf{B}^2 \otimes \mathbf{B}^2,$$

with module structure

$$[\tau] \cdot (x \otimes y) = T^2(\tau)(x \otimes y) = \tau(x) \otimes \tau(y),$$

for any $[\tau] \in \mathbf{B}[\mathbf{B}^{2\times 2}]_2$.

This should be compared with the module obtained in Example 1. Apparently this latter module contains vast amounts of superfluous data, and may be cut down in size considerably, once we take advantage of the fact that the functor is quadratic. △

§4. Homogeneous Functors

Theorem 8. — *The fundamental homogeneous functor*

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, -)$$

is a small projective generator for $\mathfrak{Hom}_n$*, through which there is a Morita equivalence*

$$\mathfrak{Hom}_n \sim {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod},$$

where $\Gamma^n(\mathbf{B}^{n\times n})$ *carries the product multiplication.*

More precisely, the functor F *corresponds to the abelian group* $F(\mathbf{B}^n)$*, with module structure given by the equation*

$$\tau^{[n]} \cdot x = F(\tau)(x).$$

Proof. The previous proof goes through exactly as before. □

Example 3. — Let $n = 2$. The functor T^2 corresponds once again to the abelian group

$$T^2(\mathbf{B}^2) = \mathbf{B}^2 \otimes \mathbf{B}^2,$$

with module structure

$$\tau^{[2]} \cdot (x \otimes y) = T^2(\tau)(x \otimes y) = \tau(x) \otimes \tau(y),$$

for any $\tau^{[2]} \in \Gamma^2(\mathbf{B}^{2\times 2})$. △

§5. Quasi-Homogeneous Functors

With all the ground-work laid in the introductory section, the module-theoretic interpretation of quasi-homogeneity is immediate.

Theorem 9. — *Let F be a numerical functor, corresponding to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M. The functor F is quasi-homogeneous of degree n iff M is a module over*

$$\operatorname{Im}\gamma_n \cong \mathbf{B}[\mathbf{B}^{n\times n}]_n/\operatorname{Ker}\gamma_n.$$

Proof. Recall that the scalar multiplication of $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ on $M = F(\mathbf{B}^n)$ is given by

$$[\sigma]x = F(\sigma)(x), \quad \sigma \in \mathbf{B}^{n\times n},\ x \in F(\mathbf{B}^n).$$

The requirement that $\operatorname{Ker}\gamma_n$ annihilate $F(\mathbf{B}^n)$ is equivalent to demanding that F itself vanish on

$$\operatorname{Ker}\gamma_n = \mathbf{Q} \otimes_{\mathbf{Z}} \left\langle [r\sigma] - r^n[\sigma] \mid r \in \mathbf{B},\ \sigma \in \mathbf{B}^{n\times n} \right\rangle \cap \mathbf{B}[\mathbf{B}^{n\times n}]_n,$$

which clearly would be a consequence of quasi-homogeneity.

To show that, conversely,

$$F(r\sigma) = r^n F(\sigma), \quad \sigma \in \mathbf{B}^{n\times n},$$

implies quasi-homogeneity, we first show that

$$F(r \cdot 1_{\mathbf{B}^q}) = r^n F(1_{\mathbf{B}^q})$$

for all natural numbers q. This is clear when $q \leqslant n$, for then $1_{\mathbf{B}^q}$ factors through $1_{\mathbf{B}^n}$. When $q > n$, split up into the canonical projections, and use induction:

$$\begin{aligned} F(r \cdot 1_{\mathbf{B}^q}) &= F(r\pi_1 + \cdots + r\pi_q) \\ &= -\sum_{I \subset [q]} (-1)^{q-|I|} F\left(\sum_{i\in I} r\pi_i\right) \\ &= -\sum_{I \subset [q]} (-1)^{q-|I|} r^n F\left(\sum_{i\in I} \pi_i\right) \\ &= r^n F(\pi_1 + \cdots + \pi_q) = r^n F(1_{\mathbf{B}^q}). \end{aligned}$$

Finally, for an arbitrary homomorphism $\alpha\colon \mathbf{B}^p \to \mathbf{B}^q$, we have

$$F(r\alpha) = F(r \cdot 1_{\mathbf{B}^q})F(\alpha) = r^n F(1_{\mathbf{B}^q})F(\alpha) = r^n F(\alpha).$$

□

Chapter 10

COMBINATORIAL REPRESENTATIONS

"Here is Ts'ui Pên's labyrinth," he said, indicating a tall lacquered desk.
"An ivory labyrinth!" I exclaimed. "A minimum labyrinth."
"A labyrinth of symbols," he corrected. "An invisible labyrinth of time. [...]"

— Jorge Luis Borges, *The Garden of Forking Paths*

At the beginning of the twenty-first century, the team Baues, Dreckmann, Franjou & Pirashvili sought methods to combinatorially encode **Z**-module functors, with a particular eye on polynomial ones. Their design was to build a clever two-way correspondence (category equivalence) between functors

$$_{\mathbf{N}}\mathfrak{XMod} \to {}_{\mathbf{Z}}\mathfrak{Mod},$$

on the one hand, and *Mackey functors* (whatever that may be) from the category $\mathfrak{Sur}$, on the other. This marvellous theorem, the first of its kind, enabled the reduction of a functor to a significantly smaller collection of data.

Passing to polynomial functors has the effect of erasing the distinction between **N**-modules and **Z**-modules, and the quartet's main result may then be stated, without resorting to Mackey functors: *Polynomial (**Z**-)functors of degree n are equivalent to linear (additive) functors from* $\widehat{\mathfrak{Sur}}_n$.

They left unresolved the interesting question of how to systematise the full category of **Z**-module functors. Moreover, attempts to generalise their argument to arbitrary base rings will encounter difficulties, as it is not clear what category should play the rôle of $\mathfrak{Sur}$. Here the labyrinth category comes to the rescue, constituting, as it does, the universal means of combinatorially encoding *all* module functors (not just polynomial ones) over *any* (unital) base ring (not just commutative ones). The generalisation to arbitrary rings reads: *Numerical functors of degree n are equivalent to linear functors from* $\mathfrak{Laby}_n$. This includes the case $n = \infty$, corresponding to general module functors. To connect with the above result, recall that we have previously exhibited a category isomorphism

$$\widehat{\mathfrak{Sur}}_n \cong {}_{\mathbf{Z}}\mathfrak{Laby}_n.$$

After the quartet had announced the publication of their results in [1], Torsten Ekedahl and Pelle Salomonsson quickly followed suite. They showed in

[20] how strict polynomial functors (over any commutative base ring) may be similarly encoded, this time as Mackey functors from the category of multisets and *multijections*. Striving, as we do, to avoid the language of Mackey functors at any cost, we have chosen to replace multijections with multations. The pair's theorem, which we reprove in this setting, then asserts the following: *Homogeneous functors of degree n are equivalent to linear functors from* $\mathfrak{MSet}_n$.

§1. Module Functors

As is our custom when dealing with general module functors, we suppose the base ring ring **B** unital only, and not necessarily commutative. As before, free and finitely generated modules are automatically bimodules, but all homomorphisms are *right module homomorphisms*.

Two kinds of functors will be considered. Module functors $\mathfrak{XMod} \to \mathfrak{Mod}$ may, of course, be of an *arbitrary* nature (linear, polynomial, numerical, and what not). On the other hand, *labyrinth modules* $\mathfrak{Laby} \to \mathfrak{Mod}$ *shall always be assumed* **B**-*linear.*

As usual, σ denotes transportation maps.

Definition 1. — Given a homomorphism

$$\alpha = \sum_{a\in A, b\in B} s_{ba}\sigma_{ba} \colon \mathbf{B}^A \to \mathbf{B}^B,$$

(a $B \times A$ matrix) its **associated maze** $S \colon A \to B$ is

$$S = \left\{ \left[\ a \xrightarrow{s_{ba}} b\ \right] \,\middle|\, a \in A, b \in B \right\}.$$

◇

Associated mazes are always simple. Note that, if but a single component s_{ba} vanish, the associated maze $S = 0$.

Theorem 1.

1. *Let P be the associated maze of* $\alpha \colon \mathbf{B}^A \to \mathbf{B}^B$, *and Q the associated maze of* $\beta \colon \mathbf{B}^A \to \mathbf{B}^B$. *Then the associated maze of* $\alpha + \beta$ *is* $P \boxplus Q$.
2. *Let P be the associated maze of* $\alpha \colon \mathbf{B}^B \to \mathbf{B}^C$, *and Q the associated maze of* $\beta \colon \mathbf{B}^A \to \mathbf{B}^B$. *Then the associated maze of* $\alpha \circ \beta$ *is* $P \boxdot Q$.

This theorem should retroactively motivate our interest in the operations $\boxplus$ and $\boxdot$, as well as our choice of notation.

Getting down to business, we wish now to define a functor

$$\Phi \colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Fun}(\mathfrak{Laby}, \mathfrak{Mod}),$$

which will eventually turn out to be an equivalence. Given a module functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$, the corresponding labyrinth functor should take a finite set to the corresponding cross-effect:

$$X \mapsto F^\dagger(\mathbf{B}|_X).$$

Mazes should be interpreted as deviations, in the following sense:

$$[P\colon X \to Y] \mapsto \left[F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right)\Big|_{F^\dagger(\mathbf{B}|_X)\to F^\dagger(\mathbf{B}|_Y)} \right].$$

Restricting action to the appropriate cross-effects is in fact an unnecessary caution, as shown by the following lemma.

Lemma 1. — *The map*

$$F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right) : F(\mathbf{B}^X) \to F(\mathbf{B}^Y)$$

is in fact a map

$$F^\dagger(\mathbf{B}|_X) \to F^\dagger(\mathbf{B}|_Y),$$

in the sense that all other components vanish.

Proof. We use Theorem 7.4. If τ_x is any insertion with $x \in X$, then $\sigma_{yx}\tau_x = 0$, and hence

$$F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right) F(\tau_x) = 0.$$

Similarly,

$$F(\rho_y) F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right) = 0,$$

when ρ_y is any retraction with $y \in Y$. □

For the functor

$$\Phi(F)\colon \mathfrak{Laby} \to \mathfrak{Mod},$$

we thus propose the following definition:

$$X \mapsto F^\dagger(\mathbf{B}|_X)$$

$$[P\colon X \to Y] \mapsto \left[F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right) : F^\dagger(\mathbf{B}|_X) \to F^\dagger(\mathbf{B}|_Y) \right].$$

Lemma 2. — $\Phi(F)$ *is a functor* $\mathfrak{Laby} \to \mathfrak{Mod}$.

Proof. That $\Phi(F)$ respects the relations in $\mathfrak{Laby}$ follows from

$$\Phi(F)\left(P \cup \{\, x \xrightarrow{0} y \,\}\right) = F(\cdots \diamond 0) = 0,$$

and

$$\begin{aligned}
\Phi(F)\left(P \cup \{\, x \xrightarrow{a+b} y \,\}\right) &= F(\cdots \diamond (a+b)\sigma_{yx}) \\
&= F(\cdots \diamond a\sigma_{yx}) + F(\cdots \diamond b\sigma_{yx}) + F(\cdots \diamond a\sigma_{yx} \diamond b\sigma_{yx}) \\
&= \Phi(F)\left(P \cup \{\, x \xrightarrow{a} y \,\}\right) + \Phi(F)\left(P \cup \{\, x \xrightarrow{b} y \,\}\right) \\
&\quad + \Phi(F)\left(P \cup \{\, x \underset{b}{\overset{a}{\rightrightarrows}} y \,\}\right).
\end{aligned}$$

Functoriality of $\Phi(F)$ follows from the Deviation Formula and the definition of maze composition. □

Let $\zeta\colon F \to G$ be a natural transformation. Define

$$\Phi(\zeta)\colon \Phi(F) \to \Phi(G)$$

by restriction to the appropriate cross-effects:

$$\Phi(\zeta)_X = \zeta^{\dagger}_{\mathbf{B}|_X}\colon F^{\dagger}(\mathbf{B}|_X) \to G^{\dagger}(\mathbf{B}|_X).$$

Lemma 3. — Φ *is a functor*

$$\mathrm{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \mathrm{Fun}(\mathfrak{Laby}, \mathfrak{Mod}).$$

Proof. By the Labyrinthine Yoneda Lemma,

$$\zeta^{\dagger}_{\mathbf{B}|_X}\colon F^{\dagger}(\mathbf{B}|_X) \to G^{\dagger}(\mathbf{B}|_X)$$

corresponds to

$$\zeta_*\colon\ \mathrm{Nat}(\Delta^X, F) \to \mathrm{Nat}(\Delta^X, G).$$

□

We now construct the inverse of Φ. Let

$$H\colon \mathfrak{Laby} \to \mathfrak{Mod}$$

be a functor. Define

$$\Phi^{-1}(H)\colon \mathfrak{XMod} \to \mathfrak{Mod}$$

by

$$\Phi^{-1}(H)(\mathbf{B}^A) = \bigoplus_{X \subseteq A} H(X).$$

Also, let

$$\alpha = \sum_{\substack{a\in A\\ b\in B}} s_{ba}\sigma_{ba} : \mathbf{B}^A \to \mathbf{B}^B$$

be a homomorphism, and let S be its associated maze. Define the component $H(X) \to H(Y)$ of $\Phi^{-1}(H)(\alpha)$ to be

$$\sum_{P \subseteq S\,|_{X\to Y}} H(P).$$

This amounts to saying that

$$\Phi^{-1}(H)(\alpha) = \sum_{\substack{X\subseteq A\\ Y\subseteq B}} \sum_{P\subseteq S\,|_{X\to Y}} H(P).$$

Note in particular that

$$\Phi^{-1}(H)(S)\,|_{H(Y)\to H(Z)} = 0$$

if

$$Y = \emptyset \neq Z \quad \text{or} \quad Y \neq \emptyset = Z,$$

but

$$\Phi^{-1}(H)(S)\,|_{H(\emptyset)\to H(\emptyset)} = H(I_\emptyset) = I_{H(\emptyset)}.$$

Lemma 4. — $\Phi^{-1}(H)$ *is a module functor.*

Proof. Let

$$\alpha = \sum_{\substack{b\in B\\ c\in C}} s_{cb}\sigma_{cb} : \mathbf{B}^B \to \mathbf{B}^C \qquad \text{and} \qquad \beta = \sum_{\substack{a\in A\\ b\in B}} t_{ba}\sigma_{ba} : \mathbf{B}^A \to \mathbf{B}^B$$

be homomorphisms, and let S and T be their associated mazes. The associated maze of $\alpha \circ \beta$ is then $S \boxdot T$.

Let $X \subseteq A$ and $Z \subseteq C$. By Theorem 3.2,

$$\begin{aligned}\Phi^{-1}(H)(S \circ T)\,|_{H(X)\to H(Z)} &= \sum_{W\subseteq (S\boxdot T)\,|_{X\to Z}} H(W)\\ &= \sum_{W\subseteq (S\boxtimes T)\,|_{X\to Z}} H(W),\end{aligned}$$

while

$$\begin{aligned}&\big(\Phi^{-1}(H)(S)\circ\Phi^{-1}(H)(T)\big)\,|_{H(X)\to H(Z)}\\ &\quad= \sum_{Y\subseteq B}\Big(\Phi^{-1}(H)(S)\,|_{H(Y)\to H(Z)}\Big)\Big(\Phi^{-1}(H)(T)\,|_{H(X)\to H(Y)}\Big)\end{aligned}$$

$$= \sum_{Y\subseteq B} \left(\sum_{P\subseteq S\big|_{Y\to Z}} H(P) \right) \left(\sum_{Q\subseteq T\big|_{X\to Y}} H(Q) \right)$$
$$= \sum_{Y\subseteq B} \sum_{P\subseteq S\big|_{Y\to Z}} \sum_{Q\subseteq T\big|_{X\to Y}} H(PQ)$$
$$= \sum_{Y\subseteq B} \sum_{P\subseteq S\big|_{Y\to Z}} \sum_{Q\subseteq T\big|_{X\to Y}} H\left(\sum_{V\sqsubseteq P\boxtimes Q} V \right)$$
$$= \sum_{V\subseteq (S\boxtimes T)\big|_{X\to Z}} H(V).$$

The last step comes from noting that every submaze of $(S\boxtimes T)\big|_{X\to Z}$ is obtained as $V \sqsubseteq P \boxtimes Q$, for some P and Q. The functoriality of $\Phi^{-1}(H)$ follows. □

Lemma 5.

$$\Phi(\Phi^{-1}(H)) = H.$$

Proof. Let $P\colon X \to Y$ be a maze, and calculate the deviation

$$\Phi^{-1}(H) \left(\lozenge_{[p:\, x\to y]\in P} \bar{p}\sigma_{yx} \right) = \sum_{S\subseteq P} (-1)^{|P|-|S|} \Phi^{-1}(H) \left(\sum_{p\in S} \bar{p}\sigma_{yx} \right). \tag{1}$$

The sum should run over all *sub-multi-sets* S of P (rather than just submazes). Beginning with

$$\Phi^{-1}(H) \left(\sum_{p\in S} \bar{p}\sigma_{yx} \right),$$

the component $H(Z_1) \to H(Z_2)$ is

$$\sum_{Q\subseteq S\big|_{Z_1\to Z_2}} H(Q).$$

The component $H(Z_1) \to H(Z_2)$ of (1) is then

$$\sum_{S\subseteq P} (-1)^{|P|-|S|} \sum_{Q\subseteq S\big|_{Z_1\to Z_2}} H(Q) = \sum_{Q\subseteq P\big|_{Z_1\to Z_2}} (-1)^{|P|} H(Q) \sum_{Q\subseteq S\subseteq P} (-1)^{|S|}.$$

The inner sum vanishes if $P \neq Q$, and it equals $(-1)^{|P|}$ if $P = Q$. In the latter case, since $Q \subseteq P\big|_{Z_1\to Z_2}$, it must be that $Z_1 = X$, $Z_2 = Y$, and $Q = P = P\big|_{X\to Y}$. Consequently,

$$\Phi^{-1}(H) \left(\lozenge_{[p:\, x\to y]\in P} \bar{p}\sigma_{yx} \right) \Bigg|_{H(Z_1)\to H(Z_2)} = \begin{cases} H(P) & \text{if } Z_1 = X \text{ and } Z_2 = Y, \\ 0 & \text{else.} \end{cases}$$

It is then an immediate consequence that

$$\Phi(\Phi^{-1}(H))(X) = \Phi^{-1}(H)^{\dagger}(\mathbf{B}|_X) = \operatorname{Im}\Phi^{-1}(H)\left(\Diamond_{x\in X}\pi_x\right)$$
$$= \operatorname{Im} H(I_X) = \operatorname{Im} 1_{H(X)} = H(X),$$

and also that

$$\Phi(\Phi^{-1}(H))(P) = \Phi^{-1}(H)\left(\Diamond_{[p\colon x\to y]\in P}\bar{p}\sigma_{yx}\right) = H(P).$$

□

LEMMA 6.

$$\Phi^{-1}(\Phi(F)) = F.$$

Proof. We have, by the Cross-Effect Decomposition,

$$\Phi^{-1}(\Phi(F))(\mathbf{B}^A) = \bigoplus_{X\subseteq A}\Phi(F)(X) = \bigoplus_{X\subseteq A}F^{\dagger}(\mathbf{B}|_X) = F(A).$$

Let a homomorphism

$$\alpha = \sum_{a\in A, b\in B} s_{ba}\sigma_{ba}\colon \mathbf{B}^A \to \mathbf{B}^B$$

be given, with associated maze S. Then

$$\begin{aligned}
\Phi^{-1}(\Phi(F))(\alpha) &= \sum_{\substack{X\subseteq A\\ Y\subseteq B}}\sum_{P\subseteq S\,\big|_{X\to Y}}\Phi(F)(P)\\
&= \sum_{\substack{X\subseteq A\\ Y\subseteq B}}\sum_{P\subseteq S\,\big|_{X\to Y}}F\left(\Diamond_{[p\colon x\to y]\in P}\bar{p}\sigma_{yx}\right)\\
&= \sum_{\substack{X\subseteq A\\ Y\subseteq B}}\sum_{P\subseteq S\,\big|_{X\to Y}}F\left(\Diamond_{[x\to y]\in P}s_{yx}\sigma_{yx}\right)\\
&= \sum_{E\subseteq A\times B}F\left(\Diamond_{(x,y)\in E}s_{yx}\sigma_{yx}\right)\\
&= F\left(\sum_{(x,y)\in A\times B}s_{yx}\sigma_{yx}\right) = F(\alpha).
\end{aligned}$$

□

Assembling these results, we obtain the following marvellous theorem. (The functoriality of Φ^{-1} is a direct consequence of the lemmata, and need not be established separately.)

THEOREM 2: LABYRINTH OF FUN. — *The functor*

$$\Phi_{\mathfrak{Laby}}\colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Fun}(\mathfrak{Laby}, \mathfrak{Mod}),$$

where $\Phi_{\mathfrak{Laby}}(F)\colon \mathfrak{Laby} \to \mathfrak{Mod}$ *takes*

$$X \mapsto F^{\dagger}(\mathbf{B}|_X)$$
$$[P\colon X \to Y] \mapsto \left[F\left(\underset{[p\colon x\to y]\in P}{\Diamond} \bar{p}\sigma_{yx} \right) : F^{\dagger}(\mathbf{B}|_X) \to F^{\dagger}(\mathbf{B}|_Y) \right],$$

is an equivalence of categories.

§2. Polynomial Functors

Since mazes correspond to deviations, the following simple characterisation of polynomiality should come as no surprise.

THEOREM 3. — *The module functor F is polynomial of degree n iff $\Phi_{\mathfrak{Laby}}(F)$ vanishes on sets with more than n elements.*

Proof. Assume first that F is polynomial of degree n. Since mazes with k passages correspond to kth deviations, $\Phi_{\mathfrak{Laby}}(F)$ will certainly vanish on mazes with more than n passages.

Suppose now, conversely, that $\Phi_{\mathfrak{Laby}}(F)$ vanishes on mazes with more than n passages. Consider $n+1$ homomorphisms

$$\alpha_1, \ldots, \alpha_{n+1}\colon \mathbf{B}^A \to \mathbf{B}^B,$$

with associated mazes

$$P_1, \ldots, P_{n+1}\colon A \to B,$$

respectively. These mazes are all similar, and we may label consistently the passages of each P_i by

$$p_{i1}, \ldots, p_{im}.$$

Let $X \subseteq A$ and $Y \subseteq B$ be sets.

Note that if

$$\{p_{ij} \mid j \in J\}$$

is a legitimate submaze of P_i for one particular i, it is so for all choices of i. When this is the case, we say that the set $J \subseteq [m]$ is *admissible*. Then also

$$\left\{ \sum_{i\in I} p_{ij} \,\middle|\, j \in J \right\}$$

is a legitimate submaze of

$$\left(\boxplus_{i\in I} P_i \right)\Big|_{X\to Y} = \boxplus_{i\in I} P_i \,\big|_{X\to Y}$$

for any $I \subseteq [n+1]$. Note that this is the associated maze of the sum $\sum_{i\in I} \alpha_i$.

We are now ready to calculate the deviation of F:

$$\begin{aligned}
F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1})\,\Big|_{F^\dagger(\mathbf{B}|_X)\to F^\dagger(\mathbf{B}|_Y)}
&= \sum_{I\subseteq[n+1]} (-1)^{n+1-|I|} F\left(\sum_{i\in I}\alpha_i\right)\Bigg|_{F^\dagger(\mathbf{B}|_X)\to F^\dagger(\mathbf{B}|_Y)} \\
&= \sum_{I\subseteq[n+1]} (-1)^{n+1-|I|} \sum_{Q\subseteq\left(\boxplus_{i\in I} P_i\right)\,|_{X\to Y}} \Phi_{\mathfrak{Laby}}(F)(Q) \\
&= \sum_{I\subseteq[n+1]} (-1)^{n+1-|I|} \sum_{J\subseteq[m]} \Phi_{\mathfrak{Laby}}(F)\left(\left\{\, \sum_{i\in I} p_{ij} \,\middle|\, j\in J \,\right\}\right),
\end{aligned}$$

where the inner sum is taken over admissible J only. For a set $K \subseteq I \times J$, let K_I and K_J denote the projections on I and J, respectively. We may use Theorem 3.3 to transform the latter sum to

$$\begin{aligned}
&F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1})\,\big|_{F^\dagger(\mathbf{B}|_X)\to F^\dagger(\mathbf{B}|_Y)} \\
&= \sum_{I\subseteq[n+1]} (-1)^{n+1-|I|} \sum_{J\subseteq[m]} \sum_{\substack{K\subseteq I\times J\\ K_J=J}} \Phi_{\mathfrak{Laby}}(F)(\{\, p_{ij} \mid (i,j)\in K \,\}) \\
&= \sum_{K\subseteq[n+1]\times[m]} \left(\sum_{K_I\subseteq I\subseteq[n+1]} (-1)^{n+1-|I|}\right)\left(\sum_{J=K_J} \Phi_{\mathfrak{Laby}}(F)(\{\, p_{ij} \mid (i,j)\in K \,\})\right) \\
&= \sum_{\substack{K\subseteq[n+1]\times[m]\\ K_I=[n+1]}} \Phi_{\mathfrak{Laby}}(F)(\{\, p_{ij} \mid (i,j)\in K \,\}).
\end{aligned}$$

The condition $K_I = [n+1]$ implies $|K| \geqslant n+1$, and so all mazes

$$\{\, p_{ij} \mid (i,j)\in K \,\}$$

will contain more than n passages. The sum will therefore equal 0, by the hypothesis on $\Phi_{\mathfrak{Laby}}(F)$. □

§3. Numerical Functors

We now investigate how to interpret numericality in the labyrinthine setting. The base ring $\mathbf{B}$ is assumed numerical.

Lemma 7. — *Let $r \in \mathbf{B}$, and let $n, w_1, \dots, w_q$ be natural numbers satisfying*

$$w_1 + \cdots + w_q \leqslant n.$$

Then

$$\prod_{j=1}^{q} \binom{r}{w_j} = \sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}.$$

Proof. We prove the formula when r is an integer, and then refer to the Numerical Transfer Principle.

Fix a subset $S \subseteq [r]$ with $|S| = m$. Suppose we wish to choose subsets $W_1, \ldots, W_q \subseteq S$, such that $|W_i| = w_i$ and $\bigcup W_i = S$. By the Principle of Inclusion and Exclusion, this can be done in

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}$$

ways. The quantites

$$\prod_{j=1}^{q} \binom{r}{w_j} = \sum_{m=0}^{\infty} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \prod_{j=1}^{q} \binom{k}{w_j}$$

will then both count the total number of ways to choose subsets $W_1, \ldots, W_q \subseteq [r]$, such that $|W_i| = w_i$. □

Theorem 4. — *The module functor F is numerical of degree n iff $\Phi_{\mathfrak{Laby}}(F)$ factors through $\mathfrak{Laby}_n$. The functor $\Phi_{\mathfrak{Laby}}$ induces an equivalence of categories*

$$\mathfrak{Num}_n \to \operatorname{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod}).$$

Proof. Recall from Chapter 3 that, when A is a multi-set supported in the maze P, we let E_A denote the maze

$$E_A = \bigcup_{[p:\, x \to y] \in A} \left\{ x \xrightarrow{1} y \right\},$$

with all passages of A reassigned the label 1.

The theorem states that the functor F is numerical of degree n iff the equation

$$\Phi_{\mathfrak{Laby}}(F)(P) = \sum_{\substack{\#A = P \\ |A| \leqslant n}} \prod_{p \in P} \binom{\bar{p}}{\deg_A p} \Phi_{\mathfrak{Laby}}(F)(E_A)$$

holds for all mazes P. It should be clear from Theorem 6.4 that numerical functors satisfy this equation.

Suppose now, conversely, that $\Phi_{\mathfrak{Laby}}(F)$ satisfies the equation. It will then certainly vanish on mazes with more than n elements, whence F is polynomial of degree n. We now wish to use Theorem 6.3, and thus seek to evaluate

$$F(r \cdot 1_{\mathbf{B}^n}) = \sum_{P \subseteq r \boxdot I_{[n]}} \Phi_{\mathfrak{Laby}}(F)(P).$$

The component

$$\Phi_{\mathfrak{Laby}}(F)(X) \to \Phi_{\mathfrak{Laby}}(F)(Y)$$

of this is 0 if $X \neq Y$. Turning to the case $X = Y$, we may without loss of generality assume $X = Y = [q]$. Then the component

$$\Phi_{\mathfrak{Laby}}(F)([q]) \to \Phi_{\mathfrak{Laby}}(F)([q])$$

is

$$\begin{aligned}\Phi_{\mathfrak{Laby}}(F)(r \boxdot I_{[q]}) &= \sum_{\substack{\#A=[q]\\|A|\leqslant n}} \prod_{j=1}^{q} \binom{r}{\deg_A j} \Phi_{\mathfrak{Laby}}(F)(E_A) \\ &= \sum_{w_1+\cdots+w_q \leqslant n} \prod_{j=1}^{q} \binom{r}{w_j} \Phi_{\mathfrak{Laby}}(F)(E_w),\end{aligned}$$

where we let $w_j = \deg_A j \geqslant 1$. Similarly, the component

$$\Phi_{\mathfrak{Laby}}(F)([q]) \to \Phi_{\mathfrak{Laby}}(F)([q])$$

of

$$\sum_{m=0}^{n} \binom{r}{m} F\left(\mathop{\Diamond}_{m} 1_{\mathbf{B}^n}\right) = \sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} F(k \cdot 1_{\mathbf{B}^n})$$

is

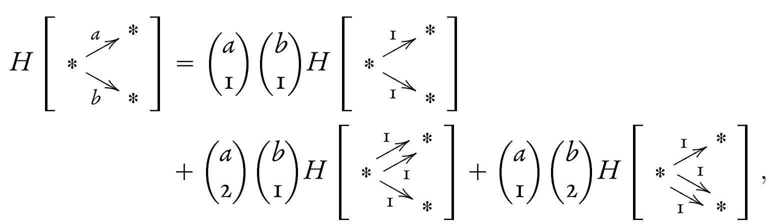
$$\sum_{m=0}^{n} \binom{r}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{w_1+\cdots+w_q \leqslant n} \prod_{j=1}^{q} \binom{k}{w_j} \Phi_{\mathfrak{Laby}}(F)(E_w).$$

It is now only a matter of using the lemma, to establish the equality

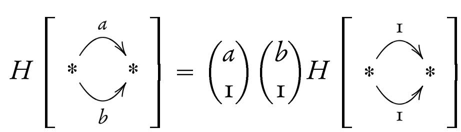
$$F(r \cdot 1_{\mathbf{B}^n}) = \sum_{m=0}^{n} \binom{r}{m} F\left(\mathop{\Diamond}_{m} 1_{\mathbf{B}^n}\right).$$

Consequently, F is numerical. □

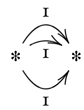
Example 1. — Let us take a simple example. If the labyrinth module H corresponds to a module functor of degree 3, it will satisfy the equation

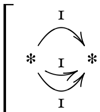


and also



$$+\binom{a}{2}\binom{b}{1}H\left[\quad\right]+\binom{a}{1}\binom{b}{2}H\left[\quad\right].$$





△

We now pay homage to our predecessors. The main result of Professor Pirashvili's team ([1]) is the following[1]: *Polynomial functors on the category of* $\mathbf{Z}$*-modules are equivalent to linear functors from* $\widehat{\mathfrak{Sur}}_n$ *to* $\mathbf{Z}$*-modules.* For the ring $\mathbf{Z}$, the concepts "polynomial" and "numerical" coincide; hence this theorem may be written as

$$_{\mathbf{Z}}\mathfrak{Num}_n \cong \mathrm{Fun}(\widehat{\mathfrak{Sur}}_n, {}_{\mathbf{Z}}\mathfrak{Mod}).$$

It is an immediate consequence of the isomorphisms

$$\widehat{\mathfrak{Sur}}_n \cong {}_{\mathbf{Z}}\mathfrak{Laby}_n \quad \text{and} \quad \mathfrak{Num}_n \cong \mathrm{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod}).$$

§4. Quasi-Homogeneous Functors

Theorem 5. — *The module functor F is quasi-homogeneous of degree n iff $\Phi_{\mathfrak{Laby}}(F)$ factors through $\mathfrak{Laby}^n$. The functor $\Phi_{\mathfrak{Laby}}$ induces an equivalence of categories*

$$\mathfrak{QHom}_n \to \mathrm{Fun}(\mathfrak{Laby}^n, \mathfrak{Mod}).$$

Proof. Let F be quasi-homogeneous, and let $a \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$. For any deviation, we have

$$F(a\alpha_1 \diamond \cdots \diamond a\alpha_k) = a^n F(\alpha_1 \diamond \cdots \diamond \alpha_k),$$

and we may calculate for a pure maze P:

$$\begin{aligned} a^n \Phi_{\mathfrak{Laby}}(F)(P) &= a^n F\left(\bigdiamond_{[x\to y]\in P} \sigma_{yx}\right) = F\left(\bigdiamond_{[x\to y]\in P} a\sigma_{yx}\right) \\ &= \Phi_{\mathfrak{Laby}}(F)(a \boxdot P). \end{aligned}$$

Conversely, assume $\Phi_{\mathfrak{Laby}}(F)$ factors via $\mathfrak{Laby}^n$. Then, for any $k \in \mathbf{N}$,

$$\begin{aligned} a^n F(1_{\mathbf{B}^k}) &= a^n \sum_{K \subseteq [k]} \Phi_{\mathfrak{Laby}}(F)(I_K) \\ &= \sum_{K \subseteq [k]} \Phi_{\mathfrak{Laby}}(F)(a \boxdot I_K) = F(a \cdot 1_{\mathbf{B}^k}). \end{aligned}$$

□

[1] We point out that they restrict their attention to *pointed* functors. We have circumvented this restriction by including $\emptyset$ among the finite sets.

According to a structure theorem for the labyrinth categories, there are isomorphisms

$$ {}_{\mathbf{B}}\mathfrak{Laby}_n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n \quad \text{and} \quad {}_{\mathbf{B}}\mathfrak{Laby}^n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}^n. $$

Their significance may be summarised thus. The category of numerical (or quasi-homogeneous) functors over an arbitrary numerical ring is identical in structure to the category of numerical (respectively, quasi-homogeneous) functors over **Z**. These are the equations to prove this:

$$ \begin{aligned} {}_{\mathbf{B}}\mathfrak{Num}_n &\cong \mathrm{Fun}_{\mathbf{B}}({}_{\mathbf{B}}\mathfrak{Laby}_n, {}_{\mathbf{B}}\mathfrak{Mod}) \\ &\cong \mathrm{Fun}_{\mathbf{B}}(\mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{Laby}_n, {}_{\mathbf{B}}\mathfrak{Mod}) \\ &\cong \mathrm{Fun}_{\mathbf{Z}}({}_{\mathbf{Z}}\mathfrak{Laby}_n, {}_{\mathbf{B}}\mathfrak{Mod}). \end{aligned} $$

The corresponding result for homogeneous functors holds more trivially. By definition, for any ring **B**,

$$ {}_{\mathbf{B}}\mathfrak{MSet}_n \cong \mathbf{B} \otimes_{\mathbf{Z}} {}_{\mathbf{Z}}\mathfrak{MSet}_n, $$

which leads to

$$ {}_{\mathbf{B}}\mathfrak{Hom}_n \cong \mathrm{Fun}_{\mathbf{Z}}({}_{\mathbf{Z}}\mathfrak{MSet}_n, {}_{\mathbf{B}}\mathfrak{Mod}). $$

§5. Quadratic Functors

A few examples of labyrinth representations are in order. Let us take $[n]$ as the canonical representative of sets of cardinality n.

Example 2. — Let $C(\mathbf{B}^n) = K$ be a constant functor. The labyrinth functor $\Phi_{\mathfrak{Laby}}(C)$ will take

$$ [0] \mapsto K, \qquad [1], [2], [3], \ldots \mapsto 0. $$

△

Example 3. — Let $F(\mathbf{B}^n) = K \oplus L^n$ be an affine functor. $\Phi_{\mathfrak{Laby}}(F)$ will take

$$ [0] \mapsto K, \qquad [1] \mapsto L, \qquad [2], [3], \ldots \mapsto 0, $$

and map the maze

$$ \left[\, 1 \xrightarrow{c} 1 \,\right] \mapsto [c \colon L \to L]. $$

△

Let us now determine the structure of $\mathfrak{Num}_2$ by classifying the quadratic numerical functors. The key point is unravelling the structure of the category $\mathfrak{Laby}_2$. It contains three non-isomorphic objects: [0], [1], and [2]. We observe the following relations:

$$ \left[\, * \xrightarrow{a} * \,\right] = \binom{a}{1} \left[\, * \xrightarrow{1} * \,\right] + \binom{a}{2} \left[\, * \overset{1}{\underset{1}{\rightrightarrows}} * \,\right] $$

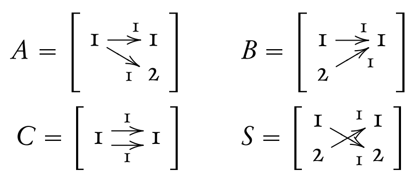
$\cdot$	A	B	C	S
A	–	$I+S$	$2A$	–
B	C	–	–	B
C	–	$2B$	$2C$	–
S	A	–	–	I

Table 10.1: Multiplication table for $\mathfrak{Laby}_2$.

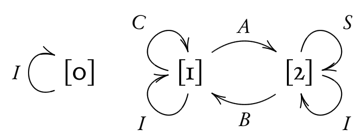
and

$$\begin{bmatrix} * \xrightarrow{a} * \\ * \xrightarrow[b]{} * \end{bmatrix} = \binom{a}{1}\binom{b}{1}\begin{bmatrix} * \xrightarrow{1} * \\ * \xrightarrow[1]{} * \end{bmatrix}.$$

Consequently, every maze in $\mathfrak{Laby}_2$ can be reduced to (linear combinations of) identity mazes and the following:



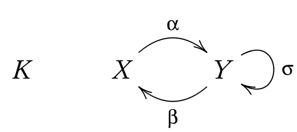
The (skeletal) structure of the category $\mathfrak{Laby}_2$ is thus reduced to the following, promptly suggesting the nickname *dogegory*:



The mazes A, B, C, and S are not independent. Their multiplication table is given in Table 10.1. Clearly we can do with only A, B and S, and we obtain the following explicit description of $\mathfrak{Num}_2$.

Theorem 6. — *A quadratic numerical functor is equivalent to the collection of modules K, X, and Y; together with homomorphisms α, β, σ as indicated, subject to the following four relations:*

$$\alpha\beta = 1 + \sigma, \qquad \beta\sigma = \beta, \qquad \sigma\alpha = \alpha, \qquad \sigma^2 = 1.$$



The reader will no doubt note that we can, in fact, also dispense with $\sigma = \alpha\beta - 1$, and let the two homomorphisms α and β be subject to a meagre two relations:

$$\beta\alpha\beta = 2\beta, \qquad \alpha\beta\alpha = 2\alpha.$$

We now describe the four classical quadratic functors. Because they are of the second degree, and because they are all reduced, the module $K = 0$. We will juggle with two isomorphic copies of $\mathbf{B}$, denoted by $\mathbf{B}_1 = \langle e_1 \rangle$ and $\mathbf{B}_2 = \langle e_2 \rangle$.

Example 4. — The functor $\Phi_{\mathfrak{Laby}}(T^2)$ has modules

$$X = (T^2)^\dagger(\mathbf{B}_1) = \langle e_1 \otimes e_1 \rangle, \qquad Y = (T^2)^\dagger(\mathbf{B}_1|\mathbf{B}_2) = \langle e_1 \otimes e_2, e_2 \otimes e_1 \rangle$$

and homomorphisms

$$\begin{aligned} \alpha\colon &\quad e_1 \otimes e_1 \mapsto e_1 \otimes e_2 + e_2 \otimes e_1 \\ \beta\colon &\quad e_1 \otimes e_2, e_2 \otimes e_1 \mapsto e_1 \otimes e_1 \\ \sigma\colon &\quad e_1 \otimes e_2 \mapsto e_2 \otimes e_1 \\ &\quad e_2 \otimes e_1 \mapsto e_1 \otimes e_2. \end{aligned}$$

△

Example 5. — The functor $\Phi_{\mathfrak{Laby}}(S^2)$ has modules

$$X = (S^2)^\dagger(\mathbf{B}_1) = \langle e_1^2 \rangle, \qquad Y = (S^2)^\dagger(\mathbf{B}_1|\mathbf{B}_2) = \langle e_1 e_2 \rangle$$

and homomorphisms

$$\begin{aligned} \alpha\colon &\quad e_1^2 \mapsto 2e_1e_2 \\ \beta\colon &\quad e_1e_2 \mapsto e_1^2 \\ \sigma\colon &\quad e_1e_2 \mapsto e_1e_2. \end{aligned}$$

△

Example 6. — The functor $\Phi_{\mathfrak{Laby}}(\Lambda^2)$ has modules

$$X = (\Lambda^2)^\dagger(\mathbf{B}_1) = \langle e_1 \wedge e_1 \rangle = 0, \qquad Y = (\Lambda^2)^\dagger(\mathbf{B}_1|\mathbf{B}_2) = \langle e_1 \wedge e_2 \rangle$$

and homomorphisms

$$\begin{aligned} \alpha\colon &\quad 0 \\ \beta\colon &\quad 0 \\ \sigma\colon &\quad e_1 \wedge e_2 \mapsto -e_1 \wedge e_2. \end{aligned}$$

△

Example 7. — The functor $\Phi_{\mathfrak{Laby}}(\Gamma^2)$ has modules

$$X = (\Gamma^2)^\dagger(\mathbf{B}_1) = \left\langle e_1^{[2]} \right\rangle, \qquad Y = (\Gamma^2)^\dagger(\mathbf{B}_1|\mathbf{B}_2) = \langle e_1 e_2 \rangle$$

and homomorphisms

$$\begin{aligned} \alpha\colon &\quad e_1^{[2]} \mapsto e_1e_2 \\ \beta\colon &\quad e_1e_2 \mapsto 2e_1^{[2]} \\ \sigma\colon &\quad e_1e_2 \mapsto e_1e_2. \end{aligned}$$

△

§6. Labyrinth Modules

Our description of the category equivalence

$$\mathfrak{Num}_n \sim \mathrm{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod}),$$

explicit though it be, has the disadvantage of obscuring the fact that this is another instance of Morita equivalence. A shorter, possibly more conceptual proof, would run as follows.

Let $n \in \mathbf{Z}^+ \cup \{\infty\}$. The decomposition

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)] = \bigoplus_{X \subseteq [n]} \Delta^X.$$

shows that the functors Δ^X constitute a family of small projective generators for $\mathfrak{Num}_n$. Letting

$$L \colon \mathfrak{Laby}_n \overset{\circ}{\to} \mathfrak{Dev}_n \to \mathfrak{Num}_n$$

denote the contra-variant inclusion functor, there is a Morita equivalence:

$$\mathfrak{Num}_n \underset{L \otimes_{\mathfrak{Laby}_n} -}{\overset{\mathrm{Nat}(L,-)}{\rightleftarrows}} \mathrm{Fun}(\mathfrak{Laby}_n, \mathfrak{Mod})$$

As an immediate consequence of this latter view, the labyrinth functor $H \colon \mathfrak{Laby}_n \to \mathfrak{Mod}$ corresponds to the module functor $F = L \otimes_{\mathfrak{Laby}_n} H$, given by

$$\begin{aligned}(L \otimes_{\mathfrak{Laby}_n} H)(M) &= \bigoplus_{X \in \mathfrak{Laby}_n} L(X)(M) \otimes H(X) \\ &= \bigoplus_{X \in \mathfrak{Laby}_n} \Delta^X(M) \otimes H(X).\end{aligned}$$

Imposed upon this module are the following relations:

$$U \otimes H(P)(z) = UP \otimes z,$$

for any $z \in H(X)$, $U \in \Delta^Y(M)$, and maze $P \colon X \to Y$. Interpreting this in terms of F, we have the following theorem, where σ denotes the usual canonical transportation maps.

Theorem 7. — *Let F be a numerical functor of degree n, where $n \in \mathbf{Z}^+ \cup \{\infty\}$. The module $F(M)$ is the quotient of*

$$\bigoplus_{X \in \mathfrak{Laby}_n} \Delta^X(M) \otimes F^\dagger(\mathbf{B}|_X)$$

by all relations

$$U \otimes F\left(\Diamond_{[p\colon x \to y] \in P} \bar{p}\sigma_{yx} \right)(z) = UP \otimes z,$$

for any $z \in F^\dagger(\mathbf{B}|_X)$, $U \in \Delta^Y(M)$, *and maze* $P\colon X \to Y$.

Moreover, when $\alpha\colon M \to N$ *is a homomorphism,*

$$F(\alpha) = \bigoplus_{X \in \mathfrak{Lady}_n} \Delta^X(\alpha) \otimes \mathrm{I}_{F^\dagger(\mathbf{B}|_X)}.$$

We will often abuse notation, and write simply

$$F(M) = \bigoplus_{X \in \mathfrak{Lady}_n} \Delta^X(M) \otimes F^\dagger(\mathbf{B}|_X),$$

with the imposed relations tacitly understood.

The advantage of this viewpoint will become evident as soon as we try to compute the composition of two module functors F and G (which will be used later when considering the plethysm). We may write

$$F(G(M)) = \bigoplus_{X \in \mathfrak{Lady}_n} \Delta^X(G(M)) \otimes F^\dagger(\mathbf{B}|_X),$$

and we obtain the following formula for the deviations of a composition.

Theorem 8.

$$(F \circ G)(\alpha_1 \diamond \cdots \diamond \alpha_m) = \sum_{X \in \mathfrak{Lady}_n} \sum_{J \lhd [m]} \left(\bigcup_{I \in J} \left\{ * \xrightarrow{G(\diamondsuit_{i \in I} \alpha_i)} * \right\} \right)_* \otimes \mathrm{I}_{F^\dagger(\mathbf{B}|_X)}.$$

Proof. Calculate, according to Theorems 7.3 and 8.7:

$$\begin{aligned}(F \circ G)(\alpha_1 \diamond \cdots \diamond \alpha_m) &= \sum_{X \in \mathfrak{Lady}_n} \Delta^X(G(\alpha_1 \diamond \cdots \diamond \alpha_m)) \otimes \mathrm{I}_{F^\dagger(\mathbf{B}|_X)} \\ &= \sum_{X \in \mathfrak{Lady}_n} \sum_{J \lhd [m]} \Delta^X \left(\diamondsuit_{I \in J} G \left(\diamondsuit_{i \in I} \alpha_i \right) \right) \otimes \mathrm{I}_{F^\dagger(\mathbf{B}|_X)} \\ &= \sum_{X \in \mathfrak{Lady}_n} \sum_{J \lhd [m]} \left(\bigcup_{I \in J} \left\{ * \xrightarrow{G(\diamondsuit_{i \in I} \alpha_i)} * \right\} \right)_* \otimes \mathrm{I}_{F^\dagger(\mathbf{B}|_X)}.\end{aligned}$$

□

§7. Homogeneous Functors

We now turn to combinatorially interpreting homogeneous polynomial functors, and cite [20] as our reference. Specifically, we do not obtain any new results in this section — they were known previously to at least two people — but simply rephrase old ones in the smooth language of multations. This allows for, we believe, the most economic formulation of the theory.

Let $\mathbf{B}$ be commutative and unital. We propose to establish an equivalence of categories

$$\mathfrak{Hom}_n \sim \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod}),$$

and we emphasise, as we did in the labyrinth case, that *functors* $\mathfrak{MSet}_n \to \mathfrak{Mod}$, *shall always be assumed linear*. These functors go by the name of **Schur modules**.

Let F be a homogeneous functor of degree n. Define

$$\Phi(F)\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

by the formulæ

$$A \mapsto F_A^\dagger(\mathbf{B}|_{\#A})$$
$$[\mu\colon A \to B] \mapsto \left[F_{\sigma[\mu]} : F_A^\dagger(\mathbf{B}|_{\#A}) \to F_B^\dagger(\mathbf{B}|_{\#B})\right].$$

Lemma 8. — $\Phi(F)$ *is a functor* $\mathfrak{MSet}_n \to \mathfrak{Mod}$.

Proof. Functoriality is clear from the Multi-Set Yoneda Lemma, as $F_{\sigma[\mu]}$ corresponds to

$$(\Gamma^\mu)^*\colon\ \mathrm{Nat}(\Gamma^A, F) \to \mathrm{Nat}(\Gamma^B, F).$$

□

Let $\zeta\colon F \to G$ be a natural transformation. Define

$$\Phi(\zeta)\colon \Phi(F) \to \Phi(G)$$

by restriction to the appropriate multi-cross-effects:

$$\Phi(\zeta)_A = (\zeta_A^\dagger)_{\mathbf{B}|_{\#A}} : F_A^\dagger(\mathbf{B}|_{\#A}) \to G_A^\dagger(\mathbf{B}|_{\#A}).$$

Lemma 9. — Φ *is a functor*

$$\mathfrak{Hom}_n \to \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod}).$$

Proof. By the Multi-Set Yoneda Lemma, $(\zeta_A^\dagger)_{\mathbf{B}|_{\#A}}$ corresponds to

$$\zeta_*\colon\ \mathrm{Nat}(\Gamma^A, F) \to \mathrm{Nat}(\Gamma^A, G).$$

□

We now construct the inverse of Φ. Let

$$J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

be a Schur module. Define

$$\Phi^{-1}(J)\colon \mathfrak{XMod} \to \mathfrak{Mod}$$

by

$$\Phi^{-1}(J)(\mathbf{B}^X) = \bigoplus_{\substack{\#A\subseteq X\\ |A|=n}} J(A)$$

(where, of course, X is a set, but A ranges over multi-sets). Also, let

$$\alpha = \sum_{\substack{x\in X\\ y\in Y}} s_{yx}\sigma_{yx} : \mathbf{B}^X \to \mathbf{B}^Y$$

be a homomorphism. When $A \subseteq X$ and $B \subseteq Y$, we define the component $J(A) \to J(B)$ of $\Phi^{-1}(J)(\alpha)$ to be

$$\sum_{\mu:\, A\to B} s^{\mu} J(\mu).$$

This amounts to saying that

$$\Phi^{-1}(J)(\alpha) = \sum_{\substack{A\subseteq X\\ B\subseteq Y}} \sum_{\mu:\, A\to B} s^{\mu} J(\mu).$$

Lemma 10. — *$\Phi^{-1}(J)$ is a homogeneous functor of degree n.*

Proof. Let

$$\alpha = \sum_{\substack{y\in Y\\ z\in Z}} s_{zy}\sigma_{zy} : \mathbf{B}^Y \to \mathbf{B}^Z \qquad \text{and} \qquad \beta = \sum_{\substack{x\in X\\ y\in Y}} t_{yx}\sigma_{yx} : \mathbf{B}^X \to \mathbf{B}^Y$$

be homomorphisms, and calculate

$$\begin{aligned}
&J\left(\left(\sum_{\substack{b\in\#B\\ c\in\#C}} s_{cb}\begin{bmatrix} b\\ c\end{bmatrix}\right)^{[n]}\right) \circ J\left(\left(\sum_{\substack{a\in\#A\\ b\in\#B}} t_{ba}\begin{bmatrix} a\\ b\end{bmatrix}\right)^{[n]}\right)\\
&\quad = J\left(\sum_{B',C}\sum_{\mu:\, B'\to C} s^{\mu}\mu\right) \circ J\left(\sum_{A,B}\sum_{\nu:\, A\to B} t^{\nu}\nu\right)\\
&\quad = \sum_{A,C}\sum_{B,B'}\sum_{\substack{\nu:\, A\to B\\ \mu:\, B'\to C}} s^{\mu}t^{\nu}J(\mu\nu)\\
&\quad = \sum_{A,C}\sum_{B}\sum_{\substack{\nu:\, A\to B\\ \mu:\, B\to C}} s^{\mu}t^{\nu}J(\mu\nu)\\
&\quad = \sum_{A,C}\sum_{B}\left(\sum_{\mu:\, B\to C} s^{\mu}J(\mu)\right)\left(\sum_{\nu:\, A\to B} t^{\nu}J(\nu)\right)
\end{aligned}$$

$$= \Phi^{-1}(J)(\alpha) \circ \Phi^{-1}(J)(\beta).$$

At the same time,

$$J\left(\left(\sum_{\substack{b\in\#B\\c\in\#C}} s_{cb}\begin{bmatrix} b\\ c\end{bmatrix}\right)^{[n]}\right) \circ J\left(\left(\sum_{\substack{a\in\#A\\b\in\#B}} t_{ba}\begin{bmatrix} a\\ b\end{bmatrix}\right)^{[n]}\right)$$
$$= J\left(\left(\sum_{\substack{a\in\#A\\b\in\#B\\c\in\#C}} s_{cb}t_{ba}\begin{bmatrix} a\\ c\end{bmatrix}\right)^{[n]}\right)$$
$$= J\left(\left(\sum_{\substack{a\in\#A\\c\in\#C}} \left(\sum_{b\in\#B} s_{cb}t_{ba}\right)\begin{bmatrix} a\\ c\end{bmatrix}\right)^{[n]}\right)$$
$$= J\left(\sum_{A,C}\sum_{\xi\colon A\to C}\left(\sum_{b\in\#B} s_{cb}t_{ba}\right)^{\xi}\xi\right)$$
$$= \sum_{A,C}\sum_{\xi\colon A\to C}\left(\sum_{b\in\#B} s_{cb}t_{ba}\right)^{\xi} J(\xi) = \Phi^{-1}(J)(\alpha\beta).$$

That $\Phi^{-1}(J)$ is strict polynomial is clear, as the defining equation

$$\Phi^{-1}(J)\left(\sum_{\substack{x\in X\\y\in Y}} s_{yx}\sigma_{yx}\right) = \sum_{A,B}\sum_{\mu\colon A\to B} s^{\mu}J(\mu)$$

works when the coefficients s_{yx} belong to any algebra. Finally, it is evident that it is homogeneous of degree n. □

Lemma 11.

$$\Phi(\Phi^{-1}(J)) = J.$$

Proof. The equation

$$\Phi^{-1}(J)\left(\sum_{\substack{x\in X\\y\in Y}} s_{yx}\sigma_{yx}\right) = \sum_{\substack{A\subseteq X\\B\subseteq Y}}\sum_{\mu\colon A\to B} s^{\mu}J(\mu)$$

implies that

$$\Phi^{-1}(J)_{\sigma[\mu]} = J(\mu).$$

Hence

$$\begin{aligned}\Phi(\Phi^{-1}(J))(A) &= \Phi^{-1}(J)^{\dagger}_{A}(\mathbf{B}|_{\#A}) = \operatorname{Im}\Phi^{-1}(J)_{\pi[A]} \\ &= \operatorname{Im} J(\iota_A) = J(A),\end{aligned}$$

and

$$\Phi(\Phi^{-1}(J))(\mu) = \Phi^{-1}(J)_{\sigma[\mu]} = J(\mu).$$

□

Lemma 12.

$$\Phi^{-1}(\Phi(F)) = F.$$

Proof. By the Multi-Cross-Effect Decomposition,

$$\begin{aligned}\Phi^{-1}(\Phi(F))(\mathbf{B}^X) &= \bigoplus_{\substack{\#A\subseteq X\\ |A|=n}} \Phi(F)(A) \\ &= \bigoplus_{\substack{\#A\subseteq X\\ |A|=n}} F^{\dagger}_{A}(\mathbf{B}|_{\#A}) = F(\mathbf{B}^X),\end{aligned}$$

and

$$\begin{aligned}\Phi^{-1}(\Phi(F))\left(\sum_{\substack{x\in X\\ y\in Y}} s_{yx}\sigma_{yx}\right) &= \sum_{\substack{A\subseteq X\\ B\subseteq Y}} \sum_{\mu\colon A\to B} s^{\mu}\Phi(F)(\mu) \\ &= \sum_{\substack{A\subseteq X\\ B\subseteq Y}} \sum_{\mu\colon A\to B} s^{\mu}F_{\sigma[\mu]} = F\left(\sum_{\substack{x\in X\\ y\in Y}} s_{yx}\sigma_{yx}\right).\end{aligned}$$

□

Collecting these results together, we obtain the following theorem.

Theorem 9. — *The functor*

$$\Phi_{\mathfrak{MSet}_n}\colon \mathfrak{Hom}_n \to \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod}),$$

where

$$\Phi_{\mathfrak{MSet}_n}(F)\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

takes

$$\begin{aligned} A &\mapsto F^{\dagger}_{A}(\mathbf{B}|_{\#A}) \\ [\mu\colon A\to B] &\mapsto \left[F_{\sigma[\mu]}: F^{\dagger}_{A}(\mathbf{B}|_{\#A}) \to F^{\dagger}_{B}(\mathbf{B}|_{\#B})\right],\end{aligned}$$

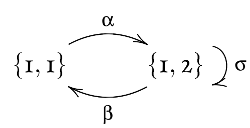
is an equivalence of categories.

·	α	β	σ
α	–	$\iota+\sigma$	–
β	2ι	–	β
σ	α	–	ι

Table 10.2: Multiplication table for $\mathfrak{MSet}_2$.

§8. Homogeneous Quadratic Functors

We here determine the structure of $\mathfrak{Hom}_2$ by classifying the homogeneous quadratic functors. To find the multi-set description of homogeneous quadratic functors, we first draw the (skeletal) structure of the category $\mathfrak{MSet}_2$:



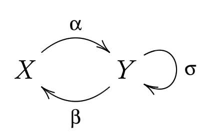
Every multation reduces to a linear combination of identity multations and the following:

$$\alpha = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \beta = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad \sigma = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The multiplication table is given in Table 10.2. Compare this with Table 10.1 — the only difference lies in the value of the product $\beta\alpha$.

Theorem 10. — *A quadratic homogeneous functor is equivalent to the collection of modules X and Y; together with homomorphisms α, β, σ as indicated, subject to the following five relations:*

$$\alpha\beta = 1+\sigma, \qquad \beta\sigma = \beta, \qquad \sigma\alpha = \alpha, \qquad \sigma^2 = 1, \qquad \beta\alpha = 2.$$



Evidently $\sigma = \alpha\beta - 1$ is dispensable. It is enough to have α and β, subject to the single relation

$$\beta\alpha = 2.$$

§9. Schur Modules

The category equivalence

$$\mathfrak{Hom}_n \cong \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod})$$

is yet another instance of Morita equivalence, and we construct a more conceptual proof as follows.

The decomposition

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, -) = \bigoplus_{\substack{\#A\subseteq[n]\\ |A|=n}} \Gamma^A$$

shows that the functors Γ^A constitute a family of small projective generators for $\mathfrak{Hom}_n$. Letting

$$L\colon \mathfrak{MSet}_n \xrightarrow{\circ} \mathfrak{Div}_n \to \mathfrak{Hom}_n$$

denote the contra-variant inclusion functor, there is a Morita equivalence:

$$\mathfrak{Hom}_n \underset{L\otimes_{\mathfrak{MSet}_n}-}{\overset{\mathrm{Nat}(L,-)}{\rightleftarrows}} \operatorname{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod})$$

As an immediate consequence of this latter view, the functor

$$J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$$

corresponds to the homogeneous functor $F = L \otimes_{\mathfrak{MSet}_n} J$, given by

$$\begin{aligned}(L \otimes_{\mathfrak{MSet}_n} J)(M) &= \bigoplus_{A\in\mathfrak{MSet}_n} L(A)(M) \otimes J(A)\\ &= \bigoplus_{A\in\mathfrak{MSet}_n} \Gamma^A(M) \otimes J(A).\end{aligned}$$

Imposed upon this module are the following relations:

$$w \otimes J(\mu)(x) = \Gamma^{\mu}(M)(w) \otimes x,$$

for any $x \in J(A)$, $w \in \Gamma^B(M)$, and multation $\mu\colon A \to B$. Interpreting this in terms of F, we have the following theorem, where σ denotes the usual canonical transportation maps.

Theorem 11. — *Let F be a homogeneous functor of degree n. The module $F(M)$ is the quotient of*

$$\bigoplus_{A\in\mathfrak{MSet}_n} \Gamma^A(M) \otimes F_A^{\dagger}(\mathbf{B}|_{\#A})$$

by all relations

$$w \otimes F_{\sigma[\mu]}(x) = \Gamma^{\mu}(M)(w) \otimes x,$$

for any $x \in F_A^{\dagger}(\mathbf{B}|_{\#A})$, $w \in \Gamma^B(M)$, and multation $\mu\colon A \to B$.

Moreover, when $\alpha\colon M \to N$ is a homomorphism,

$$F(\alpha) = \bigoplus_{A\in\mathfrak{MSet}_n} \Gamma^A(\alpha) \otimes 1_{F_A^{\dagger}(\mathbf{B}|_{\#A})}.$$

By abuse of notation, we shall often write simply

$$F(M) = \bigoplus_{A \in \mathfrak{MSet}_n} \Gamma^A(M) \otimes F_A^\dagger(\mathbf{B}|_{\#A}),$$

with the imposed relations tacitly understood. The formula may be extended to an arbitrary strict analytic functor:

$$F(M) = \bigoplus_{A \in \bigoplus \mathfrak{MSet}_n} \Gamma^A(M) \otimes F_A^\dagger(\mathbf{B}|_{\#A}).$$

The composition of two strict analytic functors F and G is expressible as

$$F(G(M)) = \bigoplus_{A \in \bigoplus \mathfrak{MSet}_n} \Gamma^A(G(M)) \otimes F_A^\dagger(\mathbf{B}|_{\#A}),$$

and we obtain the following formula for the multi-deviations of a composition.

THEOREM 12.

$$(F \circ G)_{\alpha[X]} = \sum_{A \in \bigoplus \mathfrak{MSet}_n} \left(\sum_{\omega \in \mathrm{Com}_A X} (G_\alpha)^{\otimes[\omega]} \right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\#A})}.$$

Proof. Calculate:

$$\begin{aligned}
(F \circ G)\left(\sum_{i=1}^{k} s_i \otimes \alpha_i\right) &= \sum_{A \in \mathfrak{MSet}_n} \Gamma^A\left(G\left(\sum_{i=1}^{k} s_i \otimes \alpha_i\right)\right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\#A})} \\
&= \sum_{A \in \bigoplus \mathfrak{MSet}_n} \Gamma^A\left(\sum_B s^B \otimes G_{\alpha[B]}\right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\#A})} \\
&= \sum_{A \in \bigoplus \mathfrak{MSet}_n} \left(\sum_\omega \bigotimes_{a \in \#A} \bigodot_{(a,Z)\in\omega} s^Z \otimes G_{\alpha[Z]}\right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\#A})}.
\end{aligned}$$

(The sum is extended over all $\omega\colon A \to C$, such that $\#C \subseteq [k]$ and $|C| = n$.) Identifying the coefficient of s^X yields the formula. □

Chapter 11

NUMERICAL VERSUS STRICT POLYNOMIAL FUNCTORS

> [...] le plus beau projet de notre académie,
> Une entreprise noble et dont je suis ravie,
> Un dessein plein de gloire, et qui sera vanté
> Chez tous les beaux esprits de la postérité [...].
>
> — Molière, *Les Femmes savantes*

Plainly, the notion of numerical functor is weaker than that of strict polynomial functor. It will naturally be inquired: how much weaker? This chapter is devoted to a comparison of the two species. The combinatorial and the module-theoretic viewpoint will be explored in turn, each leading to several illuminating insights.

The reader should keep in mind that there is a fundamental difference between the two types of functors, which is constantly at play. For while both kinds may be viewed as ordinary functors equipped with extra data, we know that numerical functors allow for an alternative characterisation, viz. as ordinary functors satisfying certain equations. *A fortiori*, a numerical functor is uniquely determined by its underlying functor. This is not true for strict polynomial functors, as the following example shows.

Example 1. — Let $\mathbf{B}$ be numerical, let A be an algebra (not necessarily numerical!), and let p be a prime. The ring A/pA is a bimodule over $\mathbf{B}$ in the usual way. Keeping the left module structure, equip it with another right module structure, mediated by the Frobenius map:

$$(x + pA) \cdot a = a^p x + pA.$$

That this is a module action is a consequence of Fermat's Little Theorem. Let $(A/pA)^{(1)}$ denote the bimodule thus obtained.

Define, for any algebra A (not necessarily numerical!), the functors

$$F_A : {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$$
$$M \mapsto A/pA \otimes M$$

and

$$G_A : {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$$
$$M \mapsto (A/pA)^{(1)} \otimes M.$$

These functors commute with scalar extensions; hence they give strict analytic functors $\mathfrak{XMod} \to \mathfrak{Mod}$.

Let $\alpha\colon M \to N$ be a homomorphism of A-modules, and let $a \in A$. As a homomorphism

$$A/pA \otimes M \to A/pA \otimes N,$$

we have

$$F_A(a\alpha) = 1 \otimes a\alpha = a \otimes \alpha = aF_A(\alpha),$$

which shows F is homogeneous of degree 1. As a homomorphism

$$(A/pA)^{(1)} \otimes M \to (A/pA)^{(1)} \otimes N,$$

we have

$$G_A(a\alpha) = 1 \otimes a\alpha = a^p \otimes \alpha = a^p G_A(\alpha),$$

which shows G is homogeneous of degree p.

Nonetheless, considered just as (numerical) functors $\mathfrak{XMod} \to \mathfrak{Mod}$, F and G are both linear, and are in fact isomorphic! This is again because of Fermat's Little Theorem:

$$(x + p\mathbf{B}) \cdot a = a^p x + p\mathbf{B} = ax + p\mathbf{B},$$

and consequently

$$(\mathbf{B}/p\mathbf{B})^{(1)} \cong \mathbf{B}/p\mathbf{B}$$

as $\mathbf{B}$-bimodules. △

Let us briefly sum up what is known of numerical versus strict polynomial functors. For affine functors, degree 0 and 1, the two notions coincide. This will no longer be the case in higher degrees, as there exist numerical functors which do not arise from strict polynomial ones. Even when it exists, the strict polynomial structure on a given functor is usually not unique. We saw this in the example above, where it was even possible to define strict polynomial structures of different degrees on the same underlying functor.

The situation for quadratic functors turns out to present an intermediate case, exhibiting some atypical phenomena. For example, as will be seen below, the existence of a strict polynomial structure on a quadratic functor may be inferred from a simple equation.

Another anomaly, occurring in the quadratic case only, is the following. A *quasi-homogeneous* quadratic functor may always be made homogeneous of degree 2; and not only that, but uniquely so. This is singular indeed, and far from the generic situation. Beware, however, that a quadratic functor which is *not* quasi-homogeneous need not arise from a strict polynomial functor.

§1. Quadratic Functors

We first propose to examine quadratic functors in detail. Let the following be the Schur description, as in Theorem 10.10, of a strict polynomial functor F of degree 2:

$$K \qquad X_1 \qquad X_2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$$

Here K denotes the degree 0 part, X_1 the degree 1 part, and $\beta\alpha = 2$ arises from degree 2. The labyrinthine description of F is, as may be checked:

$$K \qquad X_1 \oplus X_2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$$

We hence obtain the following characterisation of strict polynomial functors of degree 2.

Theorem 1. — *Let the following be the labyrinthine description of a quadratic functor F:*

$$K \qquad X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$$

- *F may be (non-uniquely) extended to a strict quadratic functor iff the following conditions are satisfied:*
  - *X has a direct sum decomposition $X = X_1 \oplus X_2$, such that $X_1 \subseteq \operatorname{Ker}\alpha$ and $\operatorname{Im}\beta \subseteq X_2$.*
  - $\beta\alpha = 2_{X_2}$.
- *F may be (uniquely) extended to a homogeneous quadratic functor iff the following conditions are satisfied:*
  - $K = 0$.
  - $\beta\alpha = 2$.

Example 2. — A continuation of the example above in the case $p = 2$ will serve to illustrate the theorem, and also to point out its subtlety. Both functors F and G have the following labyrinthine description:

$$0 \qquad \mathbf{B}/2\mathbf{B} \underset{0}{\overset{0}{\rightleftarrows}} 0$$

Since it evidently satisfies the conditions of the theorem, it has a unique structure of homogeneous quadratic functor — the functor G above. Yet we managed to exhibit another strict polynomial structure F on this same underlying functor, but this one linear! △

It will perhaps be illuminating to write out in detail *why* it is necessary that $\beta\alpha = 2$ in order to define a strict polynomial structure on the functor. This becomes evident when, given the labyrinthine description of a functor F, we try to calculate how F acts on the homomorphism

$$g\colon \mathbf{B} \to \mathbf{B},$$

given by multiplication by the ring element g. The maze associated to g is

$$G = \left[\; * \xrightarrow{\quad g \quad} * \;\right],$$

and hence

$$\begin{aligned} F(g)\,\big|_{F^\dagger(\mathbf{B})\to F^\dagger(\mathbf{B})} &= \sum_{P\subseteq G} \Phi_{\mathfrak{Laby}}(F)(G) = \Phi_{\mathfrak{Laby}}\left(\; * \xrightarrow{g} * \;\right) \\ &= \binom{g}{1}\Phi_{\mathfrak{Laby}}(F)\left(\; * \xrightarrow{1} * \;\right) + \binom{g}{2}\Phi_{\mathfrak{Laby}}(F)\left(\; * \overset{1}{\underset{1}{\rightrightarrows}} * \;\right) \\ &= g + \binom{g}{2}\beta\alpha. \end{aligned}$$

On the other hand, we know what the answer should be for a homogeneous quadratic functor F, namely

$$F(g) = g^2F(1) = g^2.$$

Equating these two expressions (for $g = 2$) yields $\beta\alpha = 2$.

Example 3. — Denote by $F = \mathbf{B}[\mathrm{Hom}(\mathbf{B}^2, -)]_2$ the fundamental quadratic functor. There are isomorphisms

$$F(\mathbf{B}^2) = \mathbf{B}[\mathrm{Hom}(\mathbf{B}^2, \mathbf{B}^2)]_2 = \mathbf{B}[\mathbf{B}^{2\times 2}]_2 \cong \mathbf{B}[t_{11}, t_{12}, t_{21}, t_{22}]/J_2,$$

where we have denoted

$$t_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

and J_2 is the ideal of polynomials of degree greater than 2. This module is free of rank 15, and is the direct sum of four cross-effects:

$$\begin{aligned} F^\dagger() &= \langle 1 \rangle \\ F^\dagger(\mathbf{B}_1) &= \langle t_{11}, t_{12}, t_{11}^2, t_{11}t_{12}, t_{12}^2 \rangle \end{aligned}$$

$$F^\dagger(\mathbf{B}_2) = \langle t_{21}, t_{22}, t_{21}^2, t_{21}t_{22}, t_{22}^2 \rangle$$
$$F^\dagger(\mathbf{B}_1|\mathbf{B}_2) = \langle t_{11}t_{21}, t_{11}t_{22}, t_{12}t_{21}, t_{12}t_{22} \rangle .$$

It may now be checked that the modules

$$K = \langle 1 \rangle, \quad X = \langle t_{11}, t_{12}, t_{11}^2, t_{11}t_{12}, t_{12}^2 \rangle, \quad Y = \langle t_{11}t_{21}, t_{11}t_{22}, t_{12}t_{21}, t_{12}t_{22} \rangle,$$

together with the following maps, constitute the labyrinthine description of F:

$$\alpha\colon \left\{ \begin{array}{rcl} t_{11} & \mapsto & t_{11}t_{21} \\ t_{12} & \mapsto & t_{12}t_{22} \\ t_{11}^2 & \mapsto & 2t_{11}t_{21} \\ t_{11}t_{12} & \mapsto & t_{11}t_{22} + t_{12}t_{21} \\ t_{12}^2 & \mapsto & 2t_{12}t_{22} \end{array} \right. \qquad \beta\colon \left\{ \begin{array}{rcl} t_{11}t_{21} & \mapsto & t_{11}^2 \\ t_{11}t_{22} & \mapsto & t_{11}t_{12} \\ t_{12}t_{21} & \mapsto & t_{12}t_{11} \\ t_{12}t_{22} & \mapsto & t_{12}^2. \end{array} \right.$$

We now put

$$X_1 = \langle t_{11}^2 - 2t_{11}, t_{12}^2 - 2t_{12} \rangle$$
$$X_2 = \langle t_{11}^2, t_{11}t_{12}, t_{12}^2 \rangle,$$

and observe that, if 2 is invertible in $\mathbf{B}$, X will decomposes as $X = X_1 \oplus X_2$, and F is in fact strict polynomial of degree 2.

This is an instance of a general phenomenon. We pointed out before that over a $\mathbf{Q}$-algebra, numerical and strict polynomial functors coincide. A slightly stronger statement is true: *If the integers 1 through n are invertible, then numerical and strict polynomial functors of degree n coincide.* This may not be completely obvious from the theory developed thus far, but we shall prove it presently. △

§2. The Ariadne Thread

Before Theseus entered the legendary labyrinth to fight the Minotaur, Ariadne presented him with a wonderful gift: the thread that will now forever bear her name. This device would eventually assist him in backtracking out of the frightful maze. Such is the legend. We too will be assisted on our quest by an Ariadne thread.

Let us expound our doctrine. We know that homogeneous functors of degree n correspond to Schur modules

$$J\colon \mathfrak{MSet}_n \to \mathfrak{Mod},$$

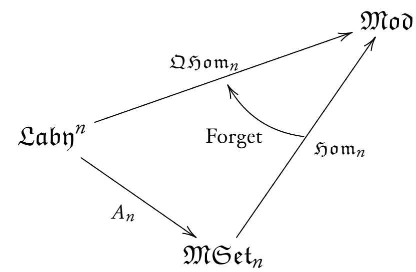
and numerical functors to labyrinth modules

$$H\colon \mathfrak{Laby}_n \to \mathfrak{Mod}.$$

Pre-composition with the Ariadne functor corresponds in effect to the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

In fact, since homogeneous functors are of course quasi-homogeneous, we have the following diagram:



Theorem 2: The Ariadne Thread.

$$\Phi_{\mathfrak{Laby}} \circ \Phi^{-1}_{\mathfrak{MSet}_n} = (A_n)^*.$$

Proof. We must show that, for a Schur module $J\colon \mathfrak{MSet}_n \to \mathfrak{Mod}$,

$$\Phi_{\mathfrak{Laby}} \Phi^{-1}_{\mathfrak{MSet}_n}(J) = JA_n.$$

Denoting $H = \Phi_{\mathfrak{Laby}} \Phi^{-1}_{\mathfrak{MSet}_n}(J)$, we have, for a finite set X,

$$\begin{aligned} H(X) &= \Phi^{-1}_{\mathfrak{MSet}_n}(J)^{\dagger}(\mathbf{B}|_X) = \operatorname{Im} \Phi^{-1}_{\mathfrak{MSet}_n}(J)\left(\Diamond_{x\in X} \pi_x\right) \\ &= \operatorname{Im} \sum_{Y\subseteq X} (-1)^{|X|-|Y|} \Phi^{-1}_{\mathfrak{MSet}_n}(J)\left(\sum_{y\in Y} \pi_y\right). \end{aligned}$$

Recalling that

$$\Phi^{-1}_{\mathfrak{MSet}_n}(J)\left(\sum s_{yx}\sigma_{yx}\right) = \sum_{A,B} \sum_{\mu\colon A\to B} s^{\mu} J(\mu),$$

we obtain

$$\begin{aligned} H(X) &= \operatorname{Im} \sum_{Y\subseteq X} (-1)^{|X|-|Y|} \sum_{\substack{\#A\subseteq Y\\ |A|=n}} J(\iota_A) \\ &= \operatorname{Im} \sum_{Y\subseteq X} (-1)^{|X|-|Y|} \sum_{\substack{\#A\subseteq Y\\ |A|=n}} 1_{J(A)} \\ &= \operatorname{Im} \sum_{\substack{\#A=X\\ |A|=n}} 1_{J(A)} = \bigoplus_{\substack{\#A=X\\ |A|=n}} J(A) = JA_n(X). \end{aligned}$$

The fourth step is due to the Principle of Inclusion and Exclusion.

We now turn to computing $H(P)$ when $P\colon X \to Y$ is a maze. It will be enough to consider the case of a simple maze, as any maze may be expressed as a sum of such. Denote the passages of P by $p_i\colon x_i \to y_i$ for $1 \leqslant i \leqslant k$, and compute:

$$\begin{aligned} H(P) &= \Phi^{-1}_{\mathfrak{M}\mathfrak{Set}_n}(J)\left(\lozenge\, \overline{p_i}\sigma_{y_i x_i}\right) \\ &= \sum_{I\subseteq[k]} (-1)^{k-|I|}\Phi^{-1}_{\mathfrak{M}\mathfrak{Set}_n}(J)\left(\sum_{i\in I}\overline{p_i}\sigma_{y_i x_i}\right), \end{aligned}$$

of which the $J(A) \to J(B)$ component is

$$\sum_{I\subseteq[k]} (-1)^{k-|I|} \sum_{\mu\colon A\to B} \left(\prod \bar{p}^I_{\mu(a)a}\right) J(\mu)$$

$$= \sum_{\mu\colon A\to B} \left(\sum_{I\subseteq[k]} (-1)^{k-|I|} \prod \bar{p}^I_{\mu(a)a}\right) J(\mu), \quad (1)$$

where we have defined

$$\bar{p}^I_{ba} = \begin{cases} \bar{p}_i & \text{if } a = x_i \text{ and } b = y_i \text{ for } i \in I \\ 0 & \text{else.} \end{cases}$$

We see that for the coefficient of $J(\mu)$ to be non-zero, all elements of the multation μ must correspond to passages in P. The converse also holds, namely, that all passages of P must be represented in μ. This is because of the following reason. If a passage p_j be "missing" from μ, sets I with and without j in (1) will give rise to terms of alternating signs, which will cancel. Hence the coefficient of $J(\mu)$ will survive only if μ is of the form

$$\mu = \prod_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}^{[m_i]},$$

for positive integers

$$m_1 + \cdots + m_k = n.$$

Then only $I = [k]$ will yield a non-zero contribution in (1), and consequently

$$H(P) = \sum_{m_1+\cdots+m_k=n} \left(\prod \bar{p}_i^{m_i}\right) J\left(\prod_i \begin{bmatrix} x_i \\ y_i \end{bmatrix}^{[m_i]}\right) = JA_n(P).$$

□

§3. Numerical versus Strict Polynomial Functors

As the crowning glory of our work, the pinnacle of the palace, let us record the exact obstruction for a numerical functor to be strict polynomial. Recall from Theorem 9.4 the homomorphism

$$\varepsilon_n : \Gamma^n(\mathbf{B}^{n\times n}) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[\mathbf{B}^{n\times n}]_n$$
$$\sigma^{[A]} \mapsto \frac{1}{\deg A} \left[\Diamond_A \sigma \right]$$

and the direct sum decomposition

$$\operatorname{Im} \varepsilon_n \cong \Gamma^n(M) \oplus (\operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n).$$

THEOREM 3: THE POLYNOMIAL FUNCTOR THEOREM. — *Let F be a quasi-homogeneous functor of degree n, corresponding to the labyrinth module*

$$H : \mathfrak{Laby}^n \to \mathfrak{Mod}$$

and the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$*-module M. The following three constructions are equivalent:*

A. *Imposing the structure of homogeneous functor upon F.*

B. *Exhibiting a factorisation of H through* $\mathfrak{Laby}^{\oplus n}$.

C. *Giving M the structure of* $\operatorname{Im} \varepsilon_n$*-module.*

Proof. The isomorphism $\mathfrak{Laby}^{\oplus n} \cong \mathfrak{MSet}_n$ shows the equivalence of A and B.
From the isomorphism

$$\operatorname{Im} \varepsilon_n \cong \Gamma^n(\mathbf{B}^{n\times n}) \times (\operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n)$$

we conclude that $\Gamma^n(\mathbf{B}^{n\times n})$-modules canonically correspond to $\operatorname{Im} \varepsilon_n$-modules, and vice versa. (The ring $\operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n$ corresponds to subfunctors of lower degree. By considering quasi-homogeneous functors only, modules over this ring will be zero.) This shows the equivalence of A and C. □

We caution the reader that, even in the case H factors through $\mathfrak{Laby}^{\oplus n}$ and M may be considered an $\operatorname{Im} \varepsilon_n$-module, the factorisation and the module structure are not unique. There are in general many strict polynomial structures on the same functor, even of different degrees!

EXAMPLE 4. — We point out one particular case when any such H will factor (uniquely) through $\mathfrak{Laby}^{\oplus n}$. When $\mathbf{B}$ is a $\mathbf{Q}$-algebra,

$$\mathfrak{Laby}^{\oplus n} \cong \mathfrak{MSet}_n$$

is simply the additive hull of $\mathfrak{Laby}^n$. This mirrors the already well-known fact that, over a $\mathbf{Q}$-algebra, numerical and strict polynomial functors coincide. In fact, as is seen from the definition of $\mathfrak{Laby}^{\oplus n}$, it is sufficient for the integers 1 through n to be invertible in $\mathbf{B}$ to guarantee such a factorisation. △

Example 5. — For *affine* functors (degree 0 and 1), numerical and strict polynomial functors coincide. This is no longer the case in higher degrees. Yet, as we saw before, the quadratic case retains some regularity, in that any *quasi-homogeneous* functor is necessarily homogeneous. Any quasi-homogeneous functor of degree 2 may be given a unique strict polynomial structure, which makes it homogeneous of degree 2.

The reason for this becomes clear when we examine the creation of $\mathfrak{Laby}^{\oplus 2}$ in detail. The localisation procedure requires us to adjoin mazes

$$\frac{1}{2}\left[\ * \overset{1}{\underset{1}{\rightrightarrows}} *\ \right],$$

but in $\mathfrak{Laby}^2$, the equation

$$\left[\ * \overset{1}{\underset{1}{\rightrightarrows}} *\ \right] = 2\left[\ * \xrightarrow{1} *\ \right]$$

holds, as may be verified. Hence $\mathfrak{Laby}^{\oplus 2}$ is simply the additive hull of $\mathfrak{Laby}^2$. △

§4. Restriction and Extension of Scalars

The divided power map

$$\gamma_n : \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n})$$

gives rise to two natural functors between the corresponding module categories, namely restriction and extension of scalars. We consider them in turn.

Restriction of scalars is the functor

$$_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod} \to {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod},$$

which takes a $\Gamma^n(\mathbf{B}^{n\times n})$-module M and views it a $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module under the multiplication

$$[\sigma]x = \gamma_n(\sigma)x = \sigma^{[n]}x = F(\sigma)x.$$

On the functorial level, this corresponds to the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

Extension of scalars is the functor

$$_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod} \to {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod},$$

which takes a $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M and transforms it into a $\Gamma^n(\mathbf{B}^{n\times n})$-module

$$\Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} M$$

through the tensor product. How does it act on the functorial level?

Let us denote by

$$P = \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n$$
$$Q = \Gamma^n \mathrm{Hom}(\mathbf{B}^n, -)$$

the projective generators of the categories $\mathfrak{Num}_n$ and $\mathfrak{Hom}_n$, respectively. A functor $F \in \mathfrak{Num}_n$ will then correspond to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module

$$M = \mathrm{Nat}(P, F).$$

Extension of scalars transforms it into the $\Gamma^n(\mathbf{B}^{n\times n})$-module

$$N = \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} M = \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F),$$

which corresponds to the homogeneous functor

$$\begin{aligned} G &= Q \otimes_{\Gamma^n(\mathbf{B}^{n\times n})} N \\ &= Q \otimes_{\Gamma^n(\mathbf{B}^{n\times n})} \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F) \\ &= Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F). \end{aligned}$$

This tensor product is interpreted in the usual way. By definition,

$$\begin{aligned} P \mapsto{} & Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, P) \\ &= Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathbf{B}[\mathbf{B}^{n\times n}]_n^{\circ} = Q, \end{aligned}$$

and we then extend by direct sums and right-exactness.

We summarise in a theorem.

Theorem 4. — *Consider the divided power map*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n}).$$

- *Restriction of scalars*

$${}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod} \to {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod}$$

  corresponds to the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

- *Extension of scalars corresponds to the functor*

$${}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod} \to {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod},$$

  which maps

$$F \mapsto \Gamma^n \mathrm{Hom}(\mathbf{B}^n, -) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, F).$$

Restriction of scalars, as we know, corresponds on the combinatorial level to the Ariadne functor. Extension of scalars, on the contrary, does not seem to admit a simple combinatorial interpretation. Evidently, this procedure will grasp *any* numerical functor, and through brute force transform it into a homogeneous functor of degree n. The functor to start with need not be quasi-homogeneous; it may very well contain parts of lower degree. Excessive amounts of violence will then be needed to transfigure it into a homogeneous functor, and it is hardly surprising that this process will wreak havoc with its internal structure.

§5. Torsion-Free Functors

In this section we consider torsion-free functors and modules. *Torsion* shall, as before, always mean **Z**-*torsion*. We let

$$M^\star$$

denote the greatest torsion-free quotient of the module M, which is simply M divided by its torsion submodule. Observe that

$$\mathbf{Q} \otimes_{\mathbf{Z}} M^\star = \mathbf{Q} \otimes_{\mathbf{Z}} M.$$

Also, when C is a category of modules or functors, we shall let

$$C^\star$$

denote the subcategory of torsion-free modules.

Definition 1. — Let $A, B \subseteq C$ be linear categories. The **category of (epic) angles** joining A and B over C is the linear category of formal angles

$$X \longrightarrow Z \longleftarrow Y\ , \qquad X \in A,\ Y \in B,\ Z \in C;$$

with both arrows epic (in C). It will be denoted by

$$A \vee_C B.$$

◇

We shall usually suppress mention of the epics, and denote the formal angle

$$X \longrightarrow Z \longleftarrow Y$$

by simply

$$[XZY],$$

when no confusion is likely to result.

An arrow in the angle category is defined in the obvious way. Namely, an arrow from $[XZY]$ to $[X'Z'Y']$ is given by a commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Z' & \longleftarrow & Y' \end{array}$$

where the three arrows $X \to X'$, $Y \to Y'$, and $Z \to Z'$ belong to A, B, and C, respectively.

A subtle fact, which has hitherto gone unmentioned, is the following. A careful examination of the proof of Theorem 9.7 will reveal that

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^{n+1}, -)]_n$$

is another projective generator for $\mathfrak{Num}_n$. As a notable consequence, the rings

$$\mathbf{B}[\mathbf{B}^{n\times n}]_{n-1} \quad \text{and} \quad \mathbf{B}[\mathbf{B}^{(n-1)\times(n-1)}]_{n-1}$$

are Morita equivalent, so that, in fact,

$$\mathfrak{Num}_{n-1} \sim {}_{\mathbf{B}[\mathbf{B}^{(n-1)\times(n-1)}]_{n-1}}\mathfrak{Mod} \sim {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_{n-1}}\mathfrak{Mod}.$$

The set-up is then as follows. We have rings

$$R = \mathbf{B}[\mathbf{B}^{n\times n}]_n$$
$$S = \mathbf{B}[\mathbf{B}^{n\times n}]_{n-1}$$
$$T = \mathrm{Im}\,\gamma_n \subseteq \Gamma^n(\mathbf{B}^{n\times n}),$$

and surjections

$$\sigma\colon R \to S, \qquad \tau\colon R \to T,$$

where σ denotes the canonical quotient map, and $\tau = \gamma_n$. By Theorem 9.2, the homomorphism

$$(\sigma, \tau)\colon R \to S \times T$$

is an injection of finite index. The goal is to show how this leads to a category equivalence

$${}_R\mathfrak{Mod}^\star \sim {}_S\mathfrak{Mod}^\star \vee_{{}_R\mathfrak{Mod}} {}_T\mathfrak{Mod}^\star.$$

(Recall the construction of the pullback categpry in Chapter 0.)

Let us begin with a very general remark. Let $\chi\colon A \to B$ be a surjective ring homomorphism, and let M be an A-module. There is an exact sequence:

$$0 \longrightarrow (\mathrm{Ker}\,\chi)M \longrightarrow M \longrightarrow B \otimes_A M \longrightarrow 0$$

The module

$$B \otimes_A M = M/(\mathrm{Ker}\,\chi)M$$

is the greatest B-quotient of M. Furthermore, $(B\otimes_A M)^\star$ is the greatest torsion-free B-quotient. Applying this observation to the homomorphisms $\sigma\colon R \to S$ and $\tau\colon R \to T$, we conclude:

Theorem 5. — *Every functor in* $\mathfrak{Num}_n$ *has a greatest quotient in both* $\mathfrak{Num}_{n-1}$ *and* $\mathfrak{QHom}_n$.

Theorem 6. — *When* $[XZY]$ *is an object of the category*

$$_S\mathfrak{Mod}^\star \vee_{_R\mathfrak{Mod}} {_T\mathfrak{Mod}^\star},$$

the module Z *is a torsion module.*

Proof. There are R-homomorphisms:

$$X \xrightarrow{\varphi} Z \xleftarrow{\psi} Y$$

Note that X and Y are also $S \times T$-modules, according to:

$$(s,t)\cdot x = sx, \qquad (s,t)y = ty.$$

Consider a $z \in Z$. Because R has finite index in $S \times T$, we can find an integer $p \neq 0$ such that $p(1,0) \in R$. Choose $x \in X$ such that

$$\varphi(x) = z,$$

and calculate

$$\begin{aligned} pz &= p\varphi(x) = \varphi(px) \\ &= \varphi((p,0)\cdot x) = (p,0)\varphi(x) = (p,0)z. \end{aligned}$$

In the fifth step, we used that φ is R-linear. Similarly, there is a non-zero integer q such that $(0,q) \in R$ and

$$qz = (0,q)z.$$

We conclude that

$$pqz = (p,0)(0,q)\cdot z = 0 \cdot z = 0,$$

so that z is a torsion element. □

We shall now construct our category equivalence:

$$_R\mathfrak{Mod}^\star \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} {_S\mathfrak{Mod}^\star} \vee_{_R\mathfrak{Mod}} {_T\mathfrak{Mod}^\star}$$

Consider first an $M \in {_R\mathfrak{Mod}^\star}$. There is a homomorphism of R-modules

$$\begin{aligned} \chi\colon M &\to (S \otimes_R M)^\star \oplus (T \otimes_R M)^\star \\ x &\mapsto (1 \otimes x, 1 \otimes x). \end{aligned}$$

The projections

$$\begin{array}{cc} M \to S \otimes_R M & M \to T \otimes_R M \\ x \mapsto 1 \otimes x & x \mapsto 1 \otimes x \end{array}$$

are onto, because σ and τ are onto.

Let us now show χ is one-to-one. Consider the following exact sequence of R-modules:

$$0 \longrightarrow \operatorname{Ker}\chi \longrightarrow M \xrightarrow{\ \chi\ } (S \otimes_R M)^\star \oplus (T \otimes_R M)^\star$$

Tensoring with the $\mathbf{Z}$-module $\mathbf{Q}$ produces:

$$\begin{array}{ccccccc} 0 \longrightarrow \operatorname{Ker}\chi \otimes_{\mathbf{Z}} \mathbf{Q} & \longrightarrow & M \otimes_{\mathbf{Z}} \mathbf{Q} & \xrightarrow{\ \chi\otimes 1\ } & \big[(S \otimes_R M)^\star \oplus (T \otimes_R M)^\star\big] \otimes_{\mathbf{Z}} \mathbf{Q} \\ & & \| & & \| \\ & & R \otimes_R M \otimes_{\mathbf{Z}} \mathbf{Q} & \longrightarrow & (S \times T) \otimes_R M \otimes_{\mathbf{Z}} \mathbf{Q} \end{array}$$

Since R has finite index in $S \times T$, the lower map is an isomorphism, and hence $\operatorname{Ker}\chi \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. We conclude that $\operatorname{Ker}\chi$ is a torsion module, and therefore, being included in the torsion-free module M, zero. Consequently, χ is one-to-one.

We thus infer, by Delsarte's Lemma, the existence of a formal angle

$$(S \otimes_R M)^\star \xrightarrow{\ \varphi\ } \big[(S \otimes_R M)^\star \oplus (T \otimes_R M)^\star\big]/M \xleftarrow{\ \psi\ } (T \otimes_R M)^\star$$

in the category

$${}_S\mathfrak{Mod}^\star \vee_{{}_R\mathfrak{Mod}} {}_T\mathfrak{Mod}^\star.$$

We define this to be $\Pi(M)$.

Theorem 7: The Splitting-Off Theorem. — *The functors*

$$\Pi\colon M \mapsto \Big[(S \otimes_R M)^\star, \big[(S \otimes_R M)^\star \oplus (T \otimes_R M)^\star\big]/M, (T \otimes_R M)^\star\Big]$$
$$\Sigma\colon [XZY] \mapsto \operatorname{Ker}(X \oplus Y \to Z)$$

provide a category equivalence:

$${}_R\mathfrak{Mod}^\star \underset{\Sigma}{\overset{\Pi}{\rightleftarrows}} {}_S\mathfrak{Mod}^\star \vee_{{}_R\mathfrak{Mod}} {}_T\mathfrak{Mod}^\star$$

Consequently, there is an equivalence of functor categories:

$$\mathfrak{Num}_n^\star \sim \mathfrak{QHom}_n^\star \vee_{\mathfrak{Num}_n} \mathfrak{Num}_{n-1}^\star.$$

Proof. With notation as above,

$$\operatorname{Ker}(\varphi + \psi) = M,$$

so we immediately have $\Sigma\Pi \cong I$.

Consider now an angle

$$[XZY] \in {}_S\mathfrak{Mod}^\star \vee_{{}_R\mathfrak{Mod}} {}_T\mathfrak{Mod}^\star.$$

Defining

$$M = \Sigma([XZY]) = \operatorname{Ker}(X \oplus Y \to Z),$$

we observe that M has finite index in $X \oplus Y$, since Z is torsion. Furthermore,

$$\Pi\Sigma([XZY]) = \Pi(M) = \Big[(S \otimes_R M)^\star, \big[(S \otimes_R M)^\star \oplus (T \otimes_R M)^\star\big]/M, (T \otimes_R M)^\star\Big].$$

By Delsarte's Lemma, X is an S-quotient of M, and, moreover, it is by assumption torsion-free. Since $(S \otimes_R M)^\star$ is by definition the *greatest* torsion-free S-quotient of M, there is a factorisation:

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \operatorname{Ker}\xi & \longrightarrow & (S \otimes_R M)^\star & \xrightarrow[\xi]{} & X & \longrightarrow & 0 \end{array}$$

Tensoring with $\mathbf{Q}$ yields:

$$\begin{array}{ccccccccc} & & & & M \otimes_{\mathbf{Z}} \mathbf{Q} & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & \operatorname{Ker}\xi \otimes_{\mathbf{Z}} \mathbf{Q} & \longrightarrow & (S \otimes_R M)^\star \otimes_{\mathbf{Z}} \mathbf{Q} & \xrightarrow[\xi \otimes 1]{} & X \otimes_{\mathbf{Z}} \mathbf{Q} & \longrightarrow & 0 \end{array}$$

Because M has finite index in $X \oplus Y$, we have

$$M \otimes_{\mathbf{Z}} \mathbf{Q} \cong (X \otimes_{\mathbf{Z}} \mathbf{Q}) \oplus (Y \otimes_{\mathbf{Z}} \mathbf{Q}).$$

We infer that the greatest torsion-free S-quotient of $M \otimes_{\mathbf{Z}} \mathbf{Q}$ is, in fact, $X \otimes_{\mathbf{Z}} \mathbf{Q}$. Therefore, the homomorphism $\xi \otimes 1$ is an isomorphism, with kernel

$$\operatorname{Ker}\xi \otimes_{\mathbf{Z}} \mathbf{Q} = 0.$$

The module $\operatorname{Ker}\xi$ is torsion, and therefore zero, being included in the torsion-free module $(S \otimes_R M)^\star$. We have thus deduced

$$(S \otimes_R M)^\star \cong X.$$

Similarly,

$$(T \otimes_R M)^\star \cong Y,$$

and there only remains the verification

$$\begin{aligned}\big[(S \otimes_R M)^{\star} \oplus (T \otimes_R M)^{\star}\big]/M &\cong (X \oplus Y)/\operatorname{Ker}(X \oplus Y \to Z) \\ &\cong \operatorname{Im}(X \oplus Y \to Z) = Z.\end{aligned}$$

The relation $\Pi\Sigma \cong I$ follows. $\square$

Chapter 12

POLYNOMIAL MONADS AND OPERADS

Han ämnar nu ge ut, till vinst för Pappersbruken,
Till vinst för hvar och en som läser i vårt land,
Et dubbelt Skalde verk ifrån sin lärda hand:
Om rynkbands nytta i peruken
Och skadan utaf mal i gamla folioband.

— Anna Maria Lenngren, *Herr Grälberg*

This closing chapter will deal with practical matters — it is the palace kitchen, if the expression be us pardoned. Let us, very briefly, indicate how polynomial functors could possibly be put to use.

The theory of operads has long suffered from its unnatural restriction to fields of characteristic 0. Polynomial functors, it turns out, allow for a natural extension to arbitrary base rings. We certainly do not aim to be encyclopædic, but merely to sketch an outline of what polynomial monads and operads look like, and how they behave under the fundamental operations of induction product and plethysm.

It is our hope that somebody, someday, might find this theory useful.

§1. Classical Operads

The reader wishing a comfortable introduction to operads is referred to the manuscript [15] by Professors Loday and Vallette (currently at the draft stage), which has served as our source of information.

Let $\mathbf{B}$ be a field of characteristic 0, and let Σ_n denote the symmetric group on n symbols. A Σ_n-module is (by definition, if you like) the same as a functor

$$\mathbf{B}[\Sigma_n] \to \mathfrak{Mod},$$

where $\mathbf{B}[\Sigma_n]$ is interpreted as a category with a single object. Already in 1995, Professor Macdonald had established a category equivalence

$$\mathfrak{Hom}_n \sim {}_{\Sigma_n}\mathfrak{Mod}.$$

A homogeneous functor F would correspond to the Σ_n-module[1]

$$\mathrm{Nat}(T^n, F),$$

and a Σ_n-module P to the homogeneous functor

$$M \mapsto T^n(M) \otimes_{\Sigma_n} P.$$

(The right action of Σ_n on $T^n(M)$ is the obvious one.) See [16], in which the result has been relegated to an "appendix"! This led to the following definitions.

Definition 1. — A **Σ-module** is a family

$$P = (P_n)_{n \in \mathbf{N}}$$

of Σ_n-modules. ◇

Definition 2. — The **Schur functor** associated to a Σ-module P is

$$\mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$$
$$M \mapsto \bigoplus_{n=0}^{\infty} T^n(M) \otimes_{\Sigma_n} P_n.$$

◇

Definition 3. — A **classical operad** is a Σ-module with an associated monadic Schur functor. ◇

Theorem 1. — *Over a field of characteristic* 0*, a classical operad is equivalent to a strict analytic monad.*

Proof. Because of the category equivalence

$$\mathfrak{Hom}_n \sim {}_{\Sigma_n}\mathfrak{Mod},$$

giving a strict analytic monad is the same as specifying a Σ-module, and requiring that its Schur functor be monadic. □

Operads and monads are thus equivalent, and we shall usually prefer the latter viewpoint. The examples we give below will all be of monads.

When considering a strict analytic monad

$$D\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod},$$

we have automatically the concept of *algebra* (as inherited from the theory of monads). The base category is (as always) $\mathfrak{Mod}$, so an algebra for our operads will first of all be a module.

[1] Evidently, $\mathrm{Nat}(T^n, F)$ is a right module over $\mathrm{Nat}\, T^n$. That indeed $\mathrm{Nat}\, T^n \cong \mathbf{B}[\Sigma_n]^\circ$ is a result that dates back to Weyl.

Example 1. — Let A be an (associative, unital) algebra, and consider the linear functor

$$D(M) = A \otimes M.$$

It is canonically a monad, with structure maps

$$\mu_A : A \otimes A \otimes M \to A \otimes M, \quad a \otimes b \otimes x \mapsto ab \otimes x$$
$$\varepsilon_A : M \to A \otimes M, \quad x \mapsto 1 \otimes x.$$

An algebra over this monad is a homomorphism

$$A \otimes M \to M$$

(satisfying some axioms), which is just an A-module. D is the **operad (or monad) of A-modules**. △

Example 2. — The tensor functor T is a strict analytic monad. An algebra over T is a homomorphism

$$T(M) \to M,$$

which is just an associative, unital algebra. T gives the **operad of associative algebras**. (To obtain non-unital algebras, simply remove the degree 0 part from T, which produces the *reduced* tensor algebra.) △

Example 3. — The symmetric functor S is a strict analytic monad. An algebra over S is an associative, commutative, and unital algebra. The operad corresponding to S is known as the **operad of commutative algebras**. △

Example 4. — The monad Λ is evidently the **operad of anti-commutative algebras**. △

Example 5. — The monad Γ is the **operad of divided power algebras**. In the current context, $\mathbf{B}$ being a field of characteristic 0, $\Gamma \cong S$. △

Example 6. — The functor L, which constructs the free Lie algebra on a module, is a monad. Being a subfunctor of T, it is strict analytic. It gives the **operad of Lie algebras**. △

There are two standard constructions on operads: the induction product and the plethysm. On the monadic level, they are simply tensor product and composition, respectively.

Definition 4. — Let P and Q be operads with Schur functors F and G, respectively. The **induction product** is the Σ-module

$$P \otimes Q$$

that has Schur functor $F \otimes G$. ◇

There is a conceivable risk of confusion here with the usual use of the symbol $\otimes$, but we shall never have occasion to consider the *tensor product* of Σ-modules.

It is contended in [15] that the induction product is given by the formula

$$P \otimes Q \colon X \mapsto \bigoplus_{A \sqcup B = X} P(A) \otimes Q(B).$$

We shall presently produce a stronger statement, from which this formula can be derived as a special case.

Definition 5. — Let P and Q be operads with Schur functors F and G, respectively. The **plethysm** is the Σ-module

$$P \circ Q$$

that has Schur functor $F \circ G$. ◇

To describe the plethysm, we recall the terminology of compositions from Chapter 2. Let X be a set, and let $\omega \colon [n] \to 2^X$ be an $[n]$-composition of X. Define

$$Q(\omega) = Q(\omega(1)) \otimes Q(\omega(2)) \otimes \cdots \otimes Q(\omega(n)).$$

According to [15], the plethysm is given by the formula

$$P \circ Q \colon X \mapsto \bigoplus_{n=0}^{\infty} \bigoplus_{\omega \in \mathrm{Com}_{[n]} X} P([n]) \otimes Q(\omega).$$

§2. Schur Operads

Arguably, the fields of characteristic 0 form a rather limited class of rings. It should be clear, then, that an extension of the operad concept is called for. Evidently, there is no problem in considering strict analytic monads over an arbitrary (commutative and unital) base ring, but it is, perhaps, not as evident what the appropriate generalisation on the operad side is.

A Σ-module is (by definition, if you like) the same as a functor

$$\mathbf{B}[\Sigma_n] \to \mathfrak{Mod},$$

where $\mathbf{B}[\Sigma_n]$ is interpreted as a category with a single object. It extends to a linear functor

$$\mathfrak{Set}_n \to \mathfrak{Mod},$$

where $\mathfrak{Set}_n$ denotes the linear category of sets of cardinality n. It was the insight of Dr. Salomonsson and Professor Ekedahl ([20]) that, over an arbitrary commutative and unital base ring, $\mathfrak{Set}_n$ is superseded by $\mathfrak{MSet}_n$. These two categories are, quite blatantly, not equivalent, but they are Morita equivalent over a field of characteristic 0.

Definition 6. — A **Schur module** is a linear functor

$$\bigoplus_{n=0}^{\infty} \mathfrak{MSet}_n \to \mathfrak{Mod}.$$

◇

Recall the category equivalence

$$\Phi_{\mathfrak{MSet}_n} \colon \mathfrak{Hom}_n \to \mathrm{Fun}(\mathfrak{MSet}_n, \mathfrak{Mod}),$$

under which a homogeneous functor F corresponds to the Schur module

$$\Phi_{\mathfrak{MSet}_n}(F) \colon X \mapsto \mathrm{Nat}(\Gamma^X, F).$$

Definition 7. — The **Schur functor** associated to a Schur module P is the homogeneous functor

$$\begin{aligned} \Phi^{-1}_{\mathfrak{MSet}_n}(P) \colon \mathfrak{XMod} &\to \mathfrak{Mod} \\ M &\mapsto \bigoplus_{X \in \bigoplus \mathfrak{MSet}_n} \Gamma^X(M) \otimes P(X). \end{aligned}$$

◇

Definition 8. — A **Schur operad** is a Schur module with an associated monadic Schur functor. ◇

We then have the following result, which appears as Theorem I.3.4 in Dr. Salomonsson's thesis [20].

Theorem 2. — *Over a commutative, unital base ring, a Schur operad is equivalent to a strict analytic monad.*

Proof. Immediate from the category equivalence above. □

Example 7. — Let $\mathbf{B}$ be a ring of characteristic p. Recall from Professor Jacobson's excellent treatise on Lie algebras, [13], the definition of a *restricted Lie algebra of characteristic p*. It is a Lie algebra A of characteristic p, equipped with a unary operation $(-)^{[p]}$, satisfying the following axioms:

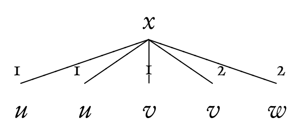
1. $(ax)^{[p]} = a^p x^{[p]}$, for $a \in \mathbf{B},\ x \in A$.
2. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$, where $is_i(x,y)$ is the coefficient of t^{i-1} in the expansion of $x(\mathrm{ad}(tx+y))^{p-1}$. Here x and y commute with the indeterminate t, but not with each other.
3. $[xy^{[p]}] = x(\mathrm{ad}\, y)^p$, for $x, y \in A$.

The monad L_p of free restricted algebras is strict analytic, and gives rise to the **operad of restricted Lie algebras of characteristic** p. It is not given by a classical operad, mainly because the characteristic is wrong. △

Example 8. — No treatment of operads with self-respect could omit mention of trees. Like their classical counterparts, also Schur operads may be illustrated by trees. An element

$$(u^{[2]}v \otimes vw) \otimes x \in \Gamma^{\{1,1,1,2,2\}}(M) \otimes P(\{1,1,1,2,2\})$$

of the Schur functor should be thought of as a **multi-tree**[2]:



From their interpretation as a tensor product, it is clear that multi-trees are linear in each vertex (top and bottom row, that is; those in the middle are just labels). But, as we recall from Theorem 10.11, they are subject to yet another relation, which we now exemplify. Let $u, v \in M$, and let $x \in P(\{1,1,2\})$. Because

$$\begin{bmatrix} 1 & 2 & 2 \\ u & u & v \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 & 2 \\ u & u & v \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ u & v & u \end{bmatrix},$$

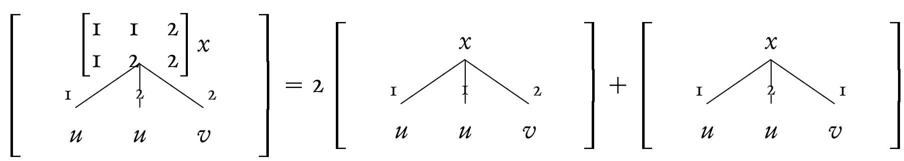
we have a relation

$$(u \otimes uv) \otimes P\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}\right)(x) = 2(u^{[2]} \otimes v) \otimes x + (uv \otimes u) \otimes x$$

on the Schur functor

$$\bigoplus_{X \in \bigoplus \mathfrak{MSet}_n} \Gamma^X(M) \otimes P(X),$$

and a corresponding equation for trees, which might be termed "vertical associativity":



△

[2] This was originally Dr. Salomonsson's device, and somewhat (genetically) modified to suit our needs.

§3. The Induction Product of Schur Operads

Induction products and plethysms of Schur modules are defined as for Σ-modules. We emphasise again that these operations correspond to tensor product and composition of the corresponding Schur *functors*.

Consider two Schur modules P and Q, corresponding to Schur functors F and G, respectively. Let π, as usual, denote projection. According to Theorem 7.8, the induction product $P \otimes Q$ maps

$$\begin{aligned}(P \otimes Q)(X) &= (F \otimes G)^{\dagger}_{X}(\mathbf{B}|_{\#X}) \\ &= \operatorname{Im}(F \otimes G)_{\pi[X]} = \bigoplus_{A \sqcup B = X} \operatorname{Im}\left(F_{\pi[A]} \otimes G_{\pi[B]}\right) \\ &= \bigoplus_{A \sqcup B = X} F^{\dagger}_{A}(\mathbf{B}|_{\#X}) \otimes G^{\dagger}_{B}(\mathbf{B}|_{\#X}) = \bigoplus_{A \sqcup B = X} P(A) \otimes Q(B).\end{aligned}$$

The action of a multation $\mu \colon X \to Y$ is found as follows. By the Multi-Set Yoneda Lemma, it corresponds to

$$(F \otimes G)_{\sigma[\mu]} \colon (F \otimes G)^{\dagger}_{X}(\mathbf{B}|_{\#X}) \to (F \otimes G)^{\dagger}_{Y}(\mathbf{B}|_{\#Y}),$$

or, as it so happens, the homomorphism

$$(F \otimes G)_{\sigma[\mu]} \colon \bigoplus_{A \sqcup B = X} F^{\dagger}_{A}(\mathbf{B}|_{\#X}) \otimes G^{\dagger}_{B}(\mathbf{B}|_{\#X}) \to \bigoplus_{C \sqcup D = Y} F^{\dagger}_{C}(\mathbf{B}|_{\#Y}) \otimes G^{\dagger}_{D}(\mathbf{B}|_{\#Y}),$$

given by the formula

$$(F \otimes G)_{\sigma[\mu]} = \sum_{\kappa \sqcup \lambda = \mu} F_{\sigma[\kappa]} \otimes G_{\sigma[\lambda]}.$$

Theorem 3. — *The induction product of two Schur modules P and Q is given by*

$$(P \otimes Q)(X) = \bigoplus_{A \sqcup B = X} P(A) \otimes Q(B).$$

For a multation $\mu \colon X \to Y$,

$$(P \otimes Q)(\mu) = \sum_{\kappa \sqcup \lambda = \mu} P(\kappa) \otimes Q(\lambda).$$

§4. The Plethysm of Schur Operads

We now turn to the plethysm. From Theorem 10.12, we recall the formula

$$(F \circ G)_{\alpha[X]} = \sum_{A} \left(\sum_{\omega \in \operatorname{Com}_A X} \bigotimes_{a \in \#A} \bigodot_{(a,Z) \in \omega} G_{\alpha[Z]} \right) \otimes 1_{F^{\dagger}_{A}(\mathbf{B}|_{\#A})}.$$

This yields the plethysm

$$\begin{aligned}(P \circ Q)(X) &= \operatorname{Im}(F \circ G)_{\pi[X]} \\ &= \operatorname{Im} \sum_{A} \left(\sum_{\omega \in \mathrm{Com}_A X} \bigotimes_{a \in \# A} \bigodot_{(a,Z) \in \omega} G_{\pi[Z]} \right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\# A})} \\ &= \operatorname{Im} \sum_{A} \left(\sum_{\omega \in \mathrm{Com}_A X} \bigotimes_{a \in \# A} \bigodot_{(a,Z) \in \omega} \pi_{G_Z^\dagger(\mathbf{B}|_{\# Z})} \right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\# A})} \\ &\cong \bigoplus_{A} \left(\bigoplus_{\omega \in \mathrm{Com}_A X} \bigodot_{(a,Z) \in \omega} Q(Z) \right) \otimes P(A),\end{aligned}$$

with certain relations divided away, as seen in the following theorem.

THEOREM 4. — *The plethysm of two Schur modules P and Q is given by*

$$(P \circ Q)(X) = \bigoplus_{A \in \mathfrak{MSet}} \left(\bigoplus_{\omega \in \mathrm{Com}_A X} \bigodot_{(a,Z) \in \omega} Q(Z) \right) \otimes P(A),$$

which is a quotient by all relations

$$\left(\bigodot_{(b,Z) \in \omega} w_Z \right) \otimes P(\nu)(x) = \left(\bigodot_{(a,Z) \in \omega\nu} w_Z \right) \otimes x,$$

for any elements $x \in P(A)$, $w_Z \in Q(Z)$, B-composition ω, and multation $\nu \colon A \to B$.

For a multation $\mu \colon X \to Y$,

$$(P \circ Q)(\mu) = \sum_{A} \left(\sum_{\xi \in \mathrm{Com}_A \mu} \bigodot_{(a,\nu) \in \xi} Q(\nu) \right) \otimes 1_{P(A)}.$$

Proof. Only the last part remains:

$$\begin{aligned}(P \circ Q)(\mu) &= (F \circ G)_{\sigma[\mu]} \\ &= \sum_{A} \left(\sum_{\xi \in \mathrm{Com}_A \mu} \bigotimes_{a \in \# A} \bigodot_{(a,\nu) \in \xi} G_{\sigma[\nu]} \right) \otimes 1_{F_A^\dagger(\mathbf{B}|_{\# A})} \\ &\cong \sum_{A} \left(\sum_{\xi \in \mathrm{Com}_A \mu} \bigodot_{(a,\nu) \in \xi} Q(\nu) \right) \otimes 1_{P(A)}.\end{aligned}$$

□

Example 9. — As an example of a relation of the above kind, we take

$$X = \{1,1,2\}$$
$$A = \{a,a,b\}$$
$$B = \{c,d,d\},$$

and

$$\nu = \begin{bmatrix} a & a & b \\ c & d & d \end{bmatrix}, \qquad \omega = \begin{bmatrix} c & d & d \\ \{1\} & \{1\} & \{2\} \end{bmatrix}.$$

Because

$$\omega\nu = 2\begin{bmatrix} a & a & b \\ \{1\} & \{1\} & \{2\} \end{bmatrix} + \begin{bmatrix} a & a & b \\ \{1\} & \{2\} & \{1\} \end{bmatrix},$$

we get

$$(w_1 \otimes w_1 w_2) \otimes P(\nu)(x) = 2(w_1^{[2]} \otimes w_2) \otimes x + (w_1 w_2 \otimes w_1) \otimes x,$$

where the left-hand side comes from

$$\bigodot_{(s,Z)\in\left[\begin{smallmatrix} c & d & d \\ \{1\} & \{1\} & \{2\} \end{smallmatrix}\right]} Q(Z) \otimes P(B) = Q(\{1\}) \otimes Q(\{1\}) \otimes Q(\{2\}) \otimes P(B),$$

and the right-hand side from

$$\left(\bigodot_{(s,Z)\in\left[\begin{smallmatrix} a & a & b \\ \{1\} & \{1\} & \{2\} \end{smallmatrix}\right]} Q(Z) \otimes P(A) \right) \oplus \left(\bigodot_{(s,Z)\in\left[\begin{smallmatrix} a & a & b \\ \{1\} & \{2\} & \{1\} \end{smallmatrix}\right]} Q(Z) \otimes P(A) \right)$$
$$= \Big(\Gamma^2(Q(\{1\})) \otimes Q(\{2\}) \otimes P(A)\Big) \oplus \Big(Q(\{1\}) \otimes Q(\{2\}) \otimes Q(\{1\}) \otimes P(A)\Big).$$

△

§5. Sur Operads

The theory of numerical functors and mazes allows for a further extension of the operad concept.

Definition 9. — A **labyrinth module** is a linear functor

$$\mathfrak{Laby} \to \mathfrak{Mod}.$$

◇

Recall the category equivalence

$$\Phi_{\mathfrak{Laby}}\colon \operatorname{Fun}(\mathfrak{XMod}, \mathfrak{Mod}) \to \operatorname{Fun}(\mathfrak{Laby}, \mathfrak{Mod}),$$

under which a module functor F corresponds to the labyrinth module

$$\Phi_{\mathfrak{Laby}}(F)\colon X \mapsto \operatorname{Nat}(\Delta^X, F).$$

Definition 10. — The **Sur functor** associated to a labyrinth module P is the module functor

$$\begin{aligned} \Phi^{-1}_{\mathfrak{Laby}}(P) \colon \mathfrak{XMod} &\to \mathfrak{Mod} \\ M &\mapsto \bigoplus_{X \in \mathfrak{Laby}} \Delta^X(M) \otimes P(X). \end{aligned}$$

◇

Definition 11. — A **Sur operad** is a labyrinth module with an associated monadic Sur functor. ◇

Quite analogous to the situation for Schur operads, we have the following result.

Theorem 5. — *Over any (unital) base ring, a Sur operad is equivalent to a monad.*

Proof. Immediate from the category equivalence above. □

For our purposes, it will not be necessary to restrict attention to *analytic* monads, but it might well be needed for any sensible applications.

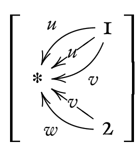
Example 10. — Sur operads manage to capture at least two additional classes of algebras. First, of course, there is the monad

$$N(M) = \mathbf{B}\binom{M}{-}$$

of free numerical rings. It gives the **operad of numerical algebras**. △

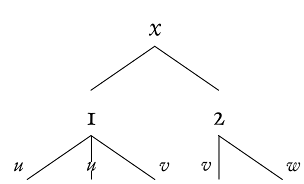
Example 11. — As another example, Sur operads succeed in encoding λ-rings. Constructing the free λ-ring on a module yields an analytic monad, which we call the **operad of λ-algebras**. △

Example 12. — Let us now describe the graphical version of Sur operads. An element



$$\otimes x \in \Delta^{\{1,2\}}(M) \otimes P(\{1,2\})$$

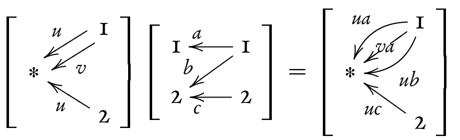
of the Sur functor should be thought of as a **qvast**[3]:



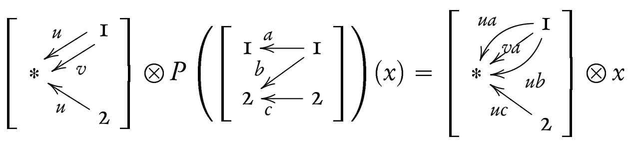
[3]The Swedish word *qvast* means *broom-stick*.

The qvast is, unlike the multi-tree, *not linear* in its vertices or edges. Rather, being a maze construct itself, it will inherit its properties from the labyrinth category, which we leave for the interested reader to find.

Let us, however, indicate an instance of the relation displayed in Theorem 10.7. Let $u, v \in M$, a *right* module over the possibly non-commutative ring $\mathbf{B}$, and let $x \in P(\{1,2\})$. Because



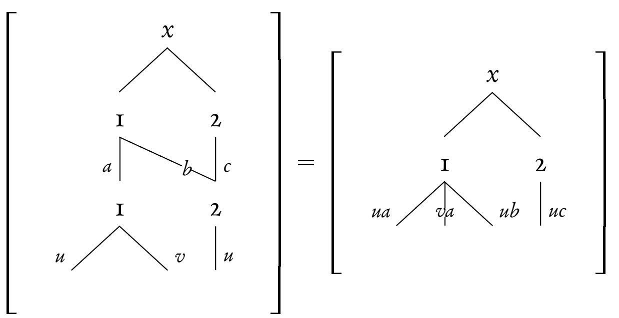
there will be a relation



on the Sur functor

$$\bigoplus_{X \in \mathfrak{Laby}} \Delta^X(M) \otimes P(X),$$

which will again translate into vertical associativity:



△

§6. The Induction Product of Sur Operads

Induction products and plethysms of labyrinth modules are defined as in the classical case.

Let us find the induction product. Consider two labyrinth modules P and Q, corresponding to Sur functors F and G, respectively. According to Theorem 7.2, the induction product $P \otimes Q$ maps

$$(P \otimes Q)(X) = (F \otimes G)^{\dagger}(\mathbf{B}|_X)$$

$$= \operatorname{Im}(F \otimes G)\left(\mathop{\Diamond}_{x \in X} \pi_x\right)$$
$$= \bigoplus_{A \cup B = X} \operatorname{Im}\left(F\left(\mathop{\Diamond}_{a \in A} \pi_a\right) \otimes G\left(\mathop{\Diamond}_{b \in B} \pi_b\right)\right)$$
$$= \bigoplus_{A \cup B = X} F^{\dagger}(\mathbf{B}|_A) \otimes G^{\dagger}(\mathbf{B}|_B) = \bigoplus_{A \cup B = X} P(A) \otimes Q(B).$$

Theorem 6. — *The induction product of two Sur modules P and Q is given by*

$$(P \otimes Q)(X) = \bigoplus_{A \cup B = X} P(A) \otimes Q(B).$$

For a maze $R\colon X \to Y$,

$$(P \otimes Q)(R) = \sum_{S \cup T = R} P(S) \otimes Q(T).$$

§7. The Plethysm of Sur Operads

One last definition, before we reach the end. When X is a set and $(M_y)_{y \in Y}$ is a (finite) family of modules, we shall let

$$\Delta^X((M_y)_{y \in Y}) \subseteq \Delta^X\left(\bigoplus_{y \in Y} M_y\right)$$

denote the module of deviations $[\Diamond_k u_k]$ such that every u_k belongs to some M_y, *each M_y being represented at least once*. Alternatively, this may be interpreted as the set of mazes $X \to *$, where the passages have been labelled with elements of the modules M_y, again with each module represented at least once.

Lemma 1. — *Let*

$$\pi_y\colon \bigoplus_{y \in Y} M_y \to \bigoplus_{y \in Y} M_y$$

be the canonical projections, and consider

$$\left(\bigcup_{y \in Y}\left\{ * \xrightarrow{\pi_y} * \right\}\right)_* : \Delta^X\left(\bigoplus_{y \in Y} M_y\right) \to \Delta^X\left(\bigoplus_{y \in Y} M_y\right).$$

Its image is

$$\operatorname{Im}\left(\bigcup_{y \in Y}\left\{ * \xrightarrow{\pi_y} * \right\}\right)_* = \Delta^X((M_y)_{y \in Y}).$$

From Theorem 10.8 we have the formula

$$(F \circ G)(\alpha_1 \diamond \cdots \diamond \alpha_n) = \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{J \lhd [n]} \left(\bigcup_{I \in J} \left\{ * \xrightarrow{G(\diamond_{i \in I} \alpha_i)} * \right\} \right)_* \otimes 1_{F^\dagger(\mathbf{B}|_A)},$$

which we use to compute the plethysm:

$$\begin{aligned} (P \circ Q)(X) &= \operatorname{Im}(F \circ G) \left(\mathop{\diamond}_{x \in X} \pi_x \right) \\ &= \operatorname{Im} \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{W \lhd X} \left(\bigcup_{Z \in W} \left\{ * \xrightarrow{G\left(\diamond_{z \in Z} \pi_z\right)} * \right\} \right)_* \otimes 1_{F^\dagger(\mathbf{B}|_A)} \\ &= \operatorname{Im} \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{W \lhd X} \left(\bigcup_{Z \in W} \left\{ * \xrightarrow{\pi_{G^\dagger(\mathbf{B}|_Z)}} * \right\} \right)_* \otimes 1_{F^\dagger(\mathbf{B}|_A)} \\ &= \bigoplus_{A \in \mathfrak{Lab}\mathfrak{y}} \bigoplus_{W \lhd X} \Delta^A \left((G^\dagger(\mathbf{B}|_Z))_{Z \in W} \right) \otimes F^\dagger(\mathbf{B}|_A) \\ &= \bigoplus_{A \in \mathfrak{Lab}\mathfrak{y}} \bigoplus_{W \lhd X} \Delta^A \left((Q(Z))_{Z \in W} \right) \otimes P(A), \end{aligned}$$

with certain relations divided away, as displayed in the following theorem.

Theorem 7. — *The plethysm of two Sur modules P and Q is given by*

$$(P \circ Q)(X) = \bigoplus_{A \in \mathfrak{Lab}\mathfrak{y}} \bigoplus_{W \lhd X} \Delta^A((Q(Z))_{Z \in W}) \otimes P(A),$$

which is a quotient by all relations

$$U \otimes P(R)(x) = UR \otimes x,$$

for any $W \lhd X$, elements $x \in P(A)$, $U \in \Delta^B((Q(Z))_{Z \in W})$, and maze $R \colon A \to B$.

For a maze $S \colon X \to Y$,

$$(P \circ Q)(S) = \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{E \lhd S} \left(\bigcup_{T \in E} \left\{ * \xrightarrow{Q(T)} * \right\} \right)_* \otimes 1_{P(A)}.$$

Proof. There remains only the last statement:

$$\begin{aligned} (P \circ Q)(S) &= (F \circ G) \left(\mathop{\diamond}_{[s \colon x \to y] \in S} \bar{s}\sigma_{yx} \right) \\ &= \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{E \lhd S} \left(\bigcup_{T \in E} \left\{ * \xrightarrow{G\left(\diamond_{[s \colon x \to y] \in T} \bar{s}\sigma_{yx}\right)} * \right\} \right)_* \otimes 1_{F^\dagger(\mathbf{B}|_A)} \\ &= \sum_{A \in \mathfrak{Lab}\mathfrak{y}} \sum_{E \lhd S} \left(\bigcup_{T \in E} \left\{ * \xrightarrow{Q(T)} * \right\} \right)_* \otimes 1_{P(A)}. \end{aligned}$$

□

Example 13. — From the classical case of operads and trees, the graphical interpretation of the induction product and the plethysm is clear. The induction product corresponds to a two-stemmed tree, and the plethysm to a grafted tree. These interpretations remain valid for multi-trees and qvastar. △

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