ALGEBRAIC STRUCTURES

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1. (a) —

   (b)
   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 9 & 3 & 1 & 2 & 6 & 8 & 7 \end{pmatrix} = (14397625).$$

   (c) $\sigma = (15)(12)(16)(17)(19)(13)(14)$ is odd.

   (d) σ is a cycle of length 8, hence it is of even order 8.

   (e) No, since $\sigma = \pi^2$ has order 8, π would have to have order 16 or more. There are no such permutations in S_9.

2. (a) —

   (b)
   $$p(x) = 1 + x + x^2 + x^3 = (1 + x)(1 + x^2),$$
   where $1 + x^2$ is irreducible since it has no zeroes.

   (c) Polynomials (residue classes) in $\mathbf{Z}_3[x]/(p(x))$ add and multiply as usual, modulo the relation $x^3 = -1 - x - x^2$. There will be zero divisors, for $(1 + x)(1 + x^2) = 0$.

3. (a) —

   (b) By the Fundamental Theorem of Finitely Generated Abelian Groups, there are exactly two groups of order $2012 = 2^2 \cdot 503$:
   $$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{503} \qquad \text{and} \qquad \mathbf{Z}_4 \times \mathbf{Z}_{503}.$$

4. (a) —

   (b) Let n be the least positive integer with $x^n = 0$. If $n = 1$, then $x = 0$. If $n \geqslant 2$, then $0 = x^n = x \cdot x^{n-1}$, and both $x, x^{n-1} \neq 0$.

(c) If x is nilpotent with $x^n = 0$ and r is arbitrary, then $(rx)^n = r^n x^n = 0$, so also rx is nilpotent. If x and y are both nilpotent with $x^n = y^n = 0$, then

$$(x+y)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k y^{2n-k} = 0,$$

since either k or $2n-k$ is at least n.

(d) The relation

$$0 + N = (x+N)^n = x^n + N$$

implies $x^n \in N$, which in turn implies $(x^n)^m = 0$ for some m. Hence x itself is nilpotent, and therefore $x \in N$, so that $x + N = 0 + N$.

5. (a) —

(b) Obviously

$$(g,g) \times (h,h) = (gh, gh) \in D$$
$$(1,1) \in D$$
$$(g,g)^{-1} = (g^{-1}, g^{-1}) \in D.$$

(c) Every subgroup of a normal subgroup is normal. The map

$$\varphi\colon G \to (G \times G)/D, \qquad x \mapsto (x,1)D$$

is homomorphic:

$$\varphi(x)\varphi(y) = (x,1)D \cdot (y,1)D = (xy,1)D = \varphi(xy).$$

It is injective since $(x,1)D = (1,1)D$ implies $(x,1) \in D$, and therefore $x = 1$. It is surjective since

$$(a,b)D = (ab^{-1},1)(b,b)D = (ab^{-1},1)D.$$

Hence it is an isomorphism, whence $G \cong (G \times G)/D$.

(d) Choose $G = S_3$. Then

$$((12),(23)) \cdot ((12),(12)) \cdot ((12),(23))^{-1} = ((12),(13)) \notin D.$$

6. (a) —

(b) One finds the multiplication table in Table 1.

(c) T is clearly commutative, since the elements 1, i, j, k all commute. T has zero divisors, for example $(1+j)(1-j) = 1 - j^2 = 0$.

(d) Define the isomorphism $T \to \mathbf{C} \times \mathbf{C}$ by

$$1 \mapsto (1,1) \quad i \mapsto (i,i) \quad j \mapsto (1,-1) \quad k \mapsto (i,-i).$$

Since these four elements constitute a basis for $\mathbf{C} \times \mathbf{C}$, and they satisfy the relations the tessarines do, this will be an isomorphism.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	k	1	i
k	k	$-j$	i	-1

TABLE 1: Multiplication table for the Tessarines.

7. (a) —

   (b) One calculates

$$\alpha^2 + 1 = 2i\sqrt{6}$$
$$(\alpha^2 + 1)^2 = -24,$$

   so that α is a zero of the polynomial $p(x) = (x^2 + 1)^2 + 24 = x^4 + 2x^2 + 25$.

   It has no integral zeroes since $p(x) > 0$ for real x (alternatively, use the Rational Root Theorem; the only possibilities are $\pm 1, \pm 5, \pm 25$).

   There are no integral quadratic factors, since the three numbers 1, $\alpha = \sqrt{2} + i\sqrt{3}$, and $\alpha^2 = 2i\sqrt{6} - 1$ obviously cannot be rationally combined to form 0. (Alternatively, one may consider the equation

$$x^4 + 2x^2 + 25 = (x^2 + ax + b)(x^2 + cx + d),$$

   which has no integral solutions a, b, c, d.)

   Hence $p(x)$ is the minimal polynomial of α.

   (c) $[\mathbf{Q}(\alpha) : \mathbf{Q}] = \deg p(x) = 4$.

   (d) Since $\mathbf{Q}(\alpha)$ contains $\sqrt{2} + i\sqrt{3}$ and

$$\frac{1}{\sqrt{2} + i\sqrt{3}} = \frac{1}{5}(\sqrt{2} - i\sqrt{3}),$$

   it will contain $\sqrt{2}$ and $i\sqrt{3}$. Consequently,

$$\mathbf{Q}(\alpha) = \mathbf{Q}(\sqrt{2}, i\sqrt{3})$$

   and all the conjugates of α, namely $\pm\sqrt{2} \pm i\sqrt{3}$, will be included. This shews $\mathbf{Q}(\alpha)$ is the splitting field of $p(x)$, and therefore a Galois extension.

   To find the Galois group, we observe that an automorphism must map $\sqrt{2} \mapsto \pm\sqrt{2}$ and $i\sqrt{3} \mapsto \pm i\sqrt{3}$. Since the extension has degree 4, the Galois group has order 4, and so these four combinations are all valid, making the automorphism group isomorphic with the Klein four-group.